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BANACH SPACE PROPERTIES OF L^1 OF A VECTOR MEASURE

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ABSTRACT. We consider the space $L^1(\nu)$ of real functions which are integrable with respect to a measure ν with values in a Banach space X. We study type and cotype for $L^1(\nu)$. We study conditions on the measure ν and the Banach space X that imply that $L^1(\nu)$ is a Hilbert space, or has the Dunford–Pettis property. We also consider weak convergence in $L^1(\nu)$.

1. INTRODUCTION

Given a vector measure ν with values in a Banach space X, $L^1(\nu)$ denotes the space of (classes of) real functions which are integrable with respect to ν in the sense of Bartle, Dunford and Schwartz [BDS] and Lewis [L-1]. This space has been studied by Kluvanek and Knowles [KK], Thomas [T] and Okada [O]. It is an order continuous Banach lattice with weak unit. In [C-1, Theorem 8] we have identified the class of spaces $L^1(\nu)$, showing that every order continuous Banach lattice with weak unit can be obtained, order isometrically, as L^1 of a suitable vector measure.

A natural question arises: what is the relation between, on the one hand, the properties of the Banach space X and the measure ν, and, on the other hand, the properties of the resulting space $L^1(\nu)$. The complexity of the situation is shown by the following example: the measures, defined over Lebesgue measurable sets of [0,1], $\nu_1(A) = m(A) \in \mathbb{R}$, $\nu_2(A) = \chi_A \in L^1([0, 1])$ and $\nu_3 = (\int_A r_n(t)dt) \in c_0$, where r_n are the Rademacher functions, generate, order isometrically, the same space, namely $L^1([0, 1])$. The translation of properties from $L^1(\nu)$ to the Banach space X is limited by the following result: *every separable order continuous Banach lattice with weak unit and no atoms can be obtained, order isomorphically, as L^1 of a c_0-valued measure* [C-2, Theorem 1]. In this paper we show that in the opposite direction there is a clear line of influence, that is, the properties of X and ν determine, to some extent, the properties of $L^1(\nu)$. We study type and cotype for $L^1(\nu)$; conditions on X and ν in order to have $L^1(\nu)$ order isomorphic to a Hilbert space; and conditions on X and ν so that $L^1(\nu)$ has the Dunford–Pettis property. We

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also study weak convergence in $L^1(\nu)$, giving an example of a measure ν such that weak convergence of sequences in $L^1(\nu)$ is not given by weak convergence of the integrals over arbitrary sets.

2. Preliminaries

Let (Ω, Σ) be a measurable space, X a Banach space with unit ball B_X and dual space X^*, and $\nu : \Sigma \longrightarrow X$ a countably additive vector measure. The semivariation of ν is the set function $\|\nu\|(A) = \sup\{|x^*\nu|(A) : x^* \in B_{X^*}\}$, where $|x^*\nu|$ is the variation of the scalar measure $x^*\nu$. A Rybakov control measure for ν is a measure $\lambda = |x^*\nu|$, such that $\lambda(A) = 0$ if and only if $\|\nu\|(A) = 0$ (see [DU, Theorem IX.2.2]).

Following Lewis [L-1] we will say that a measurable function $f : \Omega \longrightarrow \mathbb{R}$ is *integrable* with respect to ν if

(1) f is $x^*\nu$ integrable for every $x^* \in X^*$, and

(2) for each $A \in \Sigma$ there exists an element of X, denoted by $\int_A f\,d\nu$, such that

$$x^* \int_A f\,d\nu = \int_A f\,dx^*\nu \quad \text{for every} \quad x^* \in X^*.$$

Identifying two functions if the set where they differ has null semivariation, we obtain a linear space of classes of functions which, when endowed with the norm

$$\|f\|_\nu = \sup\left\{\int_\Omega |f|\,d|x^*\nu| : x^* \in B_{X^*}\right\},$$

becomes a Banach space. We will denote it by $L^1(\nu)$. It is a Banach lattice for the $\|\nu\|$–almost everywhere order. Simple functions are dense in $L^1(\nu)$ and the identity is a continuous injection of the space of $\|\nu\|$–essentially bounded functions into $L^1(\nu)$. An equivalent norm for $L^1(\nu)$ is

$$|||f|||_\nu = \sup\left\{\left\|\int_A f\,d\nu\right\| : A \in \Sigma\right\},$$

for which we have $|||f|||_\nu \le \|f\|_\nu \le 2 \cdot |||f|||_\nu$.

