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ANDERSON INEQUALITY IS STRICT FOR GAUSSIAN AND STABLE MEASURES

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ABSTRACT. Let μ be a symmetric Gaussian measure on a separable Banach space $(E, \|\cdot\|)$. Denote $U = \{x : \|x\| < 1\}$. Then for every $x \in \operatorname{supp} \mu$, $x \neq 0$, the function $t \to \mu(U + tx)$ is strictly decreasing for $t \in (0, \infty)$. The same property holds for symmetric p-stable measures on E. Using this property we answer a question of W. Linde : if $\int_{U+z} x \, d\mu(x) = 0$, then $z = 0$.

1. NOTATION AND BASIC PROPERTIES OF GAUSSIAN MEASURES

We start by recalling some basic notation and facts concerning Gaussian measures on Banach spaces. For the proofs the reader may consult [LePage] or [Bor] .

Throughout the paper $(E, \|\cdot\|)$ denotes a separable Banach space. Let μ be a symmetric Gaussian measure on E. By $\operatorname{supp} \mu$ we denote the support of μ which is a linear subspace of E. Let us mention that $x \in \operatorname{supp} \mu$ if and only if $\mu\{y \in E : \|y - x\| < \varepsilon\} > 0$ for every $\varepsilon > 0$. Let $E_2^*(\mu)$ denote a closure of E^* (topological dual of E) in $L_2(\mu)$. For every $f \in E_2^*(\mu)$ we define an operator Q by the formula $Qf = \int_E f(x)\, x \, d\mu(x)$. Then Q maps $E_2^*(\mu)$ onto a subspace of E which we denote H_μ. Let $Q' : E^* \to E_2^*(\mu)$ be the natural injection. Then the covariance operator R of μ is defined by the following formula: $R = QQ' : E^* \to H_\mu$. For every $h \in H_\mu$ there exists the unique element $\tilde{h} \in E_2^*(\mu)$ such that $Q\tilde{h} = h$. Then the formula $\langle h_1, h_2 \rangle = \int_E \tilde{h}_1(x)\, \tilde{h}_2(x)\, d\mu(x)$ defines a scalar product in H_μ and this, in turn, defines a norm on H_μ: $\|h\|_\mu = \langle h, h \rangle^{\frac{1}{2}}$. H_μ equipped with this norm is a Hilbert space, we call it the reproducing kernel Hilbert space (RKHS) of μ. H_μ is dense in the support of μ and consists of the admissible translates of the measure μ. For $y \in E$ let us denote by μ_y the translated measure defined by $\mu_y(\cdot) = \mu(\cdot + y)$. There holds the following important fact.

Proposition 1 (Cameron-Martin formula [C-M]). *For each $h \in H_\mu$ the measure μ_h is absolutely continuous with respect to μ and for every measurable set A we*

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have

$$\mu_h(A) = \exp(-\frac{1}{2}\|h\|_\mu^2) \int_A e^{-\tilde{h}(x)}\, d\mu(x).$$

Applying the above proposition we obtain the following corollary.

Corollary 1. *Let* $f \in E^*$ *with* $\int_E f^2(x)\, d\mu(x) = 1$. *Set* $z_0 = Rf \in H_\mu$ *and* $\gamma = \mathscr{L}(gz_0 + X)$, *where* $\mathscr{L}(X) = \mu$ *and g is the standard Gaussian random variable on the real line, independent of X. Then* γ *is absolutely continuous with respect to* μ *and*

$$\frac{d\gamma}{d\mu}(y) = \frac{1}{\sqrt{2}} \exp(\frac{f^2(y)}{4}).$$

The following fact may be derived from the so-called log-concavity property of Gaussian measures (see e.g. [Bor], Theorem 3.2), but it was proved for the first time by Anderson [And] for unimodal distributions on R^n, hence we call it *the Anderson property* of Gaussian measures.

Proposition 2 (Anderson property of Gaussian measures). *If* μ *is a symmetric Gaussian measure on* E *and* C *is a convex, symmetric measurable subset of* E, *then for every* $x \in E$

$$\mu(C) \geq \mu(C + x).$$

2. Main result

Our main result states that the above inequality must be strict if only $x \neq 0$. We precede the proof of this fact with a lemma. Let us denote $U_t = \{x \in E : \|x\| < t\}$ and $U_1 = U$.

