

HARMONIC POLYNOMIALS AND DIRICHLET-TYPE PROBLEMS

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ABSTRACT. We take a new approach to harmonic polynomials via differentiation.

Surprisingly powerful results about harmonic functions can be obtained simply by differentiating the function $|x|^{2-n}$ and observing the patterns that emerge. This is one of our main themes and is the route we take to Theorem 1.7, which leads to a new proof of a harmonic decomposition theorem for homogeneous polynomials (Corollary 1.8) and a new proof of the identity in Corollary 1.10. We then discuss a fast algorithm for computing the Poisson integral of any polynomial. (Note: The algorithm involves differentiation, but no integration.) We show how this algorithm can be used for many other Dirichlet-type problems with polynomial data. Finally, we show how Lemma 1.4 leads to the identity in (3.2), yielding a new simple proof that the Kelvin transform preserves harmonic functions.

1. DERIVATIVES OF $|x|^{2-n}$

Unless otherwise stated, we work in \mathbf{R}^n , $n > 2$; the function $|x|^{2-n}$ is then harmonic and nonconstant on $\mathbf{R}^n \setminus \{0\}$. (When $n = 2$ we need to replace $|x|^{2-n}$ with $\log|x|$; the minor modifications needed in this case are discussed in Section 4.)

Letting D_j denote the partial derivative with respect to the j^{th} coordinate variable, we list here some standard differentiation formulas that will be useful later:

$$\begin{aligned}D_j|x|^t &= tx_j|x|^{t-2}, \\ \Delta|x|^t &= t(t+n-2)|x|^{t-2}, \\ \Delta(uv) &= u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u.\end{aligned}$$

The first two formulas are valid on $\mathbf{R}^n \setminus \{0\}$ for every real t , while the last formula holds on any open set where u and v are twice continuously differentiable (and real valued); as usual, Δ denotes the Laplacian and ∇ denotes the gradient.

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For higher-order derivatives we will use multi-index notation. Thus for α an n -tuple $(\alpha_1, \dots, \alpha_n)$ of nonnegative integers, D^α denotes the differential operator $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, x^α is the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; the degree of x^α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. A polynomial is said to be homogeneous of degree m if it is a finite linear combination of monomials x^α of degree m ; here $m = 0, 1, \dots$. (By "polynomial" we will always mean a polynomial on \mathbf{R}^n .)

The collection of all polynomials homogeneous of degree m will be denoted by \mathcal{P}_m . The subset of \mathcal{P}_m consisting of *harmonic* homogeneous polynomials of degree m will be denoted by \mathcal{H}_m .

Set $c_0 = 1$ and for $m > 0$ define

$$(1.1) \quad c_m = \prod_{j=0}^{m-1} (2 - n - 2j).$$

Also set $\mathcal{P}_{-1} = \mathcal{P}_{-2} = \{0\}$. Our first observation on differentiating $|x|^{2-n}$ is the following lemma.

Lemma 1.2. *If $|\alpha| = m$, then*

$$(1.3) \quad D^\alpha |x|^{2-n} = |x|^{2-n-2m} (c_m x^\alpha + |x|^2 q_\alpha)$$

for some $q_\alpha \in \mathcal{P}_{m-2}$.

Proof. The proof will be by induction on m ; the lemma obviously holds if $m = 0$.

Suppose that

$$D^\alpha |x|^{2-n} = |x|^k p,$$

where $|\alpha| = m$, $k = 2 - n - 2m$, and $p = c_m x^\alpha + |x|^2 q_\alpha$ for some $q_\alpha \in \mathcal{P}_{m-2}$. Then

$$\begin{aligned} D_j(D^\alpha |x|^{2-n}) &= k x_j |x|^{k-2} p + |x|^k D_j p \\ &= |x|^{k-2} (k x_j p + |x|^2 D_j p) \\ &= |x|^{k-2} (k c_m x_j x^\alpha + |x|^2 (k x_j q_\alpha + D_j p)) \\ &= |x|^{2-n-2(m+1)} (c_{m+1} x_j x^\alpha + |x|^2 r), \end{aligned}$$

where $r = k x_j q_\alpha + D_j p$. Because $r \in \mathcal{P}_{m-1}$, the last line has the form specified on the right of (1.3), with $m+1$ in place of m . This completes the induction argument and hence the proof of the lemma. \square

