ANDERSON INEQUALITY IS STRICT FOR GAUSSIAN AND STABLE MEASURES

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Abstract. Let $\mu$ be a symmetric Gaussian measure on a separable Banach space $(E, \| \cdot \|)$. Denote $U = \{ x : \| x \| < 1 \}$. Then for every $x \in \text{supp } \mu$, $x \neq 0$, the function $t \mapsto \mu(U + tx)$ is strictly decreasing for $t \in (0, \infty)$. The same property holds for symmetric $p$-stable measures on $E$. Using this property we answer a question of W. Linde: if $\int_{U+z} x \, d\mu(x) = 0$, then $z = 0$.

1. Notation and basic properties of Gaussian measures

We start by recalling some basic notation and facts concerning Gaussian measures on Banach spaces. For the proofs the reader may consult [LePage] or [Bor].

Throughout the paper $(E, \| \cdot \|)$ denotes a separable Banach space. Let $\mu$ be a symmetric Gaussian measure on $E$. By $\text{supp } \mu$ we denote the support of $\mu$ which is a linear subspace of $E$. Let us mention that $x \in \text{supp } \mu$ if and only if $\mu(\{ y \in E : \| y - x \| < \varepsilon \} > 0$ for every $\varepsilon > 0$. Let $E^*_\mu(\mu)$ denote a closure of $E^*$ (topological dual of $E$) in $L_2(\mu)$. For every $f \in E^*_\mu(\mu)$ we define an operator $Q$ by the formula $Qf = \int f(x) \, x \, d\mu(x)$. Then $Q$ maps $E^*_\mu(\mu)$ onto a subspace of $E$ which we denote $H_\mu$. Let $Q' : E^* \to E^*_\mu(\mu)$ be the natural injection. Then the covariance operator $R$ of $\mu$ is defined by the following formula: $R = QQ' : E^* \to H_\mu$. For every $h \in H_\mu$ there exists the unique element $\hat{h} \in E^*_\mu(\mu)$ such that $Q\hat{h} = h$. Then the formula $\langle h_1, h_2 \rangle = \int_{E} \hat{h}_1(x) \hat{h}_2(x) \, d\mu(x)$ defines a scalar product in $H_\mu$ and this, in turn, defines a norm on $H_\mu$: $\| h \|_\mu = \langle h, h \rangle^{\frac{1}{2}}$. $H_\mu$ equipped with this norm is a Hilbert space, we call it the reproducing kernel Hilbert space (RKHS) of $\mu$.

$H_\mu$ is dense in the support of $\mu$ and consists of the admissible translates of the measure $\mu$. For $y \in E$ let us denote by $\mu_y$ the translated measure defined by $\mu_y(\cdot) = \mu(\cdot + y)$. There holds the following important fact.

Proposition 1 (Cameron-Martin formula [C-M]). For each $h \in H_\mu$ the measure $\mu_h$ is absolutely continuous with respect to $\mu$ and for every measurable set $A$ we
have
\[ \mu_h(A) = \exp\left(-\frac{1}{2} \|h\|^2 \right) \int_A e^{-h(x)} \, d\mu(x). \]

Applying the above proposition we obtain the following corollary.

**Corollary 1.** Let \( f \in E^* \) with \( \int f^2(x) \, d\mu(x) = 1 \). Set \( z_0 = Rf \in H_\mu \) and \( \gamma = \mathcal{L}(gz_0 + X) \), where \( \mathcal{L}(X) = \mu \) and \( g \) is the standard Gaussian random variable on the real line, independent of \( X \). Then \( \gamma \) is absolutely continuous with respect to \( \mu \) and
\[ \frac{d\gamma}{d\mu}(y) = \frac{1}{\sqrt{2}} \exp\left(\frac{f^2(y)}{4}\right). \]

The following fact may be derived from the so-called log-concavity property of Gaussian measures (see e.g. [Bor], Theorem 3.2), but it was proved for the first time by Anderson [And] for unimodal distributions on \( \mathbb{R}^n \), hence we call it the Anderson property of Gaussian measures.