Let λ be a Rybakov control measure for ν. Then $L^1(\nu)$ is an order continuous Banach function space with weak unit over the finite measure space $(\Omega, \Sigma, \lambda)$ (see [C-1, Theorem 1]). Thus it can be regarded as a lattice ideal in $L^1(\lambda)$, and $L^1(\nu)^*$ can be identified with the space of functions g in $L^1(\lambda)$ such that $fg \in L^1(\lambda)$, for all f in $L^1(\nu)$, where the action of g over $L^1(\nu)$ is given by integration with respect to λ.

The integration operator $\nu : L^1(\nu) \longrightarrow X$ is defined as $\nu(f) = \int f\,d\nu$, for $f \in L^1(\nu)$. It is a continuous linear operator with norm less than or equal to one. It is important to remark that no assumptions are made on the variation of the measure ν for the definition of the space $L^1(\nu)$.

A bounded set in a Banach lattice is L–weakly compact if for every sequence (x_n) of positive pairwise disjoint vectors such that for each n there exists y_n in the set with $x_n \le |y_n|$, we have that (x_n) converges to zero in norm [M-1, Definition II.1]. L–weakly compact sets are relatively weakly compact [M-1, Satz II.6]. In order continuous Banach function spaces over a finite measure space (S, σ, μ) L–weak compactness is equivalent to equi-integrablility: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \sigma$ with $\mu(A) < \delta$ we have $\|f\chi_A\| < \varepsilon$, for all f in the set.

For the general theory of vector measures we refer the reader to [DU]. Aspects related to Banach lattices can be seen in [AB], [LT, vol. II] and [M-3].

3. Type and cotype for $L^1(\nu)$

Recall that a Banach space has cotype q, for $2 \leq q < +\infty$ (type q, for $1 < q \leq 2$), if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for any elements $x_1, \dots, x_n$ in X we have

$$\left(\sum_1^n \|x_i\|^q\right)^{1/q} \leq (\geq)\, C\,\frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\|\sum_1^n \theta_i x_i\right\|.$$

Theorem 1. *Let X be a Banach space with cotype q, for $q \geq 2$, and ν an X-valued vector measure. Then the space $L^1(\nu)$ has cotype q.*

Proof. Let $q > 2$. As $L^1(\nu)$ is a Banach lattice, the property of having cotype $q > 2$ is equivalent to satisfying a lower q-estimate [LT, vol. II, p. 88]. Let $f_1, \dots, f_n$ be disjoint functions in $L^1(\nu)$, and $(A_i)_1^n$ disjoint measurable sets such that each A_i is contained in the support of f_i. Then $\int_{\cup A_i}(\sum_1^n f_j)\, d\nu = \sum_1^n \int_{A_i} f_i\, d\nu$. Let $(\theta_i)_1^n$ be an arbitrary choice of signs $\theta_i = \pm 1$; then

$$\left\|\sum_1^n \theta_i \int_{A_i} f_i\, d\nu\right\| \leq \left\|\sum_1^n \theta_i f_i\right\|_\nu = \left\|\sum_1^n f_i\right\|_\nu .$$

Averaging over all possible choices of signs and considering that the Banach space X has cotype q, we have

$$\frac{1}{C} \cdot \left(\sum_1^n \left\|\int_{A_i} f_i\, d\nu\right\|^q\right)^{1/q} \leq \frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\|\sum_1^n \theta_i \int_{A_i} f_i\, d\nu\right\| \leq \left\|\sum_1^n f_i\right\|_\nu .$$

Taking the supremum over all possible choices of sets $(A_i)_1^n$ and considering the equivalent norm $|||\cdot|||_\nu$ in $L^1(\nu)$, we deduce that

$$\left(\sum_1^n \|f_i\|_\nu^q\right)^{1/q} \leq 2C \cdot \left\|\sum_1^n f_i\right\|_\nu .$$

Hence $L^1(\nu)$ satisfies a lower q-estimate and thus it has cotype q.