Note that the polynomial $c_m x^\alpha + |x|^2 q_\alpha$ in (1.3) belongs to \mathcal{P}_m . Actually, this polynomial belongs to \mathcal{H}_m , as the next lemma will show.

Lemma 1.4. *If $p \in \mathcal{P}_m$, then*

$$\Delta(|x|^{2-n-2m} p) = |x|^{2-n-2m} \Delta p.$$

Proof. Let $t \in \mathbf{R}$ and $p \in \mathcal{P}_m$. Using the differentiation formulas above, we compute the Laplacian of $|x|^t p$:

$$\Delta(|x|^t p) = |x|^t \Delta p + 2t |x|^{t-2} x \cdot \nabla p + t(t+n-2) |x|^{t-2} p.$$

Because $p \in \mathcal{P}_m$, we have $x \cdot \nabla p = mp$. Thus the equation above reduces to

$$(1.5) \quad \Delta(|x|^t p) = |x|^t \Delta p + t(2m + t + n - 2)|x|^{t-2} p.$$

Taking $t = 2 - n - 2m$ now gives the conclusion of the lemma. \square

Referring again to (1.3), set $p_\alpha = c_m x^\alpha + |x|^2 q_\alpha$ for α a multi-index with $|\alpha| = m$. Because $D^\alpha |x|^{2-n}$ is harmonic (being a partial derivative of a harmonic function), and because $p_\alpha \in \mathcal{P}_m$, the last two lemmas imply

$$0 = \Delta(D^\alpha |x|^{2-n}) = \Delta(|x|^{2-n-2m} p_\alpha) = |x|^{2-n-2m} \Delta p_\alpha,$$

so that $p_\alpha \in \mathcal{H}_m$ as claimed above.

Observe that we now have a method for producing elements of \mathcal{H}_m : We simply apply D^α to $|x|^{2-n}$ to arrive at $p_\alpha \in \mathcal{H}_m$. How much of \mathcal{H}_m is obtained in this manner? A consequence of Corollary 1.10 below is that the polynomials p_α span all of \mathcal{H}_m .

We can now easily handle $p(D)|x|^{2-n}$ for any $p \in \mathcal{P}_m$. Fixing such a p and writing $p(x) = \sum a_\alpha x^\alpha$ (so that $p(D) = \sum a_\alpha D^\alpha$), we use (1.3) and linearity to obtain

$$p(D)|x|^{2-n} = |x|^{2-n-2m} \left[c_m p + |x|^2 \sum a_\alpha q_\alpha \right].$$

Note that the expression in brackets belongs to \mathcal{H}_m . Thus, setting

$$(1.6) \quad \Lambda_m(p) = \frac{1}{c_m} |x|^{n-2+2m} (p(D)|x|^{2-n}),$$

we have proved the following theorem.

Theorem 1.7. *If $p \in \mathcal{P}_m$, then*

- (a) $\Lambda_m(p) \in \mathcal{H}_m$;
- (b) $p = \Lambda_m(p) + |x|^2 q$ for some $q \in \mathcal{P}_{m-2}$.

Theorem 1.7 leads to the following corollary, which gives the well-known decomposition (1.9) and an explicit formula for p_m .

Corollary 1.8. *Every $p \in \mathcal{P}_m$ can be uniquely written in the form*

$$(1.9) \quad p = p_m + |x|^2 p_{m-2} + \cdots + |x|^{2k} p_{m-2k},$$

where $k = [\frac{m}{2}]$ and $p_j \in \mathcal{H}_j$ for each j . Furthermore, $p_m = \Lambda_m(p)$.