**Proposition 2** (Anderson property of Gaussian measures). If \( \mu \) is a symmetric Gaussian measure on \( E \) and \( C \) is a convex, symmetric measurable subset of \( E \), then for every \( x \in E \)
\[ \mu(C) \geq \mu(C + x). \]

## 2. Main result

Our main result states that the above inequality must be strict if only \( x \neq 0 \).

We precede the proof of this fact with a lemma. Let us denote \( U_t = \{ x \in E : \|x\| < t \} \) and \( U_1 = U \).

**Lemma 1.** Suppose that \( x_0 \neq 0 \) is such that \( \left( (U + x_0) \setminus U \right) \cap \text{supp} \mu \neq \emptyset \). Then there exists a functional \( f \in E^* \) such that for some \( u_0 > 0 \):
\[ U \subset \{ |f| < u_0 \}, \quad \mu((U + x_0) \cap \{ |f| < u_0 \}) > 0 \text{ and } \int_E f^2(x) \, d\mu(x) = 1. \]

**Proof.** Let \( y \in \left( (U + x_0) \setminus U \right) \cap \text{supp} \mu \). Since \( (U + x_0) \setminus U \) is open, there exists a closed ball \( \overline{U}_{\{y\}} \) with center at \( y \) such that \( \overline{U}_{\{y\}} \subset \left( (U + x_0) \setminus U \right) \). Now the conclusion of the lemma yields from the convexity of \( \overline{U} \) and \( \overline{U}_{\{y\}} \), the Hahn-Banach theorem and the fact that \( \mu(U_{\{y\}}) > 0 \).

Now we are able to state the main result of this paper.

**Theorem 1.** Let \( x_0 \in E \). Then \( \mu(U + x_0) = \mu(U) \) if and only if
\[ \mu((U + x_0) \setminus U)) + \mu(U \setminus (U + x_0)) = 0. \]

Before proving Theorem 1 we derive an important corollary.

**Corollary 2.** Let \( x_0 \neq 0 \). Then \( \mu(U + x_0) < \mu(U) \) in the following cases:

1. \( x_0 \in \text{supp} \mu \).
2. \( \| \cdot \| \) is strictly convex.
Proof of the corollary. (1) It is clear that for some $\lambda > 1$ we have
\[ \|\lambda x_0 - x_0\| = (\lambda - 1)\|x_0\| < 1 \text{ and } \lambda\|x_0\| > 1. \]
Therefore $\lambda x_0 \in (U + x_0) \setminus \overline{U}$. Since $\text{supp } \mu$ is a linear subspace of $E$, we get $\mu((U + x_0) \setminus \overline{U}) > 0$.

(2) There exists an $y \in \partial U \cap \text{supp } \mu$. (If it were not true, then $\mu$ would be $\delta_0$.) If $y \notin \partial(U + x_0)$, then either $\lambda y \in (U + x_0) \setminus \overline{U}$ for some $\lambda > 1$ or $\lambda y \in U \setminus (\overline{U} + x_0)$ for some $\lambda < 1$. In both cases $\lambda y \in \text{supp } \mu \cap [(U \setminus (U + x_0)) \cup (U + x_0) \setminus \overline{U})]$ and the last set has positive measure. Now we assume that $y \in \partial U \cap \partial(U + x_0)$. From the strict convexity of the norm $\|y + x_0\| + \|y - x_0\| > 2\|y\| = 2$ and then $\|x_0 + y\| > 1$. Clearly, for some $\lambda$, $0 < \lambda < 1$, we have $\|\lambda y - x_0\| > 1$ and $\|\lambda y\| = \lambda < 1$. This means that $-\lambda y \in (U \setminus (U + x_0)) \cap \text{supp } \mu$. Hence $\mu(U \setminus (U + x_0)) > 0$.

