BOGOMOLOV-GIESEKER INEQUALITY AND COHOMOLOGY VANISHING IN CHARACTERISTIC \( p \)

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Abstract. We prove an analogue of the Bogomolov-Gieseker inequality for rank-two bundles on varieties defined over a field of positive characteristic. We derive from this some vanishing results for cohomology of line bundles.

1. Introduction

Let \( X \) be a smooth projective variety defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). In [M2], a Bogomolov-Gieseker type inequality for rank-two bundles on \( X \) has been obtained under the assumption that \( X \) is not uniruled. In the present note we shall show that a similar inequality holds under certain stability conditions on the tangent bundle of \( X \). As a corollary, we obtain some vanishing results for cohomology of line bundles. We also consider the vanishing for bundles of higher rank.

2. Main result

In what follows, we assume that all varieties are defined over an algebraically closed field \( k \) of characteristic \( p > 0 \).

Let \( X \) be a smooth projective variety over \( k \), and let \( H \) be an ample line bundle on \( X \). Let \( E \) be a vector bundle on \( X \). Following [M1], we say that \( E \) is \( p \)-semistable with respect to \( H \) if, for all \( m > 0 \), the \( m \)-th iterated Frobenius pull-back \( (F^m)^*E \) is \( \mu \)-semistable with respect to \( H \). We have the following Bogomolov-Gieseker type inequality, which is due to A. Moriwaki.

Proposition 1 ([M1, Theorem 1]). Let \( X \) be a smooth projective variety of dimension \( d \geq 2 \), and let \( H \) be an ample line bundle on \( X \). Let \( E \) be a rank-two vector bundle on \( X \) which is \( p \)-semistable with respect to \( H \). Then we have

\[
\{ c_1(E)^2 - 4c_2(E) \}.H^{d-2} \leq 0.
\]

Let \( X, H \) be as above. For a vector bundle \( E \) on \( X \), we denote by \( \mu_H(E) \) the slope of \( E \) with respect to \( H \):

\[
\mu_H(E) = \frac{c_1(E).H^{d-1}}{\text{rk} E}.
\]
Let $T_X$ be the tangent bundle of $X$, and let

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{i-1} \subset T_i = T_X$$

be the Harder-Narasimhan filtration of $T_X$ with respect to $H$. We set $\mu_H(X) := \mu_H(T_X/T_{i-1})$. The following is a generalization of [L-S, 2.4.Satz].

**Lemma 1.** Let $X$ be a smooth projective variety of dimension $d \geq 1$ with an ample line bundle $H$. Let $E$ be a rank-two vector bundle on $X$. Assume that $E$ is $\mu$-semistable and $F^*E$ is not $\mu$-semistable with respect to $H$. If $M \subset F^*E$ denotes the maximal destabilizing subline bundle, then we have

$$M.H^{d-1} \leq \frac{1}{2} \left\{ pc_1(E).H^{d-1} - \mu_H(X) \right\}.$$ 

**Proof.** By the radical descent theory in [G], we have an $\mathcal{O}_X$-homomorphism

$$f : T_X \rightarrow \text{End}_{\mathcal{O}_X}(F^*E)$$

where $X^{(p)}$ is the scheme obtained from the base change by the Frobenius map of $k$. Composing $f$ with the inclusion $M \hookrightarrow F^*E$ and with the projection $F^*E \rightarrow F^*E/M$, we obtain the following $\mathcal{O}_X$-homomorphism

$$\tilde{f} : T_X \rightarrow \text{Hom}_{\mathcal{O}_X}(M, F^*E/M).$$

We claim that $\tilde{f} \neq 0$. Otherwise, there would exist a subsheaf $M' \subset E$ such that $F^*M' = M$, contradicting the semistability of $E$. Hence the desired inequality follows immediately. $\square$

**Theorem 1.** Let $X$ be a smooth projective variety of dimension $d \geq 2$ with an ample line bundle $H$. Let $E$ be a rank-two vector bundle on $X$, which is $\mu$-semistable with respect to $H$.

