RIEMANNIAN METRICS WITH LARGE FIRST EIGENVALUE
ON FORMS OF DEGREE \( p \)

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Abstract. Let \((M, g)\) be a compact, connected, \(C^\infty\) Riemannian manifold of \(n\) dimensions. Denote by \(\lambda_{1,p}(M, g)\) the first nonzero eigenvalue of the Laplace operator acting on differential forms of degree \(p\). We prove that for \(n > 4\) and \(2 \leq p \leq n - 2\), there exists a family of metrics \(g_t\) of volume one, such that \(\lambda_{1,p}(M, g_t) \to \infty\) as \(t \to \infty\).

1. Introduction

Let \((M^n, g)\) be a compact, connected Riemannian manifold of \(n\) dimensions. The Laplacian \(\Delta_{g,p}\) acting on differential forms of degree \(p\) on \(M\) has discrete spectrum. Let \(\lambda_{1,p}(g)\) denote the smallest positive eigenvalue of \(\Delta_{g,p}\). For functions we set as usual \(\lambda_1(g) = \lambda_{1,0}(g)\). Hersch [4] has proved, that for functions on \(S^2\) we have:

\[ \lambda_1(g) \text{Vol}(S^2, g) \leq 8\pi \]

for every Riemannian metric \(g\).

In connection with this result, M. Berger [1] asked whether there exists a constant \(k(M)\) such that:

\[ \lambda_1(g) \text{Vol}(M^n, g)^{2/n} \leq k(M) \]

for any Riemannian metric \(g\) on \(M\). Yang and Yau [9] have proved that the inequality above holds for a compact surface \(S\) of genus \(\gamma\) with \(k(S) = 8\pi(\gamma + 1)\).

Subsequently, Bleecker [2], Urakawa [7] and others constructed examples of manifolds of dimension \(n \geq 3\) for which (1) was false. Finally Xu [8] and Colbois and Dodziuk [3] showed that (1) was false for every Riemannian manifold of dimension \(n \geq 3\).

The same questions was posed by S. Tanno [6] for forms of degree \(p\): does there exist a constant \(k(M)\) such that:

\[ \lambda_{1,p}(g) \text{Vol}(M^n, g)^{2/n} \leq k(M) \]

for any Riemannian metric \(g\) on \(M\)?

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2. Results

In this note we show that (2) is false for \( n \geq 4 \) and \( 2 \leq p \leq n - 2 \). The proof uses a generalization of a technical lemma figuring in J. McGowan [5]. This lemma allows one to estimate from below the first eigenvalue for exact forms, in terms of first eigenvalues for exact forms on parts of \( M \), with respect to absolute boundary conditions.

**Theorem 1.** Every compact, connected manifold \( M^n \) of dimension \( n \geq 4 \) admits metrics \( g \) of volume one with arbitrarily large \( \lambda_{1,p}(g) \) for all \( 2 \leq p \leq n - 2 \).

**Proof.** We take a topological sphere \( S^n \) and choose a metric \( g_0 \) on it, such that \( S^n \) looks like a cigar, where the middle part has length 3. In particular this middle part is a product for the metric \( g_0 \), i.e. a cylinder \( I \times S^{n-1} \) (see Figure 1).

Remove the half-sphere \( H_2 \) at one end of the cigar and form a connected sum with \( M \). The resulting manifold is diffeomorphic to \( M \) and has a submanifold \( \Omega \), with smooth boundary, naturally identified with \( S^n \setminus H_2 \).

Let \( g_1 \) be an arbitrary metric on \( M \) whose restriction to \( \Omega \) is equal to \( g_0|\Omega \). \( \Omega \) contains an open cylinder of length 3. We subdivide this cylinder into 3 cylinders \( Z_1, Z_2, Z_3 \) of length 1 (see Figure 2).

Let \( g_t \) be a metric on \( M \) such that \( g_t|_{(M \setminus Z_2)} = g_1|_{(M \setminus Z_2)} \) and such that \( Z_2 = I \times S^{n-1} \) becomes a cylinder of length \( t \). This is accomplished by replacing the unit interval by the interval \( [0, t] \) and using the product metric on \( Z_2 \). Now \( \text{Vol}(M, g_t) = a + bt \) where \( a \) and \( b \) are positive real constants (see Figure 3).
We take the following open cover of \( M \):
\[
\begin{align*}
U_1 &= H_1 \cup Z_1 \Rightarrow U_1 \cap U_2 = \emptyset, \\
U_2 &= M \setminus H_1 \cup Z_1 \cup Z_2 \Rightarrow U_1 \cap U_3 = Z_1, \quad U_1 \cap U_2 \cap U_3 = \emptyset, \\
U_3 &= Z_1 \cup Z_2 \cup Z_3 \Rightarrow U_2 \cap U_3 = Z_3.
\end{align*}
\]