Lemma 1. *Suppose that* $x_0 \neq 0$ *is such that* $((U + x_0) \setminus U) \cap \operatorname{supp} \mu \neq \varnothing$. *Then there exists a functional* $f \in E^*$ *such that for some* $u_0 > 0$:

$$U \subset \{|f| < u_0\}, \quad \mu((U + x_0) \cap \{|f| < u_0\}) > 0 \textit{ and } \int_E f^2(x)\, d\mu(x) = 1.$$

Proof. Let $y \in ((U + x_0) \setminus \overline{U}) \cap \operatorname{supp} \mu$. Since $(U + x_0) \setminus \overline{U}$ is open, there exists a closed ball $\overline{U}_{\{y\}}$ with center at y such that $\overline{U}_{\{y\}} \subset ((U + x_0) \setminus \overline{U})$. Now the conclusion of the lemma yields from the convexity of $\overline{U}$ and $\overline{U}_{\{y\}}$, the Hahn-Banach theorem and the fact that $\mu(\overline{U}_{\{y\}}) > 0$.

Now we are able to state the main result of this paper.

Theorem 1. *Let* $x_0 \in E$. *Then* $\mu(U + x_0) = \mu(U)$ *if and only if*

$$\mu((U + x_0) \setminus U)) + \mu(U \setminus (U + x_0)) = 0.$$

Before proving Theorem 1 we derive an important corollary.

Corollary 2. *Let* $x_0 \neq 0$. *Then* $\mu(U + x_0) < \mu(U)$ *in the following cases*:

(1) $x_0 \in \operatorname{supp} \mu$.
(2) $\|\cdot\|$ *is strictly convex.*

Proof of the corollary. (1) It is clear that for some $\lambda > 1$ we have

$$\|\lambda x_0 - x_0\| = (\lambda - 1)\|x_0\| < 1 \textit{ and } \lambda\|x_0\| > 1.$$

Therefore $\lambda x_0 \in (U + x_0) \setminus \overline{U}$. Since $\operatorname{supp}\mu$ is a linear subspace of E, we get $\mu((U + x_0) \setminus U) > 0$.

(2) There exists an $y \in \partial U \cap \operatorname{supp}\mu$. (If it were not true, then μ would be δ_0.) If $y \notin \partial(U + x_0)$, then either $\lambda y \in (U + x_0) \setminus \overline{U}$ for some $\lambda > 1$ or $\lambda y \in U \setminus (\overline{U + x_0})$ for some $\lambda < 1$. In both cases $\lambda y \in \operatorname{supp}\mu \cap [(U \setminus (\overline{U + x_0})) \cup (U + x_0) \setminus \overline{U})]$ and the last set has positive measure. Now we assume that $y \in \partial U \cap \partial(U + x_0)$. From the strict convexity of the norm $\|y + x_0\| + \|y - x_0\| > 2\|y\| = 2$ and then $\|x_0 + y\| > 1$. Clearly, for some λ, $0 < \lambda < 1$, we have $\| - \lambda y - x_0\| > 1$ and $\| - \lambda y\| = \lambda < 1$. This means that $-\lambda y \in (U \setminus \overline{(U + x_0)}) \cap \operatorname{supp}\mu$. Hence $\mu(U \setminus (U + x_0)) > 0$.

Proof of Theorem 1. Suppose that $\mu((U + x_0) \setminus U) > 0$. Then $((U + x_0) \setminus \overline{U}) \cap \operatorname{supp}\mu \neq \varnothing$ and, by Lemma 1, we find an $f \in E^*$ such that $\int_E f^2(x)\, d\mu(x) = 1$ and for some $u_0 > 0$

$$U \subset \{|f| \leq u_0\} \text{ and } \mu((U + x_0) \cap \{|f| > u_0\}) > 0. \tag{1}$$

Let γ be a symmetric Gaussian measure defined in Corollary 1, and let $h(y) = \frac{d\gamma}{d\mu}(y) = \frac{1}{\sqrt{2}}\exp(\frac{f^2(y)}{4})$. Denote $D_t = \{y \in E : h(y) \leq t\}$, $t > 0$. From the form of h it is clear that $D_t \subset D_{t'}$ for $t \leq t'$; Dt are symmetric, convex sets (strips in E) and there exists some $T_0 > 0$ such that

$$(U + x_0) \subset D_{T_0} \text{ and } U \subset D_{T_0}. \tag{2}$$

Let us consider the following distribution functions:

$$F_{x_0}(t) = \mu((U + x_0) \cap D_t) \text{ and } F(t) = \mu(U \cap D_t), \quad t > 0.$$