1. If $\mu_H(X) \geq 0$, then $\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} \leq 0$.
2. If $\mu_H(X) < 0$, then

$$\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} \leq \frac{\mu_H(X)^2}{p^2H^d}.$$ 

**Proof.** Let $m$ be the smallest integer such that $(F^m)^*E$ is not $\mu$-semistable. We claim that if $\mu_H(X) \geq 0$, then we have $m = \infty$. Indeed, assume that $m < \infty$ and let $M'$ be the maximal destabilizing subsheaf of $(F^m)^*E$. If we define the Q-line bundle $M := M'/p^m$, then

$$\frac{c_1(E).H^{d-1}}{2} < M.H^{d-1}.$$ 

Applying Lemma 1 to $(F^{m-1})^*E$, we obtain

$$M.H^{d-1} \leq \frac{1}{2} \left\{ c_1(E).H^{d-1} - \frac{\mu_H(X)}{p^m} \right\},$$

which is a contradiction if $\mu_H(X) \geq 0$. Hence in case (1) we are done by Proposition 1.

Assume that $\mu_H(X) < 0$. If we set

$$\alpha = \frac{(2M - c_1(E)).H^{d-1}}{H^d},$$

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then we have
\[ 0 < \alpha \leq -\frac{\mu_H(X)}{p^mH^d}. \]
Since \((2M - c_1(E) - \alpha H).H^{d-1} = 0\), the Hodge index theorem yields
\[ (2M - c_1(E) - \alpha H)^2.H^{d-2} \leq 0. \]
Therefore we have
\[
\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} \leq (2M - c_1(E))^2.H^{d-2} \\
\leq \{2\alpha(2M - c_1(E)).H - \alpha^2H^2\}.H^{d-2} \\
= \alpha^2H^d \\
\leq \frac{\mu_H(X)^2}{p^2mH^d} \leq \frac{\mu_H(X)^2}{p^2H^d}.
\]
This completes the proof. \(\square\)

The above theorem implies vanishing results for the cohomology of line bundles as in [M2].

**Corollary 1.** Let \(X\) be a smooth projective variety of dimension \(d \geq 2\) with an ample line bundle \(H\). Let \(L\) be a nef line bundle on \(X\). Assume that either
1. \(\mu_H(X) \geq 0\) and \(L^2.H^{d-2} > 0\), or
2. \(\mu_H(X) < 0\) and
\[
L^2.H^{d-2} > \frac{\mu_H(X)^2}{p^2H^d}.
\]
Then we have \(H^1(X, L^{-1}) = 0\).

**Proof.** If \(H^1(X, L^{-1}) \neq 0\), then we obtain a non-split extension
\[ 0 \to \mathcal{O}_X \to E \to L \to 0. \]
If we show that \(E\) is \(\mu\)-semistable with respect to \(H\), then we obtain a contradiction by Theorem 1, since we have \(\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} = L^2.H^{d-2} \). To show the \(\mu\)-semistability of \(E\), assume that there exist a subline bundle \(M \hookrightarrow E\) with \((2M - L).H^{d-1} > 0\) and an exact sequence
\[ 0 \to M \to E \to \mathcal{I}_Z(L - M) \to 0 \]
where \(Z\) is a codimension-two subscheme. The composition map \(M \hookrightarrow E \to L\) is not zero, since otherwise we would obtain a nontrivial map \(M \to \mathcal{O}_X\), which is impossible. Hence there exists an effective divisor \(D\) which is linearly equivalent to \(L - M\) and satisfies \((L - 2D).H^{d-2} > 0\). Then, by the Hodge index theorem, we have \((L - 2D).D.H^{d-2} > 0\). On the other hand, we have \(Z.H^{d-2} = c_2(E(-M)).H^{d-2} = (D - L).D.H^{d-2} \geq 0\). It follows that \(D = 0\) or, equivalently, \(L = M\), hence the original sequence must split. This contradiction proves that \(E\) is \(\mu\)-semistable. \(\square\)

We say that \(T_X\) is nef if the tautological line bundle \(\mathcal{O}(1)\) on \(\mathbb{P}(T_X)\) is nef. For example, \(T_X\) is nef if it is globally generated.
Corollary 2. Let $X$ be a smooth projective variety of dimension $d \geq 2$ with nef tangent bundle $T_X$. If $L$ is a nef and big line bundle on $X$, then we have $H^1(X, L^{-1}) = 0$.