Proof. Theorem 1.7 implies that $\mathcal{P}_m = \mathcal{H}_m + |x|^2 \mathcal{P}_{m-2}$ (as vector spaces). By induction we thus obtain

$$\mathcal{P}_m = \mathcal{H}_m + |x|^2 \mathcal{H}_{m-2} + \cdots + |x|^{2k} \mathcal{H}_{m-2k},$$

where $k = [\frac{m}{2}]$ (note that $\mathcal{P}_m = \mathcal{H}_m$ when $m = 0$ or 1). This establishes the existence of the representation (1.9).

To prove uniqueness, suppose we have two representations of p as in (1.9). Setting $|x| = 1$, we obtain two harmonic polynomials that agree on the unit sphere of \mathbf{R}^n , and hence agree on all of \mathbf{R}^n . Equating homogeneous terms of like degree then shows that the two representations are identical.

That $p_m = \Lambda_m(p)$ now follows from Theorem 1.7(b). \square

The following result is an immediate consequence of Corollary 1.8.

Corollary 1.10. *If $p \in \mathcal{H}_m$, then $p = \Lambda_m(p)$.*

The last corollary implies that the linear operator Λ_m is a projection of \mathcal{P}_m onto \mathcal{H}_m .

We do not claim to have the shortest proof of the direct sum decomposition $\mathcal{P}_m = \mathcal{H}_m + |x|^2 \mathcal{H}_{m-2} + \cdots$ given by (1.9); that distinction probably goes to the proof of Theorem 2.1 in Chapter IV of [6]. The more constructive approach taken here, however, gives Theorem 1.7, Corollary 1.8, and Corollary 1.10 in one stroke. Corollary 1.10 has been proved by various methods; see [3] (sections 79–80), [1] (Theorem 5.32), and [5].

2. FAST ALGORITHMS

We now show how results in the last section lead to fast algorithms for computing exact solutions to the Dirichlet and certain related problems with polynomial data. All of these problems are linear, so it suffices to treat the case where the data function is a homogeneous polynomial. Our setting is the open unit ball in \mathbf{R}^n , which we denote by B . Given $f \in C(\partial B)$, the *Dirichlet problem with boundary data* f asks for a continuous function on \bar{B} that is harmonic on B and agrees with f on ∂B . The solution, as is well known, is the function whose value at points $x \in B$ is given by the Poisson integral

$$(2.1) \quad \int_{\partial B} \frac{1 - |x|^2}{|x - \zeta|^n} f(\zeta) d\sigma(\zeta),$$

where $d\sigma$ denotes normalized surface area measure on ∂B .

Even if f is the restriction to ∂B of a polynomial, the integral in (2.1) is difficult to compute directly; here we are referring to exact computations, not numeric approximations. The algorithm we describe below avoids integration over ∂B altogether. Our starting point is the following well-known consequence of the decomposition given by Corollary 1.8.

Corollary 2.2. *If $p \in \mathcal{P}_m$, then the solution to the Dirichlet problem with boundary data $p|_{\partial B}$ is*

$$(2.3) \quad p_m + p_{m-2} + \cdots + p_{m-2k},$$

where $k = \lfloor \frac{m}{2} \rfloor$ and $p_m, p_{m-2}, \dots, p_{m-2k}$ are the harmonic polynomials given by (1.9).