Let $q = 2$. We will prove that $L^1(\nu)$ has cotype 2 by showing that it is 2-concave [LT, vol. II, Theorem 1.f.16]. Let $f_1, \dots, f_n$ be in $L^1(\nu)$. Set $f = \left(\sum_1^n |f_i|^2\right)^{1/2}$. Consider the lattice ideal generated by f in $L^1(\nu)$

$$I(f) = \left\{g \in L^1(\nu) : \exists \lambda > 0,\ |g| \leq \lambda f\right\}$$

with the norm $\|g\|_\infty = \inf\{\lambda \geq 0 : |g| \leq \lambda \cdot f/\|f\|_\nu\}$. Its completion is an AM-space with unit, so by a result of Kakutani it is order isometric to a space $C(K)$, for K a compact topological space [LT, vol. II, Theorem 1.b.6]. The injection $j: C(K) \longrightarrow L^1(\nu)$ has norm one and $\|f\|_\infty = \|f\|_\nu$. Consider the composition of this injection with the integration operator $\nu: L^1(\nu) \longrightarrow X$. As X has cotype 2, by Grothendieck's Theorem the operator $\nu \circ j$ is 2-summing

[P-2, Theorem 5.14]. Thus there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for any functions $g_1, \dots, g_n$ in $C(K)$ we have

$$\left(\sum_1^n \|\nu \circ j(g_i)\|^2\right)^{1/2} \leq C \cdot \sup\left\{\left(\sum_1^n |\langle \mu, g_i\rangle|^2\right)^{1/2} : \mu \in C(K)^*, \|\mu\| \leq 1\right\}.$$

This last supremum is $\|(\sum_1^n |g_i|^2)^{1/2}\|_\infty$. Consider measurable sets $(A_i)_1^n$ and set $g_i = f_i \cdot \chi_{A_i}$. From the previous expression we have

$$\left(\sum_1^n \left\|\int_{A_i} f_i \, d\nu\right\|^2\right)^{1/2} \leq C \cdot \left\|\left(\sum_1^n |f_i \cdot \chi_{A_i}|^2\right)^{1/2}\right\|_\infty \leq C \cdot \left\|\left(\sum_1^n |f_i|^2\right)^{1/2}\right\|_\nu .$$

Taking the supremum over all possible choices of sets $(A_i)_1^n$ and considering the equivalent norm $|||\cdot|||_\nu$ in $L^1(\nu)$, it follows that

$$\left(\sum_1^n \|f_i\|_\nu^2\right)^{1/2} \leq 2C \cdot \left\|\left(\sum_1^n |f_i|^2\right)^{1/2}\right\|_\nu .$$

Thus $L^1(\nu)$ is 2–concave and so it has cotype 2. □

$L^1(\nu)$ does not inherit type from the Banach space X: consider the Lebesgue measure restricted to [0,1], the space $L^1(\nu)$ obtained is $L^1[0, 1]$ which has no type.

