RIEMANNIAN METRICS WITH LARGE FIRST EIGENVALUE ON FORMS OF DEGREE $p$

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Abstract. Let $(M, g)$ be a compact, connected, $C^\infty$ Riemannian manifold of $n$ dimensions. Denote by $\lambda_{1,p}(M, g)$ the first nonzero eigenvalue of the Laplace operator acting on differential forms of degree $p$. We prove that for $n > 4$ and $2 \leq p \leq n - 2$, there exists a family of metrics $g_t$ of volume one, such that $\lambda_{1,p}(M, g_t) \to \infty$ as $t \to \infty$.

1. Introduction

Let $(M^n, g)$ be a compact, connected Riemannian manifold of $n$ dimensions. The Laplacian $\Delta_{g,p}$ acting on differential forms of degree $p$ on $M$ has discrete spectrum. Let $\lambda_{1,p}(g)$ denote the smallest positive eigenvalue of $\Delta_{g,p}$. For functions we set as usual $\lambda_1(g) = \lambda_{1,0}(g)$. Hersch [4] has proved, that for functions on $S^2$ we have:

$$\lambda_1(g) \text{Vol}(S^2, g) \leq 8\pi$$

for every Riemannian metric $g$.

In connection with this result, M. Berger [1] asked whether there exists a constant $k(M)$ such that:

$$\lambda_1(g) \text{Vol}(M^n, g)^{2/n} \leq k(M)$$

for any Riemannian metric $g$ on $M$. Yang and Yau [9] have proved that the inequality above holds for a compact surface $S$ of genus $\gamma$ with $k(S) = 8\pi(\gamma + 1)$.

Subsequently, Bleecker [2], Urakawa [7] and others constructed examples of manifolds of dimension $n \geq 3$ for which (1) was false. Finally Xu [8] and Colbois and Dodziuk [3] showed that (1) was false for every Riemannian manifold of dimension $n \geq 3$.

The same questions was posed by S. Tanno [6] for forms of degree $p$: does there exist a constant $k(M)$ such that:

$$\lambda_{1,p}(g) \text{Vol}(M^n, g)^{2/n} \leq k(M)$$

for any Riemannian metric $g$ on $M$?

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2. Results

In this note we show that (2) is false for \( n \geq 4 \) and \( 2 < p < n - 2 \). The proof uses a generalization of a technical lemma figuring in J. McGowan [5]. This lemma allows one to estimate from below the first eigenvalue for exact forms, in terms of first eigenvalues for exact forms on parts of \( M \), with respect to absolute boundary conditions.

**Theorem 1.** Every compact, connected manifold \( M^n \) of dimension \( n \geq 4 \) admits metrics \( g \) of volume one with arbitrarily large \( \lambda_{1,p}(g) \) for all \( 2 \leq p \leq n - 2 \).

**Proof.** We take a topological sphere \( S^n \) and choose a metric \( g_0 \) on it, such that \( S^n \) looks like a cigar, where the middle part has length 3. In particular this middle part is a product for the metric \( g_0 \), i.e. a cylinder \( I \times S^{n-1} \) (see Figure 1).

Remove the half-sphere \( H_2 \) at one end of the cigar and form a connected sum with \( M \). The resulting manifold is diffeomorphic to \( M \) and has a submanifold \( \Omega \), with smooth boundary, naturally identified with \( S^n \backslash H_2 \).

Let \( g_1 \) be an arbitrary metric on \( M \) whose restriction to \( \Omega \) is equal to \( g_0|_\Omega \). \( \Omega \) contains an open cylinder of length 3. We subdivide this cylinder into 3 cylinders \( Z_1, Z_2, Z_3 \) of length 1 (see Figure 2).

Let \( g_t \) be a metric on \( M \) such that \( g_t|(M\setminus Z_2) = g_1|(M\setminus Z_2) \) and such that \( Z_2 = I \times S^{n-1} \) becomes a cylinder of length \( t \). This is accomplished by replacing the unit interval by the interval \([0, t]\) and using the product metric on \( Z_2 \). Now \( \text{Vol}(M, g_t) = a + bt \) where \( a \) and \( b \) are positive real constants (see Figure 3).

