THE BEREZIN SYMBOL AND MULTIPLIERS
OF FUNCTIONAL HILBERT SPACES

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Abstract. This paper focuses on a multiplicative property of the Berezin symbol \( \tilde{A} \), of a given linear map \( A : \mathcal{H} \rightarrow \mathcal{H} \), where \( \mathcal{H} \) is a functional Hilbert space of analytic functions. We show \( \tilde{AB} = \tilde{A}B \) for all \( B \) in \( \mathfrak{B}(\mathcal{H}) \) if and only if \( A \) is a multiplication operator \( M_\varphi \), where \( \varphi \) is a multiplier. We also present a version of this result for vector-valued functional Hilbert spaces.

1. INTRODUCTION

Let \( n \) be a fixed positive integer and let \( \Omega \) be a region in \( \mathbb{C}^n \). A functional Hilbert space \( \mathcal{H} \) is a Hilbert space of analytic functions on \( \Omega \) such that the point evaluations are bounded, linear functionals. By the Riesz-representation theorem there exists, for each \( z \) in \( \Omega \), a unique element \( K_z \) of \( \mathcal{H} \) such that \( f(z) = \langle f, K_z \rangle \) for all \( f \) in \( \mathcal{H} \). The function \( K \) on \( \Omega \times \Omega \), defined by \( K(z, w) = K_w(z) \), is called the reproducing kernel function of \( \mathcal{H} \). Let \( k_z = K_z/\|K_z\| \) be the normalized reproducing kernel function. For a given linear map \( A : \mathcal{H} \rightarrow \mathcal{H} \), the Berezin symbol \( \tilde{A} \) (see [1]) of a map \( A \) of \( \mathcal{H} \) into itself is defined by

\[ \tilde{A}(z) = \langle Ak_z, k_z \rangle. \]

It is known that the map \( A \mapsto \tilde{A} \) is injective (see [3]). A function \( \varphi \) defined on \( \Omega \) is a multiplier of \( \mathcal{H} \) if \( \varphi \cdot f \) is in \( \mathcal{H} \), for all \( f \) in \( \mathcal{H} \). Let \( \mathfrak{B}(\mathcal{H}) \) denote the set of all bounded, linear operators from \( \mathcal{H} \) into \( \mathcal{H} \). The multiplication operator \( M_\varphi : \mathcal{H} \rightarrow \mathcal{H} \) defined by \( M_\varphi f = \varphi \cdot f \) is in \( \mathfrak{B}(\mathcal{H}) \), when \( \varphi \) is a multiplier of \( \mathcal{H} \).

2. THE MULTIPLICATIVE PROPERTY OF THE BEREZIN SYMBOL
ON A FUNCTIONAL HILBERT SPACE

Theorem 1. Let \( A \) be a bounded operator on \( \mathcal{H} \). Then

\[ \tilde{AB}(z) = \tilde{A}(z)\tilde{B}(z) \]

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for all $B$ in $\mathcal{B}(H)$ if and only if $A$ is a multiplication operator, $M_\varphi$, where $\varphi$ is a multiplier. Moreover, $\varphi = \tilde{\varphi}$.

Before proceeding with the proof, we need the following:

**Lemma 1.** When $\varphi$ is a multiplier of $H$, $\widetilde{M_\varphi}(z) = \varphi(z)$.

**Proof.** $\widetilde{M_\varphi}(z) = (M_\varphi k_z, k_z) = (\varphi k_z, k_z) = \varphi(z)$.

**Lemma 2.** The Berezin symbol of $f \otimes g$, for $f, g$ in $H$, is

$$(f \otimes g)(z) = \frac{g(z)}{\|K_z\|^2} f(z), \quad z \in \Omega.$$

**Proof.** For $f$ and $g$ in $H$ and $z$ in $\Omega$,

$$(f \otimes g)(z) = \frac{(f \otimes g) K_z K_z}{\|K_z\|^2} = \frac{1}{\|K_z\|^2} (K_z, g)(f, K_z).$$