Theorem 2. *Let ν be a measure with values in ℓ^p, for $1 \leq p < 2$. Then the space $L^1(\nu)$ has type less than or equal to p.*

Proof. Suppose $L^1(\nu)$ has type q for some $p < q \leq 2$. Then the integration operator $\nu : L^1(\nu) \longrightarrow \ell^p$ is compact. To prove it, assume by way of contradiction that this is not the case, then there exists a sequence (f_n) of norm one elements in $L^1(\nu)$ and $\varepsilon > 0$ such that $\|\nu(f_n)\| > \varepsilon$ and the sequence $(\nu(f_n))$ is weakly null in ℓ^p. Then there is a subsequence that we will still denote by $(\nu(f_n))$, which is a basic sequence equivalent to a block basis of ℓ^p. For $n \in \mathbb{N}$ and scalars $a_1, \dots, a_n$, we have:

$$\left(\sum_1^n |a_i|^p\right)^{1/p} \sim \frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\|\sum_1^n \theta_i a_i \nu(f_i)\right\| \leq \frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\|\sum_1^n \theta_i a_i f_i\right\|_\nu .$$

Since $L^1(\nu)$ has type q, there exists a constant $C > 0$ such that

$$\frac{1}{2^n} \cdot \sum_{\theta_i \in \{1,-1\}} \left\|\sum_1^n \theta_i a_i f_i\right\|_\nu \leq 1/C \cdot \left(\sum_1^n \|a_i f_i\|_\nu{}^q\right)^{1/q} = 1/C \cdot \left(\sum_1^n |a_i|^q\right)^{1/q} .$$

Combining the previous inequalities we arrive at a contradiction, as $p < q$. Hence the operator ν is compact.

The result follows from the next claim, which implies that $L^1(\nu)$ has a subspace isomorphic to ℓ^1, contradicting $L^1(\nu)$ having type $q > 1$.

Claim. *Let ν be an X-valued measure such that the integration operator ν : $L^1(\nu) \longrightarrow X$, is compact. Then the space $L^1(\nu)$ has a complemented subspace isomorphic to ℓ^1.*

Proof of the Claim. Let λ be a Rybakov control measure for ν. Consider the transpose of the integration operator $\nu^*: X^* \longrightarrow L^1(\nu)^*$. For $x^* \in X^*$, $\nu^*(x^*)$ can be identified with the Radon–Nikodym derivative of the measure $x^*\nu$ with respect to λ. Thus the norm in $L^1(\nu)$ can be written in the following way:

$$\|f\|_\nu = \sup\left\{\int |f||h|\,d\lambda : h \in \nu^*(B_{X^*})\right\}.$$

Let f be in $L^1(\nu)$ and let A be a measurable set. We have

$$(1)\qquad \begin{aligned}\|f\cdot\chi_A\|_\nu &= \sup\left\{\int_A |f||h|\,d\lambda : h \in \nu^*(B_{X^*})\right\}\\ &\le \|f\|_\nu \cdot \sup\left\{\|h\cdot\chi_A\|_{L^1(\nu)^*} : h \in \nu^*(B_{X^*})\right\}.\end{aligned}$$

Suppose $L^1(\nu)$ has no complemented subspace isomorphic to ℓ^1. Then $L^1(\nu)^*$ has no subspace isomorphic to ℓ_∞ [BP, Theorem 4]. As $L^1(\nu)^*$ is a dual Banach lattice it is order complete; this fact combined with $\ell_\infty \not\subset L^1(\nu)^*$ implies that $L^1(\nu)^*$ is order continuous [AB, Theorem 14.9]. In order continuous Banach lattices relatively compact sets are L–weakly compact [M-1, Korollar II.4]. Hence since $\nu^*(B_{X^*})$ is compact in $L^1(\nu)^*$, it is L–weakly compact, so equi–integrable; thus

$$(2)\qquad \lim_{\lambda(A)\to 0}\ \sup\left\{\|h\cdot\chi_A\|_{L^1(\nu)^*} : h \in \nu^*(B_{X^*})\right\} = 0.$$

From equations (1) and (2) it follows that in $L^1(\nu)$ norm bounded sets are equi–integrable, so L–weakly compact. Then, on the one hand, relatively weakly compact sets are L–weakly compact, which implies that every infinite-dimensional sublattice contains a subspace isomorphic to ℓ^1 [M-2, Satz 14]. On the other hand, the unit ball of $L^1(\nu)$ being bounded is L–weakly compact so relatively weakly compact. Thus $L^1(\nu)$ is reflexive. The contradiction establishes the claim. □

4. $L^1(\nu)$ A HILBERT SPACE

Theorem 3. *Let X be a Banach space with cotype 2. Let ν be an X–valued measure satisfying that for every partition $(A_n)_1^\infty$ of the measure space, the sequence*

$$(*)\qquad \left(\frac{\nu(A_n)}{\|\nu\|(A_n)}\right)$$

is 2–lacunary in X. Then $L^1(\nu)$ is order isomorphic to a Hilbert space.