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**Figure 1**

**Figure 2**

**Figure 3**
We take the following open cover of $M$:

$$
\begin{align*}
U_1 &= H_1 \cup Z_1, \\
U_2 &= M \setminus H_1 \cup Z_1 \cup Z_2, \\
U_3 &= Z_1 \cup Z_2 \cup Z_3,
\end{align*}
$$

$U_1 \cap U_2 = \emptyset,$

$U_1 \cap U_3 = Z_1,$

$U_1 \cap U_2 \cap U_3 = \emptyset.$

Let $\mu_{1,p}$ be the first positive eigenvalue of the Laplacian on exact forms of degree $p$ on $(M, g_t)$. To estimate $\mu_{1,p}$ we can use a generalization of a lemma by McGowan [5]. Suppose we are given an open cover $\{U_i\}_{i=1}^K$ of $M$ without intersections of order bigger than two and which satisfies $\sum_{i<j} \dim H^{p-1}(U_{ij}, \mathbb{R}) = 0$ ($U_{ij} = U_i \cap U_j$); then the following holds:

**Lemma 1.** Given an open cover of $M$ as above, denote by $\mu(U_i)$, resp. $\mu(U_{ij})$, the smallest positive eigenvalue of the Laplacian acting on exact forms of degree $p$ on $U_i$, resp. of degree $p-1$ on $U_{ij}$, satisfying absolute boundary conditions. Then

$$
\mu_{1,p}(M) \geq \sum_{i=1}^K \left( \frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} \left( \frac{w_{n-p} \cdot c_p}{\mu(U_{ij})} + 1 \right) \left( \frac{1}{\mu(U_j)} + \frac{1}{\mu(U_i)} \right) \right)
$$

where $m_i$ is the number of $j$, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$, $w_{n,p}$ a combinatorial constant which depends on $p$ and $n$. $c_p = (\max)_{i}(\max)_{x \in U_i} |\nabla \rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{i=1}^K$ subordinate to the given cover.

The proof of this lemma uses the same arguments as the proof of the lemma by McGowan [5]. The generalization is trivial, because we made special assumptions on the cover of $M$. (See remarks at the end of paragraph 2, p. 735 of [5].)

Denote by $\lambda_{r,s}(N)$ the $r$th eigenvalue of the Laplacian on $s$-forms on $N$ with respect to absolute boundary conditions, in case $n$ has a boundary. If $0$ is an eigenvalue with multiplicity, we denote it by $\lambda_{0,s}$. We apply Lemma 1 to $M_t = (M, g_t)$ and the cover $\{U_1, U_2, U_3\}$. $\mu(U_1), \mu(U_2), \mu(U_{13}), \mu(U_{23})$ are independent of $t$. By using the K"unneth formula, we get the following inequality for $\mu(U_3)$:

$$
\begin{align*}
\mu(U_3) &\geq \lambda_{1,p}(U_3) = \lambda_{1,p}(I_t \times S^{n-1}) \\
&\geq \min_{i,j,k,l} \{ \lambda_i,0(I_t) + \lambda_j,p(S^{n-1}), \lambda_k,1(I_t) + \lambda_l,p-1(S^{n-1}) \} \\
&\geq \min_{j,l} \{ \lambda_j,p(S^{n-1}), \lambda_l,p-1(S^{n-1}) \} = c(p) \\
&\geq \min_{2 \leq p \leq n-2} c(p) =: c > 0 \text{ independent of } t,
\end{align*}
$$

$c(p) > 0$, because for $1 \leq p \leq n-2$ there are no non-trivial harmonic forms of degree $p$ on $S^{n-1}(H^p(S^{n-1}) = 0$ for $p \neq 0, n-1$).

By applying the lemma above to $M_t = (M, g_t)$ we get that

$$
\mu_{1,p}(M_t) \geq \delta > 0
$$

independent of $t$.

The volume of $M_t$ is given by $\text{Vol}(M_t) = a + bt$ with constants $a, b > 0$. Set $\overline{g}_t = (a+bt)^{-2/n}g_t$ and $\overline{M}_t = (M, \overline{g}_t)$; then $\text{Vol}(\overline{M}_t) = 1$ and $\mu_{1,p}(\overline{M}_t) = (a+bt)^{2/n} \mu_{1,p}(M_t)$. This implies that

$$
\mu_{1,p}(\overline{M}_t) \geq \delta(a+bt)^{2/n}, \quad \text{with } \delta > 0.
$$
Therefore $\mu_{1,p}(\overline{M}_t) \to \infty$ when $t \to \infty$. Since the full spectrum of $\Delta_{g,p}$ is obtained from the spectra of exact forms of degree $p$ and $p+1$, the theorem follows. □

Remarks. Berger's problem still is open for 1-forms. Tanno [6] showed that on $S^3$, $\lambda_{1,1}$ is bounded for the 1-parameter family of metrics in the examples of Bleecker and Urakawa, which gave unbounded first eigenvalue for functions.

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