By the reproducing property of the kernel function, we have

$$(f \otimes g)(z) = \frac{g(z)}{\|K_z\|^2} f(z), \quad f, g \in H.$$

**Proof of Theorem 1.** Suppose $\widetilde{AB}(z) = \tilde{A}(z)\tilde{B}(z)$ for all $B$ in $\mathcal{B}(H)$. Let $B = f \otimes g$ for $f$ and $g$ in $H$. Then, by Lemma 2,

$$\widetilde{AB}(z) = (Af \otimes g)(z) = \frac{g(z)}{\|K_z\|^2} (Af)(z).$$

By the hypothesis, we have

$$\left(\frac{g(z)}{\|K_z\|^2} (Af)(z) = \frac{g(z)}{\|K_z\|^2} \tilde{A}(z)f(z),$$

which reduces to

$$(Af)(z) = \tilde{A}(z)f(z)$$

for all $f$ in $H$. Hence $A = M_{\tilde{\varphi}}$.

Conversely, if $A$ is a multiplication operator, $M_\varphi$, where $\varphi$ is a multiplier,

$$\widetilde{M_\varphi B} = (M_\varphi Bk_z, k_z) = \varphi(z)\frac{(Bk_z)(z)}{\|K_z\|}$$

for all $B$ in $\mathcal{B}(H)$. By Lemma 1, we have

$$\widetilde{M_\varphi B}(z) = \widetilde{M_\varphi}(z)\tilde{B}(z)$$

for all $B$ in $\mathcal{B}(H)$.

**Corollary 1.** Let $B$ be in $\mathcal{B}(H)$. Then

$$\widetilde{AB}(z) = \tilde{A}(z)\tilde{B}(z)$$

for all $A$ in $\mathcal{B}(H)$ if and only if $B = M_{\psi}^*$, where $\psi$ is a multiplier.

**Proof.** The assertion follows from Theorem 1 and the fact that $\widetilde{T^*}(z) = \widetilde{\tilde{T}}(z)$, for all $T$ in $\mathcal{B}(H)$. 

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The Hardy space $H^2$ consists of the complex-valued analytic functions on the unit disk $D$ such that the Taylor coefficients are square summable. A calculation shows that $K_z = \frac{1}{1 - z\overline{w}}$ has the reproducing property (see [4]). Let $P$ denote the orthogonal projection of $L^2(\partial D)$ onto $H^2$, and let $\varphi$ be a bounded measurable function. Then the Toeplitz operator, $T_\varphi$, induced by $\varphi$ is defined by $T_\varphi f = P(\varphi f)$, for all $f$ in $H^2$.

**Corollary 2.** Let $A$ be a bounded operator on $H^2$. Then

$$\widetilde{AB}(z) = \tilde{A}(z)\tilde{B}(z)$$

for all $B$ in $\mathcal{B}(H^2)$ if and only if $A$ is a Toeplitz operator, $T_\varphi$, induced by $\varphi$ in $H^\infty$. Moreover $\varphi = \tilde{A}$.

**Proof.** The multiplication operators on $H^2$ are the analytic Toeplitz operators.

We should mention that Corollary 2 is also true if one replaces $H^2$ by the Bergman space or any of the weighted Bergman spaces. (For analytic Toeplitz operators on weighted Bergman spaces see [6].)

3. **The multiplicative property of the Berezin symbol on the analytic reproducing kernel space, $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{E}$**

Let $\mathcal{H}_0$ be a functional Hilbert space of (scalar-valued) analytic functions on $\Omega$ with the reproducing kernel function $K_z$, for each fixed $z$ in $\Omega$. Let $\mathcal{E}$ be a separable Hilbert space, and let $\mathcal{H}$ be the functional Hilbert space of $\mathcal{E}$-valued functions, $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{E}$. The reproducing kernel function of $\mathcal{H}$, $J_z : \mathcal{E} \mapsto \mathcal{H}$, is defined by $J_z(u) = K_z \otimes u$, where $u$ is in $\mathcal{E}$.