Proof. Suppose $p \in \mathcal{P}_m$. Take $|x| = 1$ in (1.9) to show that (2.3) equals p on ∂B . Obviously (2.3) is harmonic on \mathbf{R}^n , and hence its restriction to \bar{B} is the solution to the Dirichlet problem with boundary data $p|_{\partial B}$. \square

So given $p \in \mathcal{P}_m$, we need an algorithm for computing the polynomials p_m, p_{m-2}, \dots of Corollary 2.2. The main idea for the algorithm we are about to describe comes from [2] (see page 43). We start with an observation based on repeated application of (1.5): If i, j are nonnegative integers, then the operator $|x|^{2i} \Delta^i$ equals a constant times the identity operator on the space $|x|^{2j} H_{m-2j}$. Denoting this constant by c_{ij} , note that $c_{ij} = 0$ if and only if $i > j$. (We can easily compute c_{ij} exactly using (1.5), but we will only need the diagonal terms c_{jj} .) Applying the above operators to both sides of (1.9), we obtain the

upper-triangular system of equations

$$|x|^{2i} \Delta^i p = \sum_{j=i}^k c_{ij} |x|^{2j} p_{m-2j},$$

$i = 0, \dots, k$. Letting (d_{ij}) denote the matrix inverse of (c_{ij}) , observe that (d_{ij}) is also upper-triangular. Apply this inverse matrix to the system above to solve for $|x|^{2i} p_{m-2i}$. After dividing by $|x|^{2i}$, we obtain

$$(2.4) \quad p_{m-2i} = \sum_{j=i}^k d_{ij} |x|^{2(j-i)} \Delta^j p$$

for $i = 0, \dots, k$.

To find (d_{ij}) we start with the diagonal terms:

$$(2.5) \quad d_{jj} = \frac{1}{c_{jj}} = \frac{1}{2^j j! \prod_{l=1}^j (2m + n - 2j - 2l)}.$$

Similarly, the other d_{ij} could be computed from the c_{ij} . However, an iterative formula for the d_{ij} gives faster computations. We obtain this formula by taking the Laplacian of both sides of (2.4) (use (1.5)) and recalling that p_{m-2i} is harmonic. This leads to (we spare the reader the computation details)

$$d_{ij} = \frac{-d_{i,j-1}}{2(j-i)(2m+n-2-2i-2j)}$$

for $j = i + 1, \dots, k$. The last equation, together with (2.4) and (2.5), gives us an algorithm for computing the polynomials p_{m-2i} , as desired.

The *Mathematica*¹ software package² that accompanies [1] now uses the algorithm described above. To illustrate the output of the package, it gives the following solution to the Dirichlet problem in \mathbb{R}^5 with boundary data $x_1^5 x_2$:

$$\frac{65x_1 x_2 - 110|x|^2 x_1 x_2 + 45|x|^4 x_1 x_2 + 330x_1^3 x_2 - 330|x|^2 x_1^3 x_2 + 429x_1^5 x_2}{429}.$$

This algorithm is considerably faster than the best method we previously knew for computing solutions to Dirichlet problems with polynomial boundary data. The old algorithm, based on Theorems 5.19 and 5.24 of [1], required explicit formulas for zonal harmonics and for integrating polynomials over the unit sphere. To compare the two methods, consider the following case: A computer using the old algorithm spent more than a day on the Dirichlet problem in \mathbb{R}^6 with boundary data x_1^{20} without finishing. Using the new algorithm, the same machine calculated the solution in about 10 seconds.

The new algorithm gives rise to fast algorithms for solving many Dirichlet-type problems with polynomial data. We outline some of these techniques below; all of them have been implemented in the software described above. We fix a homogeneous polynomial $p \in \mathcal{P}_m$ for the rest of this section, and

¹ *Mathematica* is a registered trademark of Wolfram Research.

² This *Mathematica* package and its documentation are electronically available free of charge. To obtain them, send an e-mail request to axler@math.msu.edu, or retrieve them using a [www](http://math.msu.edu/~axler) browser from <http://math.msu.edu/~axler>. This software, written by the authors and Paul Bourdon, will work on any computer that runs *Mathematica*.

assume that the harmonic polynomials p_m, p_{m-2}, \dots of Corollary 2.2 have been computed as above. For convenience (so we don't have to worry about whether m is even or odd), we set p_{m-1}, p_{m-3}, \dots all equal to 0. Thus we can write

$$p = \sum_{j=0}^m p_j$$

on ∂B .