Proof. Given any partition $(A_n)_1^\infty$ there exists a constant $K = K(A_n)$, depending on the partition, such that for every sequence (α_n) in ℓ^2 we have

$$\left\|\sum_1^\infty \alpha_n \frac{\nu(A_n)}{\|\nu\|(A_n)}\right\| \le K\cdot\left(\sum_1^\infty \alpha_n^2\right)^{1/2}.$$

Let λ be a control measure for ν. For a measurable set B with $\lambda(B) > 0$ define $\mathscr{K}(B) = \sup\{K(B_n) : (B_n)$ is a partition of $B\}$. Then for every $A \in \Sigma$

with $\lambda(A) > 0$ there exists a measurable set $B \subset A$ with $\lambda(B) > 0$ such that $\mathscr{K}(B) < +\infty$. Assume by way of contradiction that this is not the case. Then there exists a measurable set A with $\lambda(A) > 0$ such that for every $B \subset A$ with $\lambda(B) > 0$ we have $\mathscr{K}(B) = +\infty$. Let (A_n) be a partition of A such that $\lambda(A_n) > 0$. As $\mathscr{K}(A_n) = +\infty$, for every $n \in \mathbb{N}$ there exists a partition (A_i^n) of A_n such that $K(A_i^n) > n$. So there exist real numbers $\alpha_1^n, \alpha_2^n, \dots, \alpha_{i(n)}^n$ such that

$$\left\| \sum_{i=1}^{i(n)} \alpha_i^n \frac{\nu(A_i^n)}{\|\nu\|(A_i^n)} \right\| > n \cdot \left(\sum_{i=1}^{i(n)} |\alpha_i^n|^2 \right)^{1/2}.$$

Consider the following partition of A

$$A_1^1,\ A_2^1,\ \dots,\ A_{i(1)}^1,\ \bigcup_{i(1)}^{\infty} A_i^1,\ A_1^2,\ A_2^2,\ \dots,\ A_{i(2)}^2,\ \bigcup_{i(2)}^{\infty} A_i^2,\ \dots.$$

The associated sequence is not 2–lacunary in X. Applying an exhaustion argument [DU, Lemma III.2.4] we deduce that there exists a partition (B_n) of Ω such that $\mathscr{K}(B_n) < +\infty$, for every $n \in \mathbb{N}$. A similar argument shows that in fact we have $K = \sup_n \mathscr{K}(B_n) < +\infty$.

It follows that for every partition (A_n) and for every sequence $(a_n) \in \ell^2$ we have

$$\left\| \sum_1^{\infty} a_n \nu(A_n) \right\| \le K \cdot \left(\sum_1^{\infty} a_n^2 \|\nu\|(A_n)^2 \right)^{1/2}.$$

Let g be a simple function $g = \sum_1^n a_i \chi_{A_i}$, where the sets A_i are disjoint. Let $B \in \Sigma$. From the previous inequality we have

$$\left\| \int_B g\, d\nu \right\| = \left\| \sum_1^n a_i \nu(A_i \cap B) \right\| \le K \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2}.$$

Considering the equivalent norm $|||\cdot|||_\nu$ in $L^1(\nu)$ we deduce that

$$\left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu \le 2K \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2}. \tag{3}$$

