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RIEMANNIAN METRICS WITH LARGE FIRST EIGENVALUE ON FORMS OF DEGREE p

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ABSTRACT. Let (M, g) be a compact, connected, C^∞ Riemannian manifold of n dimensions. Denote by $\lambda_{1,p}(M, g)$ the first nonzero eigenvalue of the Laplace operator acting on differential forms of degree p. We prove that for $n \geq 4$ and $2 \leq p \leq n-2$, there exists a family of metrics g_t of volume one, such that $\lambda_{1,p}(M, g_t) \to \infty$ as $t \to \infty$.

1. INTRODUCTION

Let (M^n, g) be a compact, connected Riemannian manifold of n dimensions. The Laplacian $\Delta_{g,p}$ acting on differential forms of degree p on M has discrete spectrum. Let $\lambda_{1,p}(g)$ denote the smallest positive eigenvalue of $\Delta_{g,p}$. For functions we set as usual $\lambda_1(g) = \lambda_{1,0}(g)$. Hersch [4] has proved, that for functions on S^2 we have:

$$\lambda_1(g)\mathrm{Vol}(S^2, g) \leq 8\pi$$

for every Riemannian metric g.

In connection with this result, M. Berger [1] asked whether there exists a constant $k(M)$ such that:

$$\lambda_1(g)\mathrm{Vol}(M^n, g)^{2/n} \leq k(M) \tag{1}$$

for any Riemannian metric g on M. Yang and Yau [9] have proved that the inequality above holds for a compact surface S of genus γ with $k(S) = 8\pi(\gamma + 1)$.

Subsequently, Bleecker [2], Urakawa [7] and others constructed examples of manifolds of dimension $n \geq 3$ for which (1) was false. Finally Xu [8] and Colbois and Dodziuk [3] showed that (1) was false for every Riemannian manifold of dimension $n \geq 3$.

The same questions was posed by S. Tanno [6] for forms of degree p: does there exist a constant $k(M)$ such that

$$\lambda_{1,p}(g)\mathrm{Vol}(M^n, g)^{2/n} \leq k(M) \tag{2}$$

for any Riemannian metric g on M?

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2. Results

In this note we show that (2) is false for $n \geq 4$ and $2 \leq p \leq n-2$. The proof uses a generalization of a technical lemma figuring in J. McGowan [5]. This lemma allows one to estimate from below the first eigenvalue for exact forms, in terms of first eigenvalues for exact forms on parts of M, with respect to absolute boundary conditions.

Theorem 1. *Every compact, connected manifold M^n of dimension $n \geq 4$ admits metrics g of volume one with arbitrarily large $\lambda_{1,p}(g)$ for all $2 \leq p \leq n-2$.*

Proof. We take a topological sphere S^n and choose a metric g_0 on it, such that S^n looks like a cigar, where the middle part has length 3. In particular this middle part is a product for the metric g_0, i.e. a cylinder $I \times S^{n-1}$ (see Figure 1).

Remove the half-sphere H_2 at one end of the cigar and form a connected sum with M. The resulting manifold is diffeomorphic to M and has a submanifold Ω, with smooth boundary, naturally identified with $S^n \backslash H_2$.

Let g_1 be an arbitrary metric on M whose restriction to Ω is equal to $g_0|_\Omega$. Ω contains an open cylinder of length 3. We subdivide this cylinder into 3 cylinders Z_1, Z_2, Z_3 of length 1 (see Figure 2).

Let g_t be a metric on M such that $g_t|_{(M \backslash Z_2)} = g_1|_{(M \backslash Z_2)}$ and such that $Z_2 = I \times S^{n-1}$ becomes a cylinder of length t. This is accomplished by replacing the unit interval by the interval $[0, t]$ and using the product metric on Z_2. Now $\operatorname{Vol}(M, g_t) = a + bt$ where a and b are positive real constants (see Figure 3).

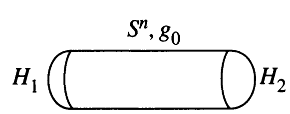

Figure 1

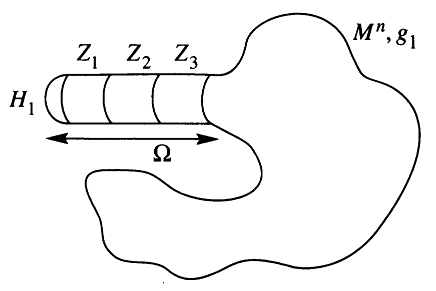

Figure 2

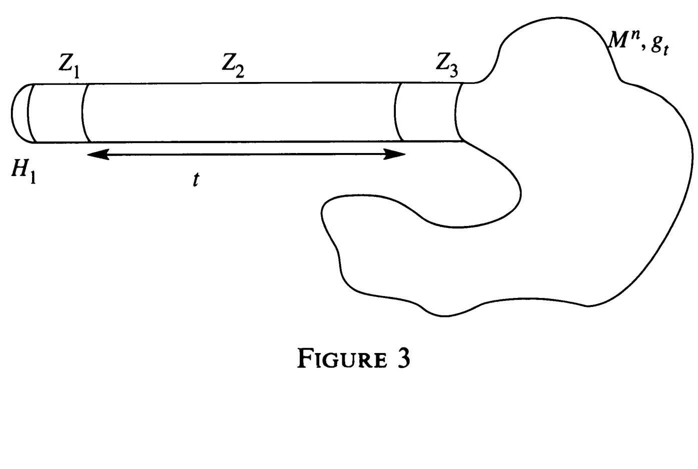

Figure 3

We take the following open cover of M:

$$\left.\begin{array}{l} U_1 = H_1 \cup Z_1 \\ U_2 = M\backslash\overline{H_1 \cup Z_1 \cup Z_2} \\ U_3 = Z_1 \cup \overline{Z}_2 \cup Z_3 \end{array}\right\} \Rightarrow \begin{array}{l} U_1 \cap U_2 = \varnothing, \\ U_1 \cap U_3 = Z_1, \\ U_2 \cap U_3 = Z_3, \end{array} \qquad U_1 \cap U_2 \cap U_3 = \varnothing.$$

Let $\mu_{1,p}$ be the first positive eigenvalue of the Laplacian on exact forms of degree p on (M, g_t). To estimate $\mu_{1,p}$ we can use a generalization of a lemma by McGowan [5]. Suppose we are given an open cover $\{U_i\}_{i=1}^K$ of M without intersections of order bigger than two and which satisfies $\Sigma_{i<j} \dim H^{p-1}(U_{ij}, \mathbb{R}) = 0$ $(U_{ij} = U_i \cap U_j)$; then the following holds:

Lemma 1. *Given an open cover of M as above, denote by $\mu(U_i)$, resp. $\mu(U_{ij})$, the smallest positive eigenvalue of the Laplacian acting on exact forms of degree p on U_i, resp. of degree $p-1$ on U_{ij}, satisfying absolute boundary conditions. Then*

$$\mu_{1,p}(M) \geq \frac{2^{-3}}{\sum_{i=1}^K \left(\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i}\left(\frac{w_{n,p} \cdot c_p}{\mu(U_{ij})} + 1\right)\left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}\right)\right)} \tag{3}$$

where m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \varnothing$, $w_{n,p}$ a combinatorial constant which depends on p and n. $c_p = (\max)_i(\max)_{x\in U_i}|\nabla \rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{j=1}^K$ subordinate to the given cover.

The proof of this lemma uses the same arguments as the proof of the lemma by McGowan [5]. The generalization is trivial, because we made special assumptions on the cover of M. (See remarks at the end of paragraph 2, p. 735 of [5].)

Denote by $\lambda_{r,s}(N)$ the rth eigenvalue of the Laplacian on s-forms on N with respect to absolute boundary conditions, in case n has a boundary. If 0 is an eigenvalue with multiplicity, we denote it by $\lambda_{0,s}$. We apply Lemma 1 to $M_t = (M, g_t)$ and the cover $\{U_1, U_2, U_3\}$.

$\mu(U_1)$, $\mu(U_2)$, $\mu(U_{13})$, $\mu(U_{23})$ are independent of t. By using the Künneth formula, we get the following inequality for $\mu(U_3)$:

$$\begin{aligned} \mu(U_3) &\geq \lambda_{1,p}(U_3) = \lambda_{1,p}(I_t \times S^{n-1}) \\ &\geq \min_{i,j,k,l}\{\lambda_{i,0}(I_t) + \lambda_{j,p}(S^{n-1}), \lambda_{k,1}(I_t) + \lambda_{l,p-1}(S^{n-1})\} \\ &\geq \min_{j,l}\{\lambda_{j,p}(S^{n-1}), \lambda_{l,p-1}(S^{n-1})\} = c(p) \\ &\geq \min_{2\leq p\leq n-2} c(p) =: c > 0 \quad \text{independent of } t, \end{aligned}$$

$c(p) > 0$, because for $1 \leq p \leq n-2$ there are no non-trivial harmonic forms of degree p on $S^{n-1}(H^p(S^{n-1}) = 0$ for $p \neq 0, n-1)$.

By applying the lemma above to $M_t = (M, g_t)$ we get that

$$\mu_{1,p}(M_t) \geq \delta > 0$$

independent of t.

The volume of M_t is given by $\text{Vol}(M_t) = a + bt$ with constants $a, b > 0$. Set $\overline{g}_t = (a+bt)^{-2/n} g_t$ and $\overline{M}_t = (M, \overline{g}_t)$; then $\text{Vol}(\overline{M}_t) = 1$ and $\mu_{1,p}(\overline{M}_t) = (a+bt)^{2/n}\mu_{1,p}(M_t)$. This implies that

$$\mu_{1,p}(\overline{M}_t) \geq \delta(a+bt)^{2/n}, \quad \text{with } \delta > 0.$$

Therefore $\mu_{1,p}(\overline{M}_t) \to \infty$ when $t \to \infty$. Since the full spectrum of $\Delta_{g,p}$ is obtained from the spectra of exact forms of degree p and $p+1$, the theorem follows. □

Remarks. Berger's problem still is open for 1-forms. Tanno [6] showed that on S^3, $\lambda_{1,1}$ is bounded for the 1-parameter family of metrics in the examples of Bleecker and Urakawa, which gave unbounded first eigenvalue for functions.

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