From the convexity of D_t and U it follows that

$$\frac{1}{2}((U + x_0) \cap D_t) + \frac{1}{2}((U - x_0) \cap D_t) \subset U \cap D_t,$$

and then we can apply the log-concavity property ([Bor], Theorem 3.2) to conclude that

$$\mu(U \cap D_t) \geq \mu^{\frac{1}{2}}((U + x_0) \cap D_t)\mu^{\frac{1}{2}}((U - x_0) \cap D_t).$$

Next, by the symmetry of U and D_t, the last statement is equivalent to

$$F_{x_0}(t) \leq F(t). \tag{3}$$

Using Proposition 2 and Corollary 1 and integrating by parts we get

$$\begin{aligned} 0 \geq \gamma(U + x_0) - \gamma(U) &= \int_{U + x_0} h(y)\, d\mu(y) - \int_U h(y)\, d\mu(y) \\ &= \int_0^{T_0} t\, dF_{x_0}(t) - \int_0^{T_0} t\, dF(t) \\ &= T_0[F_{x_0}(T_0) - F(T_0)] - \int_0^{T_0} (F_{x_0}(t) - F(t))\, dt. \end{aligned}$$

Knowing that $F_{x_0}(T_0) = \mu(U + x_0)$ and $F(T_0) = \mu(U)$ we can rewrite the last inequality as

$$\mu(U) - \mu(U + x_0) \geq \frac{1}{T_0} \int_0^{T_0} (F(t) - F_{x_0}(t))\, dt.$$

From (1) and (3) it is clear that the integral must be strictly positive. To complete the proof we have to consider the situation when $\mu(U \setminus (U + x_0)) > 0$ and $\mu((U + x_0) \setminus U) = 0$. But then

$$\mu(U) - \mu(U + x_0) = \mu(U) - \mu(U \setminus (U + x_0)) > 0.$$

The proof is complete.

The same property holds for measures which are mixtures of Gaussian ones. For example we have the following.

Corollary 3. *Let μ be a symmetric p-stable measure on E. Then for every $x \in \operatorname{supp} \mu$ we have*

$$\mu(U) > \mu(U + x).$$

Proof. By the well-known representation of symmetric stable measures as a mixture of Gaussian (compare e.g. [LP-W-Z] or [Szt]) we have for measurable set A: $\mu(A) = \int_T \gamma_t(A)\, dm(t)$, where γ_t are symmetric Gaussian, m is a finite measure on some measurable space T, and $\operatorname{supp} \gamma_t = \operatorname{supp} \mu$ for m-almost all t. Because $\gamma_t(U) - \gamma_t(U + x) > 0$, it follows that $\mu(U) - \mu(U + x) > 0$ as we claimed.

Remark. In order to get the strict Anderson inequality we must assume that the translate x or the norm $\|\cdot\|$ have some additional properties.

Example. Let $E = R^2$ be equipped with the maximum norm $\|(x, y)\| = \max(|x|, |y|)$. Let μ be the one-dimensional standard Gaussian measure that is regarded as a measure on R^2 and has the axis Ox as its support. Then for every t, $0 < t < 1$, $\mu(U_1 + (0, t)) = \mu(U_1)$, because $U_1 \cap \operatorname{supp} \mu = [U_1 + (0, t)] \cap \operatorname{supp} \mu$. Observe that in this example neither the norm is strictly convex nor $(0, t)$ belongs to the $\operatorname{supp} \mu$. However, when z has a non-zero second coordinate, then $\mu(U + z) < \mu(U)$.

3. Solution of a problem of Linde

In his paper [Lin] Linde examined the smoothness properties of the function $x \to \mu(\overline{U}_s + x)$ for μ Gaussian. Namely, he showed that this function is Gateaux differentiable at every $x \in \operatorname{supp} \mu$, that is, there exists a continuous linear functional $d(s, x)(\cdot)$ such that

$$d(s, x)(y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mu(\overline{U}_s + x + \varepsilon y) - \mu(\overline{U}_s + x)].$$

By virtue of the log-concavity, in the separable Banach spaces the Gaussian measure of $\partial(U_s + x)$ is zero for every $x \in \operatorname{supp} \mu$ and $s > 0$ (compare [HJ-S-D] for the proof of this fact if $x = 0$), hence Linde's result is also valid for U_s instead of $\overline{U}_s$. Linde showed that the differential $d(s, x)(\cdot)$ is non-trivial if $\|x\| > s$ and asked about the case $\|x\| \le s$. Now we show that $d(s, x)$ is a non-zero functional if only $x \neq 0$, what is more, we show that $d(s, x)(x) < 0$.