Since X has cotype 2, by Theorem 1 $L^1(\nu)$ has cotype 2, so it satisfies a lower–2 estimate: there exists $C > 0$ such that for any scalars $a_1, \dots, a_n$ and disjoint measurable sets $A_1, \dots, A_n$ we have

$$\left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2} \le C \cdot \left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu. \tag{4}$$

From (3) and (4) it follows that for a simple function $g = \sum_1^n a_i \chi_{A_i}$ where the sets A_i are disjoint, we have

$$1/C \cdot \left(\sum_1^n a_i^2 \|\nu\|(A_i)^2 \right)^{1/2} \le \left\| \sum_1^n a_i \chi_{A_i} \right\|_\nu \le 2K \cdot \left(\sum_1^n a_n^2 \|\nu\|(A_i)^2 \right)^{1/2}. \tag{5}$$

This inequality evaluated at $a_1 = \cdots = a_n = 1$ gives, for disjoint measurable sets $(A_i)_1^n$,

$$(1/C)^2 \cdot \sum_1^n \|\nu\|(A_i)^2 \leq \|\nu\| \left(\bigcup_1^n A_i \right)^2 \leq (2K)^2 \cdot \sum_1^n \|\nu\|(A_i)^2.$$

Consider the set function

$$A \in \Sigma \longmapsto \mu(A) = \inf \left\{ \sum_1^n \tau(A_i) : (A_i)_1^n \text{ is a partition of } A \right\} \in \mathbb{R},$$

where

$$\tau(A) = \sup \left\{ \sum_1^n \|\nu\|(A_i)^2 : (A_i)_1^n \text{ is a partition of } A \right\},$$

for $A \in \Sigma$. μ is a countably additive measure that satisfies $(1/C)^2\mu(A) \leq \|\nu\|(A)^2 \leq (2K)^2\mu(A)$ for every $A \in \Sigma$. From (5) it follows that $L^1(\nu)$ is order isomorphic to the space $L^2(\Omega, \Sigma, \mu)$. □

Remarks. 1. Condition $(*)$ is necessary for $L^1(\nu)$ to be a Hilbert space since the integration operator is continuous and, in a Hilbert lattice, a sequence of normalized disjoint functions is 2–lacunary. Condition $(*)$ does not imply that $L^1(\nu)$ or X have type 2. To see this consider the measure defined over the subsets of natural numbers, such that $\nu(\{n\}) = a_n \cdot e_n \in c_0$ where (a_n) is a positive null sequence, and e_n is the nth vector of the canonical basis of c_0. Then $L^1(\nu)$ is c_0 and ν satisfies $(*)$. The requirement of X having cotype 2 is not necessary, as the space $L^2[0, 1]$ obtained from a c_0–valued measure shows [C-2, Theorem 1].

2. A condition, stronger than $(*)$, but easier to verify as it deals with the norm instead of the semivariation, is the following: *for every partition* $(A_n)_1^\infty$ *of the measure space, such that* $\nu(A_n) \neq 0$ *for every* $n \in \mathbb{N}$, *the sequence in* X $\left(\frac{\nu(A_n)}{\|\nu(A_n)\|}\right)$, *is* 2*–lacunary.*

5. $L^1(\nu)$ A DUNFORD–PETTIS SPACE

A Banach space has the Dunford–Pettis property if weakly compact operators defined on it map relatively weakly compact sets into relatively compact sets. We study sufficient conditions on the measure ν and the Banach space X in order to obtain $L^1(\nu)$ with the Dunford–Pettis property. Recall that a Banach space has the Schur property if weak convergence of a sequence implies its norm convergence.

Theorem 4. *Let X be a Banach space with the Schur property and ν an X–valued measure with σ–finite variation. Then the space $L^1(\nu)$ has the Dunford–Pettis property.*

Proof. The result follows from the next two claims.