Neumann problem. Find the harmonic function on \bar{B} whose outward normal derivative on ∂B equals p and whose value at the origin equals 0.

The solution to the Neumann problem is the function

$$\sum_{j=1}^m \frac{p_j}{j},$$

provided that $\int_{\partial B} p \, d\sigma = 0$ (otherwise no solution exists, by Green's identity). This function is obviously harmonic. To verify that it solves the Neumann problem, note that the outward normal derivative on ∂B of a function $p_j \in \mathcal{P}_j$ equals $j p_j$.

Exterior Dirichlet problem. Find the harmonic function on $\{|x| \geq 1\}$ that equals p on ∂B and is harmonic at ∞ .

The solution to the exterior Dirichlet problem is the function

$$\sum_{j=0}^m |x|^{2-n-2j} p_j.$$

This function is harmonic by Lemma 1.4. It obviously equals p on ∂B . Its harmonicity at ∞ , which is needed to insure uniqueness, follows from the definition in [1], Chapter 4.

Bergman projection problem. Find the harmonic function closest to p in the $L^2(B)$ -norm.

Here we use Lebesgue volume measure on B . The solution to the Bergman projection problem is the function

$$\sum_{j=0}^m \frac{2j+n}{j+m+n} p_j.$$

This function is obviously harmonic. That it is the orthogonal projection of p into the harmonic functions in $L^2(B)$ follows from Theorem 8.14 of [1].

The boundary data for the next two problems consists of two polynomials. Thus in addition to $p \in P_m$, we fix $q \in P_M$; here $M \geq 0$. As with p , the algorithm for solving the Dirichlet problem gives harmonic polynomials $q_j \in \mathcal{H}_j$ such that

$$q = \sum_{j=0}^M q_j$$

on ∂B .

Annular Dirichlet problem. Let $0 < r < s < \infty$. Find the harmonic function on the annular region $\{r \leq |x| \leq s\}$ that equals p on $\{|x| = r\}$ and equals q on $\{|x| = s\}$.

The solution to the annular Dirichlet problem is the function

$$\sum_{j=0}^m \frac{|x|^{2-n-2j} - s^{2-n-2j}}{r^{2-n-2j} - s^{2-n-2j}} r^{m-j} p_j + \sum_{j=0}^M \frac{|x|^{2-n-2j} - r^{2-n-2j}}{s^{2-n-2j} - r^{2-n-2j}} s^{M-j} q_j.$$

This function is harmonic by Lemma 1.4. At a point x with $|x| = r$, it equals

$$\sum_{j=0}^m r^{m-j} p_j(x) = r^m \sum_{j=0}^m p_j(x/r) = r^m p(x/r) = p(x).$$

A similar calculation shows that it equals q on the outer boundary. The idea for this solution comes from Chapter 10 of [1].

BiDirichlet problem. Find the biharmonic function on \bar{B} that equals p on ∂B and whose outward normal derivative on ∂B equals q .

A function u is called biharmonic if $\Delta(\Delta u) = 0$. The solution to the biDirichlet problem is the function

$$(2.6) \quad \frac{(|x|^2 - 1)}{2} \left(\sum_{j=0}^M q_j - \sum_{j=1}^m j p_j \right) + \sum_{j=0}^m p_j.$$

A straightforward calculation shows that the Laplacian of this function equals

$$\sum_{j=0}^M (n + 2j) q_j - \sum_{j=1}^m j(n + 2j) p_j,$$

which is harmonic, and hence (2.6) is biharmonic. The function (2.6) obviously equals p on ∂B . An easy calculation shows that the outward normal derivative on ∂B of (2.6) equals q .