The next theorem answers in positive the question of Linde we mentioned earlier.

Theorem 2. *In the above setting, for every* $x \in \operatorname{supp}\mu$, $x \neq 0$,

$$d(s, x)(x) < 0.$$

Proof. Consider a function $f_x(t) = \frac{1}{\mu(U_s+tx)}$ for $x \neq 0$. Then f_x is even, convex and strictly increasing on $(0, \infty)$. It is clear that f_x is even and tends to infinity as $t \to \infty$. By log-concavity of the measure μ we conclude that the function $t \to \log\mu(U_s + tx)$ is concave, hence $\frac{1}{\mu(U_s+tx)} = \exp(-\log\mu(U_s + tx))$ is convex. But Theorem 1 implies that $f_x(t) > f_x(0)$ for every $t > 0$, hence f_x is strictly increasing (because it is convex).

Next, by easy computations we get for $x \in \operatorname{supp}\mu$:

$$d(s, x)(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\mu(U_s + x + \varepsilon x) - \mu(U_s + x)] = (f_x(t)^{-1})'_{t=1}.$$

But f_x is strictly increasing and convex, hence $(f_x)'_{t=1} > 0$, which implies that

$$d(s, x)(x) = -\frac{f_x'(1)}{f_x^2(1)} < 0.$$

As it was shown in Linde's paper [Lin] it is easy to compute $d(s, z)(h)$ for $h \in H(\mu)$. Namely, using the Cameron-Martin formula (Proposition 1) we get

$$d(s, z)(h) = -\int_{U_s+z} \tilde{h}(x)\, d\mu(x) = -\tilde{h}\Big(\int_{U_s+z} x\, d\mu(x)\Big).$$

Now we show that for Gaussian measure μ the condition $\int_{U_s+z} x\, d\mu(x) = 0$ implies $z = 0$.

Theorem 3. *Let* μ *be a symmetric Gaussian measure on E. If* $z \in \operatorname{supp}\mu$, *then* $\int_{U_s+z} x\, d\mu(x) = 0$ *implies* $z = 0$.

Proof. Arguing as at the beginning of the proof of Theorem 2 we infer that the function $g_h(t) = \frac{1}{\mu(U_s+z+th)}$ is convex for all $h \in H_\mu$. If $z \neq 0$, then, by Theorem 1, $\mu(U_s+z) < \mu(U_s)$, hence for at least one $h_0 \in H_\mu$, the derivative $g'_{h_0}(0)$ is not equal to zero (g_{h_0} attains its minimum at some $t \neq 0$), so that $0 \neq d(s, z)(h_0) = -\tilde{h}_0(\int_{U_s+z} x\, d\mu(x))$ which, of course, is equivalent to the condition $\int_{U_s+z} x\, d\mu(x) \neq 0$.

References

[And] T. W. Anderson, *The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. **6** (1955), 170–176.

[Bor] C. Borell, *Gaussian Radon measures on locally convex spaces*, Math. Scand. **38** (1976), 265–284.

[C-M] R. H. Cameron and W. T. Martin, *The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals*, Ann. of Math. **48** (1947), 385–392.

[HJ-S-D] J. Hoffmann-Jørgensen, L. A. Shepp, and R. M. Dudley, *On the lower tail of Gaussian seminorms*, Ann. Probab. **7** (1979), 319–342.

[LePage] R. Le Page, *Note relating Bochner integrals and reproducing kernels to series expansions on a Gaussian Banach space*, Proc. Amer. Math. Soc. **32** (1972), 285–288.

[LP-W-Z] R. Le Page, M. Woodroofe, and J. Zinn, *Convergence to a stable distribution via order statistics*, Ann. Probab. **9** (1981), 624–632.

[Lin] W. Linde, *Gaussian measure of translated balls in a Banach space*, Theory Probab. Appl. **34** (1989), 349–359.

[Szt] R. Sztencel, *Absolute continuity of the lower tail of stable seminorms*, Bull. Polish Acad. Sci. Math. **34** (1986), 231–234.

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