Claim 1. *If ν takes its values in a Banach space with the Schur property, then in $L^1(\nu)$ relatively weakly compact sets coincide with L–weakly compact sets.*

Claim 2. *If ν has σ-finite variation, Y is a Banach space and $T: L^1(\nu) \longrightarrow Y$ is a weakly compact operator, then T maps L-weakly compact sets into relatively norm compact sets.*

Proof of Claim 1. Suppose that there exists a set in $L^1(\nu)$ that is relatively weakly compact but it is not L-weakly compact: then there exist functions f_n, disjoint measurable sets A_n and $\varepsilon > 0$ such that $\|f_n \chi_{A_n}\|_\nu \geq \varepsilon$ for every $n \in \mathbb{N}$. As the set $\{f_n : n \in \mathbb{N}\}$ is relatively weakly compact, there exists a subsequence, that we still denote by (f_n), which converges weakly in $L^1(\nu)$ to a function $f \in L^1(\nu)$. It follows that for every $A \in \Sigma$ the sequence $(\int_A f_n \, d\nu)$ converges weakly in X to $\int_A f \, d\nu$. As X is a Schur space, the convergence is in norm.

Let μ and μ_n be the measures with densities f and f_n with respect to ν, respectively. They are countably additive and absolutely continuous with respect to a control measure [L-1, Theorem 2.2]. Since $(\mu_n(A))$ converges in norm to $\mu(A)$, for every $A \in \Sigma$, by the Vitali–Hahn–Saks theorem it follows that $\{\mu_n\}$ is uniformly countably additive [DU, Corollary I.5.6]. This implies that $\lim_k \sup_n \|\mu_n\|(A_k) = 0$. Since $\|\mu_n\|(A_n) = \|f_n \chi_{A_n}\|_\nu$, we arrived at a contradiction.

Proof of Claim 2. Let λ be a Rybakov control measure for ν. Considering the measure defined by $A \in \Sigma \longmapsto T(\chi_A) \in Y$, the σ–finiteness of the variation of ν and the weak compactness of T, we obtain, by the vector Radon–Nikodym Theorem (see [DU, Theorem III.2.18]), a function $g: \Omega \longrightarrow Y$ λ–measurable and Pettis integrable with respect to λ, such that the operator T can be represented as

$$T(f) = \text{Pettis–} \int f g \, d\lambda.$$

Let K be an L–weakly compact set in $L^1(\nu)$ and $\varepsilon > 0$. Since K is equi–integrable and g is λ–measurable, there is a simple function φ and a measurable set A such that $\|f \cdot \chi_A\|_\nu < \varepsilon$ for every $f \in K$, and $\|g(\omega) - \varphi(\omega)\| < \varepsilon$ for every $\omega \in \Omega \setminus A$. For f in K we have

$$Tf = T(f \cdot \chi_A) + \int_{\Omega \setminus A} f \varphi \, d\lambda + \int_{\Omega \setminus A} f(g - \varphi) \, d\lambda,$$

where $\|T(f \cdot \chi_A)\| < \varepsilon \|T\|$. Since K is bounded $\|f\|_\nu \leq M$, so

$$\left\| \int_{\Omega \setminus A} f(g - \varphi) \, d\lambda \right\| \leq \int_{\Omega \setminus A} |f| \cdot \|g - \varphi\| \, d\lambda \leq \varepsilon \int |f| \, d\lambda \leq \varepsilon \, \|f\|_\nu \leq \varepsilon M.$$

It follows that the distance between the sets $T(K)$ and $\{\int_{\Omega \setminus A} f \varphi \, d\lambda : f \in K\}$ is less than $\varepsilon(\|T\| + M)$. This last set is compact, hence $T(K)$ is relatively compact in Y. □

Remark. From Claim 2 in the previous theorem we can derive the following consequence: *If ν has no atoms and σ-finite variation, then $L^1(\nu)$ is not reflexive.* To see this let λ be a Rybakov control measure for ν and K be the unit ball of $L_\infty(\lambda)$, which is an L–weakly compact set in $L^1(\nu)$. If $L^1(\nu)$ is reflexive, then K is relatively compact in $L^1(\nu)$, hence in $L^1(\lambda)$. But this cannot be if λ is nonatomic, since in K we can build a Rademacher type sequence. It follows, for example, that in order to have $L^1(\nu)$ order isomorphic to $L^p([0, 1])$, for $1 < p < +\infty$, it is necessary that the variation of ν be infinite on every measurable set where it is non null.