3. THE KELVIN TRANSFORM

If u is a function on a subset of $\mathbf{R}^n \setminus \{0\}$, then the Kelvin transform of u is the function $K[u]$ defined by

$$K[u](x) = |x|^{2-n} u \left(\frac{x}{|x|^2} \right).$$

(For more information on the Kelvin transform, see [1], Chapter 4.) Corollary 1.10 can be reformulated in terms of the Kelvin transform to state that

$$p = \frac{1}{c_m} K[p(D)|x|^{2-n}]$$

whenever $p \in \mathcal{H}_m$ (to see this, multiply both sides of the equation $p = \Lambda_m(p)$ by $|x|^{2-n-2m}$, then take Kelvin transforms of both sides). The equation above is the form in which Corollary 1.10 appears as Theorem 5.32 in [1].

The Kelvin transform is important because it comes close enough to commuting with the Laplacian to preserve harmonic functions. The next proposition gives the precise result. Closely related formulas can be found in [4], page 221, and [7], Theorem 13.1, where the Laplacian of $K[u]$ is computed by a straightforward but long calculation. We take advantage of Lemma 1.4 to give a short and simple proof.

Proposition 3.1. *If u is a C^2 function on an open subset of $\mathbf{R}^n \setminus \{0\}$, then*

$$(3.2) \quad \Delta(K[u]) = K[|x|^4 \Delta u].$$

Proof. Suppose $p \in \mathcal{P}_m$. Then

$$(3.3) \quad \begin{aligned} \Delta(K[p]) &= \Delta(|x|^{2-n-2m} p) \\ &= |x|^{2-n-2m} \Delta p \\ &= |x|^{2-n} |x|^{-4} \frac{\Delta p}{|x|^{2(m-2)}} \end{aligned}$$

$$(3.4) \quad = K[|x|^4 \Delta p],$$

where (3.3) follows from Lemma 1.4 and (3.4) holds because Δp is homogeneous of degree $m - 2$.

The paragraph above shows that (3.2) holds whenever $u \in \mathcal{P}_m$, and hence whenever u is a polynomial (by linearity). Because polynomials are locally dense in the C^2 -norm, (3.2) holds for arbitrary C^2 functions u , as desired. \square

The last proposition is the simplest way we know to see that a function is harmonic if and only if its Kelvin transform is harmonic. Another proof of this can be found in [1], Theorem 4.4.

4. $n = 2$

So far we have been assuming that $n > 2$. In this section we assume $n = 2$ and discuss the modifications needed to make our results carry over to two dimensions.

Of course, the first order of business is to replace $|x|^{2-n}$ with $\log|x|$. We also need to redefine c_m ; thus for $m > 0$ we set

$$c_m = (-2)^{m-1} (m-1)!.$$

We then have the following analogue of Lemma 1.2.

Lemma 4.1. *Suppose $n = 2$ and $m > 0$. If $|\alpha| = m$, then*

$$D^\alpha \log|x| = |x|^{-2m} (c_m x^\alpha + |x|^2 q)$$

for some $q \in \mathcal{P}_{m-2}$.

The proof follows the same pattern as that of Lemma 1.2, except that the induction now begins with $m = 1$ instead of $m = 0$. We leave the details to the reader.

The only other change we need to make is in the definition of the projection operator Λ_m . Here we set $\Lambda_0(p) = p$ and for $m > 0$ define

$$\Lambda_m(p) = \frac{1}{c_m} |x|^{2m} (p(D) \log|x|).$$

With these modifications, all the other results in the paper carry over to the two-dimensional setting without change, with one exception: The solution to the two-dimensional annular Dirichlet problem is

$$\begin{aligned} & \frac{\log|x| - \log s}{\log r - \log s} r^m p_0 + \sum_{j=1}^m \frac{|x|^{-2j} - s^{-2j}}{r^{-2j} - s^{-2j}} r^{m-j} p_j \\ & + \frac{\log|x| - \log r}{\log s - \log r} s^M q_0 + \sum_{j=1}^M \frac{|x|^{-2j} - r^{-2j}}{s^{-2j} - r^{-2j}} s^{M-j} q_j. \end{aligned}$$

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