Proof of Theorem 1. Suppose that $\mu((U + x_0) \setminus U) > 0$. Then $(U + x_0) \setminus \text{supp } \mu \neq \emptyset$ and, by Lemma 1, we find an $f \in E^*$ such that $\int_E f^2(x) \, d\mu(x) = 1$ and for some $u_0 > 0$
\[ (1) \quad U \subset \{|f| \leq u_0\} \text{ and } \mu((U + x_0) \cap \{|f| > u_0\}) > 0. \]

Let $y$ be a symmetric Gaussian measure defined in Corollary 1, and let $h(y) = \frac{d}{d\mu}(y) = \frac{1}{\sqrt{2}} \exp\left(\frac{L^2\|y\|}{4}\right)$. Denote $D_t = \{y \in E : h(y) \leq t\}$, $t > 0$. From the form of $h$ it is clear that $D_t \subset D_{t'}$ for $t \leq t'$; $D_t$ are symmetric, convex sets (strips in $E$) and there exists some $T_0 > 0$ such that
\[ (2) \quad (U + x_0) \subset D_{T_0} \text{ and } U \subset D_{T_0}. \]

Let us consider the following distribution functions:
\[ F_{x_0}(t) = \mu((U + x_0) \cap D_t) \text{ and } F(t) = \mu(U \cap D_t), \quad t > 0. \]

From the convexity of $D_t$ and $U$ it follows that
\[ \frac{1}{2}((U + x_0) \cap D_t) + \frac{1}{2}((U - x_0) \cap D_t) \subset U \cap D_t, \]
and then we can apply the log-concavity property ([Bor], Theorem 3.2) to conclude that
\[ \mu(U \cap D_t) \geq \mu^\lambda((U + x_0) \cap D_t)\mu^{1\lambda}((U - x_0) \cap D_t). \]

Next, by the symmetry of $U$ and $D_t$, the last statement is equivalent to
\[ (3) \quad F_{x_0}(t) \leq F(t). \]

Using Proposition 2 and Corollary 1 and integrating by parts we get
\[ 0 \geq \gamma(U + x_0) - \gamma(U) = \int_{U + x_0} h(y) \, d\mu(y) - \int_U h(y) \, d\mu(y) \]
\[ = \left[ t \, dF_{x_0}(t) \right]_0^T_0 - \left[ t \, dF(t) \right]_0^T_0 \]
\[ = T_0[F_{x_0}(T_0) - F(T_0)] - \int_0^{T_0} (F_{x_0}(t) - F(t)) \, dt. \]
Knowing that $F_{x_0}(T_0) = \mu(U + x_0)$ and $F(T_0) = \mu(U)$ we can rewrite the last inequality as
\[ \mu(U) - \mu(U + x_0) \geq \frac{1}{T_0} \int_0^{T_0} (F(t) - F_{x_0}(t)) \, dt. \]

From (1) and (3) it is clear that the integral must be strictly positive. To complete the proof we have to consider the situation when $\mu(U \setminus (U + x_0)) > 0$ and $\mu((U + x_0) \setminus U) = 0$. But then
\[ \mu(U) - \mu(U + x_0) = \mu(U) - \mu(U \setminus (U + x_0)) > 0. \]
The proof is complete.

The same property holds for measures which are mixtures of Gaussian ones. For example we have the following.

**Corollary 3.** Let $\mu$ be a symmetric $p$-stable measure on $E$. Then for every $x \in \text{supp} \, \mu$ we have
\[ \mu(U) > \mu(U + x). \]

**Proof.** By the well-known representation of symmetric stable measures as a mixture of Gaussian (compare e.g. [LP-W-Z] or [Szt]) we have for measurable set $A$: $\mu(A) = \int_T \gamma_t(A) \, dm(t)$, where $\gamma_t$ are symmetric Gaussian, $m$ is a finite measure on some measurable space $T$, and $\text{supp} \, \gamma_t = \text{supp} \, \mu$ for $m$-almost all $t$. Because $\gamma_t(U) - \gamma_t(U + x) > 0$, it follows that $\mu(U) - \mu(U + x) > 0$ as we claimed.

**Remark.** In order to get the strict Anderson inequality we must assume that the translate $x$ or the norm $\| \cdot \|$ have some additional properties.

**Example.** Let $E = \mathbb{R}^2$ be equipped with the maximum norm $\|(x, y)\| = \max(|x|, |y|)$. Let $\mu$ be the one-dimensional standard Gaussian measure that is regarded as a measure on $\mathbb{R}^2$ and has the axis $Ox$ as its support. Then for every $t$, $0 < t < 1$, $\mu(U_1 + (0, t)) = \mu(U_1)$, because $U_1 \cap \text{supp} \, \mu = [U_1 + (0, t)] \cap \text{supp} \, \mu$. Observe that in this example neither the norm is strictly convex nor $(0, t)$ belongs to the $\text{supp} \, \mu$. However, when $z$ has a non-zero second coordinate, then $\mu(U + z) < \mu(U)$.