The evaluation functional $E_z : \mathcal{H} \mapsto \mathcal{E}$, defined by $E_z f = f(z)$, for $z$ in $\Omega$, is bounded (see [2], Lemma 3.2). For $f \in \mathcal{H}$, $u$ in $\mathcal{E}$, we have

$$\langle f, E_z^* u \rangle_{\mathcal{H}} = \langle f(z), u \rangle_{\mathcal{E}}.$$ 

We also have the reproducing property of the kernel function, that is

$$\langle f, J_z(u) \rangle_{\mathcal{H}} = \langle f(z), u \rangle_{\mathcal{E}}.$$ 

Therefore, $E_z^* u = J_z(u)$, for all $u$ in $\mathcal{E}$. By the reproducing property of the kernel function, we have $\|J_z(u)\|^2 = K_z(z)\|u\|^2$, where $u$ is in $\mathcal{E}$, and hence

$$\|J_z\| = \sqrt{K_z(z)} = \|E_z\|.$$ 

Let $H_z = \frac{1}{\|J_z\|}$ be the normalized reproducing kernel function, and let $A$ be a bounded linear operator on $\mathcal{H}$. Then the Berezin symbol $\widetilde{A}$ of $A$ is defined by

$$\widetilde{A}(z) = H_z^* A H_z.$$

**Lemma 3.** An operator $A$ is a multiplication operator if and only if, for each fixed $z$ in $\Omega$, $A^* E_z^* = E_z^* \Phi(z)^* \Phi(z)$ for some operator $\Phi(z)$ in $\mathcal{B}(\mathcal{E})$. Moreover, in this case, $A$ is the operator of multiplication by the function $z \mapsto \Phi(z)$.

**Proof.** Let $z$ be fixed in $\Omega$. Suppose $A$ is a multiplication operator, $M_\Phi$, induced by $\Phi : \Omega \mapsto \mathcal{B}(\mathcal{E})$. We observe that

$$E_z M_\Phi f = M_\Phi f(z) = \Phi(z) f(z) = \Phi(z) E_z f$$

for all $f$ in $\mathcal{H}$.

Then we have $E_z M_\Phi = \Phi(z) E_z$, for some operator $\Phi(z)$ in $\mathcal{B}(\mathcal{E})$. 


Conversely, let $A$ be a bounded operator on $\mathcal{H}$ such that $A^*E_z^* = E_z^*\Phi(z)^*$ for some operator $\Phi(z)$ in $\mathcal{B}(\mathcal{H})$. For $u$ in $\mathcal{B}$, we have

$$\langle f, A^*E_z^*u \rangle_{\mathcal{H}} = \langle Af, E_z^*u \rangle_{\mathcal{H}} = \langle (Af)(z), u \rangle_{\mathcal{H}}$$

for all $f$ in $\mathcal{H}$. On the other hand, for $u$ in $\mathcal{B}$, we have $\langle f, E_z^*\Phi(z)^*u \rangle = \langle \Phi(z)f(z), u \rangle$, for all $f$ in $\mathcal{H}$. Then $\langle (Af)(z), u \rangle = \langle \Phi(z)f(z), u \rangle$, for all $f$ in $\mathcal{H}$ and $u$ in $\mathcal{B}$. Therefore, $(Af)(z) = \Phi(z)f(z)$, for all $f$ in $\mathcal{H}$.

**Theorem 2.** Let $A$ be a bounded operator on $\mathcal{H}$. Then

$$\overline{AB}(z) = \overline{A(z)B(z)}$$

for all $B$ in $\mathcal{B}(\mathcal{H})$ if and only if $A = M_{\Phi}$, where $\Phi: \Omega \mapsto \mathcal{B}(\mathcal{H})$.

**Proof.** We observe that $E_zM_{\Phi}f = \Phi(z)f(z)$, for all $f$ in $\mathcal{H}$. Then $E_zM_{\Phi}E_z^* = \Phi(z)E_zE_z^*$ and $E_zM_{\Phi}BE_z^* = \Phi(z)E_zBE_z^*$, for all $B$ in $\mathcal{B}(\mathcal{H})$. Since $E_zE_z^* = K_z(z)I_{\mathcal{B}}$, we have $M_{\Phi} = \Phi(z)$ and

$$\overline{M_{\Phi}B(z)} = \overline{\Phi(z)E_zBE_z^*} = \overline{\Phi(z)}\overline{B(z)}$$

for all $B$ in $\mathcal{B}(\mathcal{H})$.