6. Weak convergence in $L^1(\nu)$

In [C-2, Theorem 4] we showed that if $L^1(\nu)$ has no complemented subspace isomorphic to ℓ^1, then weak convergence of bounded nets in $L^1(\nu)$ is characterized by weak convergence of the integrals over arbitrary sets. This was proved independently by Okada [O, Corollary 16]. In [O] the author mentions the question, raised by Professor J. Diestel, as to whether or not the above characterization of weak convergence in $L^1(\nu)$ holds, in general, for sequences. The following example shows that this is not the case.

Let $\mathscr{M}$ be the σ–algebra of Lebesgue measurable sets of the interval $[0, +\infty)$ and let m be the Lebesgue measure on the interval. Let r_n be the Rademacher functions, defined on $[0, +\infty)$ by $r_n(t) = \operatorname{sign}(\sin(2^n \pi t))$. Consider the measure

$$A \in \mathscr{M} \longmapsto \nu(A) = \sum_1^\infty \frac{1}{2^k} \nu_k(A) \in \ell^2 ,$$

where the measures ν_k are defined as

$$\nu_k(A) = \left(\overbrace{0, \ldots, 0}^{k-1}, \int_{A\cap[k-1, k]} r_k(t)\, dt, \int_{A\cap[k-1, k]} r_{k+1}(t)\, dt, \ldots \right).$$

Each measure ν_k is well defined, countably additive and satisfies

$$\|\nu_k(A)\|_2 \le \|\chi_{A\cap[k-1, k]}\|_{L^2([k-1, k])} = m(A \cap [k-1, k])^{1/2}.$$

Thus the measure ν is well defined and countably additive. Consider in $L^1(\nu)$ the sequence (f_n) where $f_n = 2^n \cdot \chi_{[n-1, n]}$. As the function f_n is supported on the interval $[n-1, n]$, we have $\|f_n\|_\nu = \|\nu_n\|([n-1, n]) \le 1$.

For every $A \in \mathscr{M}$ we have $\int_A f_n \, d\nu = \nu_n(A \cap [n-1, n])$. This vector, with norm less than or equal to one, belongs to the subspace generated by the vectors $e_n, e_{n+1}, \ldots$ of the canonical basis of ℓ^2. Thus the sequence $(\int_A f_n \, d\nu)$ tends weakly to zero in ℓ^2.

Let $a_1, \ldots, a_N$ be scalars. For every n, $1 \le n \le N$, consider the set $A_n = \{t \in [n-1, n] : r_N(t) = \operatorname{sign}(a_n)\}$. Then we have

$$\int_{A_n} r_k \, dm = \begin{cases} 0 & \text{if } k \ne N, \\ (1/2) \cdot \operatorname{sign}(a_n) & \text{if } k = N. \end{cases}$$

Thus $\nu_n(A_n) = (1/2) \cdot \operatorname{sign}(a_n) \cdot e_N$. Let $A = \bigcup_1^N A_n$. Then

$$\left\| \sum_1^N a_n f_n \right\|_\nu \ge \left\| \sum_1^N a_n \int_A f_n \, d\nu \right\| = \left\| \sum_1^N a_n \nu_n(A_n) \right\|$$
$$= \left\| \sum_1^N a_n (1/2) \cdot \operatorname{sign}(a_n) \cdot e_N \right\| = (1/2) \sum_1^N |a_n|.$$

As the sequence (f_n) is bounded, it follows that (f_n) is equivalent in $L^1(\nu)$ to the canonical basis of ℓ^1. Thus (f_n) does not tend weakly to zero in $L^1(\nu)$.

The measure ν has unbounded variation. This is not relevant, as the same construction can be done with values in c_0 and the resulting measure has bounded variation.

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