3. Solution of a problem of Linde

In his paper [Lin] Linde examined the smoothness properties of the function $x \to \mu(U_s + x)$ for $\mu$ Gaussian. Namely, he showed that this function is Gateaux differentiable at every $x \in \text{supp} \, \mu$, that is, there exists a continuous linear functional $d(s, x)(\cdot)$ such that
\[ d(s, x)(y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mu(U_s + x + \varepsilon y) - \mu(U_s + x)]. \]

By virtue of the log-concavity, in the separable Banach spaces the Gaussian measure of $\partial(U_s + x)$ is zero for every $x \in \text{supp} \, \mu$ and $s > 0$ (compare [HJ-S-D] for the proof of this fact if $x = 0$), hence Linde's result is also valid for $U_s$ instead of $\overline{U_s}$. Linde showed that the differential $d(s, x)(\cdot)$ is non-trivial.
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if \( \|x\| > s \) and asked about the case \( \|x\| \leq s \). Now we show that \( d(s, x) \) is a non-zero functional if only \( x \neq 0 \), what is more, we show that \( d(s, x)(x) < 0 \).

The next theorem answers in positive the question of Linde we mentioned earlier.

**Theorem 2.** In the above setting, for every \( x \in \text{supp} \mu, \ x \neq 0 \),

\[
d(s, x)(x) < 0.
\]

**Proof.** Consider a function \( f_x(t) = \frac{1}{\mu(U_s + tx)} \) for \( x \neq 0 \). Then \( f_x \) is even, convex and strictly increasing on \((0, \infty)\). It is clear that \( f_x \) is even and tends to infinity as \( t \to \infty \). By log-concavity of the measure \( \mu \) we conclude that the function \( t \to \log \mu(U_s + tx) \) is concave, hence \( \frac{1}{\mu(U_s + tx)} = \exp(-\log \mu(U_s + tx)) \) is convex. But Theorem 1 implies that \( f_x(t) > f_x(0) \) for every \( t > 0 \), hence \( f_x \) is strictly increasing (because it is convex).

Next, by easy computations we get for \( x \in \text{supp} \mu \):

\[
d(s, x)(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\mu(U_s + x + \epsilon x) - \mu(U_s + x)] = (f_x(t)^{-1})_{t=1}.
\]

But \( f_x \) is strictly increasing and convex, hence \( (f_x)_{t=1} > 0 \), which implies that

\[
d(s, x)(x) = -\frac{f_x'(1)}{f_x^2(1)} < 0.
\]

As it was shown in Linde's paper [Lin] it is easy to compute \( d(s, z)(h) \) for \( h \in H(p) \). Namely, using the Cameron-Martin formula (Proposition 1) we get

\[
d(s, z)(h) = -\int_{U_s + z} h(x) d\mu(x) = -h(\int_{U_s + z} x d\mu(x)).
\]

Now we show that for Gaussian measure \( \mu \) the condition \( \int_{U_s + z} x d\mu(x) = 0 \) implies \( z = 0 \).

**Theorem 3.** Let \( \mu \) be a symmetric Gaussian measure on \( E \). If \( z \in \text{supp} \mu \), then \( \int_{U_s + z} x d\mu(x) = 0 \) implies \( z = 0 \).

**Proof.** Arguing as at the beginning of the proof of Theorem 2 we infer that the function \( g_h(t) = \frac{1}{\mu(U_s + z + th)} \) is convex for all \( h \in H_\mu \). If \( z \neq 0 \), then, by Theorem 1, \( \mu(U_s + z) < \mu(U_s) \), hence for at least one \( h_0 \in H_\mu \), the derivative \( g_{h_0}'(0) \) is not equal to zero ( \( g_{h_0} \) attains its minimum at some \( t \neq 0 \)), so that

\[
0 \neq d(s, z)(h_0) = -h_0(\int_{U_s + z} x d\mu(x)) \quad \text{which, of course, is equivalent to the condition} \quad \int_{U_s + z} x d\mu(x) \neq 0.
\]

**References**


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