Conversely, suppose that $A$ is a bounded operator such that $\overline{AB}(z) = \overline{A(z)B(z)}$ for all $B$ in $\mathcal{B}(\mathcal{H})$. Then from the definitions, we get

$$E_zA^*BE_z^* = \frac{1}{\|E_z\|^2}E_zAE_z^*E_zBE_z^*$$

for all $B$ in $\mathcal{B}(\mathcal{H})$.

For $u$ and $v$ in $\mathcal{B}$, we have

$$\langle E_zA^*BE_z^*u, v \rangle = \left\langle \frac{E_zAE_z^*}{\|E_z\|^2}E_zBE_z^*u, v \right\rangle = \langle \overline{A(z)}E_zBE_z^*u, v \rangle.$$

Then we have

$$\langle BE_z^*u, A^*E_z^*v \rangle = \langle BE_z^*u, E_z^*\overline{A(z)}v \rangle.$$

For each fixed nonzero $u$, $BE_z^*u$ runs through all vectors in $\mathcal{H}$ as $B$ runs through all elements of $\mathcal{B}(\mathcal{H})$. Thus we see that $A^*E_z^* = E_z^*\overline{A(z)}^*$, for all $z$ in $\Omega$. Therefore $A$ is a multiplication operator, $M_{\overline{A}}$, by Lemma 3.

Let us note that if we take $\mathcal{B}$ to be $\mathcal{C}$ and define $\overline{\mathcal{H}} = K_z\otimes 1$, the sufficiency proof of Theorem 2 will also work for Theorem 1, the scalar-valued case.

Let $N = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers. The set $N^n$ is partially ordered by setting $I = (i_1, i_2, \ldots, i_n) \succeq (j_1, j_2, \ldots, j_n) = J$ if and only if $i_k \geq j_k$ for $k = 1, 2, \ldots, n$. If $z = (z_1, z_2, \ldots, z_n)$ is in $\Omega$, then we set $z^I = z_1^{i_1}z_2^{i_2}\cdots z_n^{i_n}$. We denote by $H^2(n) \otimes \mathcal{B}$, where $H^2(n) = H^2 \otimes H^2 \otimes \cdots \otimes H^2$ ($n$ copies), the set of all vector-valued analytic functions $f: D^n \mapsto \mathcal{B}$ with power series expansion $f(z) = \sum_{I \in N^n} z^Iv_I$, with $v_I$ in $\mathcal{B}$ and $z$ in $D^n$, such that $\sum_{I \in N^n} \|v_I\|_{\mathcal{B}}^2 < \infty$.

The space $H^2(n) \otimes \mathcal{B}$ is a Hilbert space with the reproducing kernel function, $K_z: \mathcal{B} \mapsto H^2(n) \otimes \mathcal{B}$, for $z$ in $D^n$, defined by $K_z(u) = z^I \otimes u$, where $u$ is in $\mathcal{B}$ and $K_z(u) = \sum_{I \in N^n} z^Iw_I$ is the reproducing kernel function for $H^2(n)$ (see [5]). Let $H^\infty(n)(\mathcal{B}(\mathcal{H}))$ denote the Banach space of all bounded analytic functions $\Phi: D^n \mapsto \mathcal{B}(\mathcal{H})$ with the norm $\|\Phi\|_\infty = \sup\{\|\Phi(z)\|, \text{for } z \in D^n\}$. 

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For every $\Phi$ in $H^\infty(n)(\mathcal{B}(\mathbb{C}))$, we can define the analytic Toeplitz operator $T_\Phi$ in $\mathcal{B}(H^2(n) \otimes \mathbb{C})$ as follows:

$$(T_\Phi f)(z) = \Phi(z)f(z), \quad z \in D^n, \ f \in H^2(n) \otimes \mathbb{C}.$$ 

For the boundedness of the map $T_\Phi$ see [2].

**Corollary 3.** Let $A$ be a bounded operator on $H^2(n) \otimes \mathbb{C}$. Then

$$\hat{AB}(z) = \hat{A}(z)\hat{B}(z)$$

for all $B$ in $\mathcal{B}(H^2(n) \otimes \mathbb{C})$ if and only if $A = T_\Phi$, where $\Phi$ is in $H^\infty(n)(\mathcal{B}(\mathbb{C}))$.

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**References**

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