Proof. Let $Q$ be a quotient bundle of $T_X$. It can be easily seen that if $T_X$ is nef, then $c_1(Q).H^{d-1} \geq 0$ for every ample line bundle $H$. In particular, we have $\mu_H(X) \geq 0$. Hence the claim follows from Corollary 1. □

Corollary 3. Let $X$ be a Fano variety of dimension $d \geq 2$ such that $T_X$ is $\mu$-semistable with respect to $-K_X$. Then we have $H^1(X, mK_X) = 0$ for all $m > 0$.

3. Vanishing for bundles of higher rank

Let $X$ be a smooth projective variety. A vector bundle $E$ on $X$ is said to be cohomologically $p$-ample if, for every coherent sheaf $\mathcal{F}$ on $X$, there exists an integer $m_0 = m_0(\mathcal{F})$ such that for all $m \geq m_0$ and all $i > 0$ we have $H^i(X, \mathcal{F} \otimes (Fm)^*E) = 0$. It is known that every cohomologically $p$-ample vector bundle is ample. A line bundle is cohomologically $p$-ample if and only if it is ample (cf. [K]).

Proposition 2. Let $X$ be a smooth projective variety of dimension $d \geq 1$ with an ample line bundle $H$. Let $E$ be a vector bundle on $X$ which is cohomologically $p$-ample and $p$-semistable with respect to $H$. Assume that either

1. $\mu_H(X) \geq 0$, or
2. $\mu_H(X) < 0$ and $\mu_H(E) > -\mu_H(X)$.

Then we have $H^1(X, E^\vee) = 0$.

Proof. We define

$$B_X := \text{Im} \left( d : F_*\mathcal{O}_X \to F_*\Omega_X^1 \right)$$

where $d$ is the differential map. Let $\mathcal{O}_X \to F_*\mathcal{O}_X$ be the natural map which sends $f$ to $f^p$. Then we have the exact sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B_X \to 0.$$ (*)&

By assumption, it is easy to see that $\mu_H(X) > \mu_H((Fm)^*E^\vee)$ for all $m \geq 0$. It follows that $H^0(X, (Fm)^*E^\vee \otimes \Omega_X^1) = \text{Hom}(T_X, (Fm)^*E^\vee) = 0$, hence we have $H^0(X, (Fm)^*E^\vee \otimes B_X) = 0$. Tensoring (*) with $(Fm)^*E^\vee$ and taking cohomology, we obtain injections $H^1(X, E^\vee) \hookrightarrow H^1(X, (Fm)^*E^\vee)$ for all $m \geq 0$. On the other hand, since $E$ is cohomologically $p$-ample, we have $H^1(X, (Fm)^*E^\vee) = H^{d-1}(X, (Fm)^*E \otimes \omega_X) = 0$ for sufficiently large $m$. Therefore, by descending induction on $m$, we obtain $H^1(X, E^\vee) = 0$. □

If the exact sequence (*) splits, then $X$ is called Frobenius split. It has been proved that Schubert varieties are Frobenius split ([M-R]). We have a stronger vanishing result for varieties which are Frobenius split.

Proposition 3. Assume that $X$ is Frobenius split and $E$ is a cohomologically $p$-ample vector bundle on $X$. Then we have $H^i(X, E^\vee) = 0$ for $i < d = \dim X$.

Proof. Since the exact sequence

$$0 \to (Fm)^*E^\vee \to (Fm)^*E^\vee \otimes F_*\mathcal{O}_X \to (Fm)^*E^\vee \otimes B_X \to 0$$
splits, we obtain injections $H^i(X, (F^m)^*E^\vee) \hookrightarrow H^i(X, (F^{m+1})^*E^\vee)$ for all $m$ and $i$. Then a similar argument as in Proposition 2 completes the proof. □

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