Let \( \mu_{1,p} \) be the first positive eigenvalue of the Laplacian on exact forms of degree \( p \) on \( (M, g_t) \). To estimate \( \mu_{1,p} \) we can use a generalization of a lemma by McGowan [5]. Suppose we are given an open cover \( \{U_i\}_{i=1}^K \) of \( M \) without intersections of order bigger than two and which satisfies \( \sum_{i<j} \dim H^{p-1}(U_{ij}, \mathbb{R}) = 0 \) \( (U_{ij} = U_i \cap U_j) \); then the following holds:

**Lemma 1.** Given an open cover of \( M \) as above, denote by \( \mu(U_i) \), resp. \( \mu(U_{ij}) \), the smallest positive eigenvalue of the Laplacian acting on exact forms of degree \( p \) on \( U_i \), resp. of degree \( p - 1 \) on \( U_{ij} \), satisfying absolute boundary conditions. Then

\[
\mu_{1,p}(M) \geq \frac{2^{-3}}{\sum_{i=1}^K \left( \frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} \left( \frac{w_{n,p}}{\mu(U_{ij})} + 1 \right) \right) \left( \frac{1}{\mu(U_j)} + \frac{1}{\mu(U_{ij})} \right)}
\]

where \( m_i \) is the number of \( j, j \neq i \), for which \( U_i \cap U_j \neq \emptyset \), \( w_{n,p} \) a combinatorial constant which depends on \( p \) and \( n \). \( c_p = (\max_{x \in U_i} \max_{t \in U_j} \| \nabla \rho_t(x) \|^2 \) for a fixed partition of unity \( \{\rho_t\}_{t=1}^K \) subordinate to the given cover.

The proof of this lemma uses the same arguments as the proof of the lemma by McGowan [5]. The generalization is trivial, because we made special assumptions on the cover of \( M \). (See remarks at the end of paragraph 2, p. 735 of [5].)

Denote by \( \lambda_{r,s}(N) \) the \( r \)th eigenvalue of the Laplacian on \( s \)-forms on \( N \) with respect to absolute boundary conditions, in case \( N \) has a boundary. If 0 is an eigenvalue with multiplicity, we denote it by \( \lambda_{0,s} \). We apply Lemma 1 to \( M_t = (M, g_t) \) and the cover \( \{U_1, U_2, U_3\} \).

\( \mu(U_1), \mu(U_2), \mu(U_{13}), \mu(U_{23}) \) are independent of \( t \). By using the Künneth formula, we get the following inequality for \( \mu(U_3) \):

\[
\mu(U_3) \geq \lambda_{1,p}(U_3) = \lambda_{1,p}(I_t \times S^{n-1})
\]

\[
\geq \min_{i,j,k,l} \{ \lambda_{i,0}(I_t) + \lambda_{j,p}(S^{n-1}), \lambda_{k,1}(I_t) + \lambda_{l,p-1}(S^{n-1}) \}
\]

\[
\geq \min_{j,l} \{ \lambda_{j,p}(S^{n-1}), \lambda_{l,p-1}(S^{n-1}) \} = c(p)
\]

\[
\geq \min_{2 \leq p \leq n-2} c(p) =: c > 0 \quad \text{independent of } t,
\]

\( c(p) > 0 \), because for \( 1 \leq p \leq n-2 \) there are no non-trivial harmonic forms of degree \( p \) on \( S^{n-1}(H^p(S^{n-1}) = 0 \) for \( p \neq 0, n-1 \).

By applying the lemma above to \( M_t = (M, g_t) \) we get that

\[
\mu_{1,p}(M_t) \geq \delta > 0
\]

independent of \( t \).

The volume of \( M_t \) is given by \( \text{Vol}(M_t) = a + bt \) with constants \( a, b > 0 \). Set \( \overline{g}_t = (a + bt)^{-2/n} g_t \) and \( \overline{M}_t = (M, \overline{g}_t) \); then \( \text{Vol}(\overline{M}_t) = 1 \) and \( \mu_{1,p}(\overline{M}_t) = (a + bt)^{2/n} \mu_{1,p}(M_t) \). This implies that

\[
\mu_{1,p}(\overline{M}_t) \geq \delta(a + bt)^{2/n}, \quad \text{with } \delta > 0.
\]
Therefore $\mu_{1,p}(\overline{M}_t) \to \infty$ when $t \to \infty$. Since the full spectrum of $\Delta_{g_t}$ is obtained from the spectra of exact forms of degree $p$ and $p + 1$, the theorem follows. □

Remarks. Berger's problem still is open for 1-forms. Tanno [6] showed that on $S^3$, $\lambda_{1,1}$ is bounded for the 1-parameter family of metrics in the examples of Bleecker and Urakawa, which gave unbounded first eigenvalue for functions.

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