PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 12, December 1995

BOGOMOLOV-GIESEKER INEQUALITY AND COHOMOLOGY VANISHING IN CHARACTERISTIC p

TOHRU NAKASHIMA

(Communicated by Eric Friedlander)

ABSTRACT. We prove an analogue of the Bogomolov-Gieseker inequality for rank-two bundles on varieties defined over a field of positive characteristic. We derive from this some vanishing results for cohomology of line bundles.

1. INTRODUCTION

Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. In [M2], a Bogomolov-Gieseker type inequality for rank-two bundles on X has been obtained under the assumption that X is not uniruled. In the present note we shall show that a similar inequality holds under certain stability conditions on the tangent bundle of X. As a corollary, we obtain some vanishing results for cohomology of line bundles. We also consider the vanishing for bundles of higher rank.

2. MAIN RESULT

In what follows, we assume that all varieties are defined over an algebraically closed field k of characteristic $p > 0$.

Let X be a smooth projective variety over k, and let H be an ample line bundle on X. Let E be a vector bundle on X. Following [M1], we say that E is *p-semistable with respect to* H if, for all $m \geq 0$, the m-th iterated Frobenius pull-back $(F^m)^*E$ is μ-semistable with respect to H. We have the following Bogomolov-Gieseker type inequality, which is due to A. Moriwaki.

Proposition 1 ([M1,Theorem 1]). *Let X be a smooth projective variety of dimension $d \geq 2$, and let H be an ample line bundle on X. Let E be a rank-two vector bundle on X which is p-semistable with respect to H. Then we have*

$$\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} \leq 0.$$

Let X, H be as above. For a vector bundle E on X, we denote by $\mu_H(E)$ the slope of E with respect to H:

$$\mu_H(E) = \frac{c_1(E).H^{d-1}}{\operatorname{rk} E}.$$

Received by the editors June 7, 1994.
1991 *Mathematics Subject Classification*. Primary 14F17; Secondary 14J60.

Let T_X be the tangent bundle of X, and let

$$0 = T_0 \subset T_1 \subset \cdots \subset T_{l-1} \subset T_l = T_X$$

be the Harder-Narasimhan filtration of T_X with respect to H. We set $\mu_H(X) := \mu_H(T_X/T_{l-1})$. The following is a generalization of [L-S, 2.4.Satz].

Lemma 1. *Let X be a smooth projective variety of dimension $d \geq 1$ with an ample line bundle H. Let E be a rank-two vector bundle on X. Assume that E is μ-semistable and F^*E is not μ-semistable with respect to H. If $M \subset F^*E$ denotes the maximal destabilizing subline bundle, then we have*

$$M.H^{d-1} \leq \frac{1}{2}\left\{pc_1(E).H^{d-1} - \mu_H(X)\right\}.$$

Proof. By the radical descent theory in [G], we have an $\mathscr{O}_X$-homomorphism

$$f : T_X \to \mathrm{End}_{\mathscr{O}_{X^{(p)}}}(F^*E)$$

where $X^{(p)}$ is the scheme obtained from the base change by the Frobenius map of k. Composing f with the inclusion $M \hookrightarrow F^*E$ and with the projection $F^*E \to F^*E/M$, we obtain the following $\mathscr{O}_X$-homomorphism

$$\tilde{f} : T_X \to \mathrm{Hom}_{\mathscr{O}_X}(M, F^*E/M).$$

We claim that $\tilde{f} \neq 0$. Otherwise, there would exist a subsheaf $M' \subset E$ such that $F^*M' = M$, contradicting the semistability of E. Hence the desired inequality follows immediately. □

Theorem 1. *Let X be a smooth projective variety of dimension $d \geq 2$ with an ample line bundle H. Let E be a rank-two vector bundle on X, which is μ-semistable with respect to H.*

(1) *If $\mu_H(X) \geq 0$, then $\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} \leq 0$.*
(2) *If $\mu_H(X) < 0$, then*

$$\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} \leq \frac{\mu_H(X)^2}{p^2 H^d}.$$

Proof. Let m be the smallest integer such that $(F^m)^*E$ is not μ-semistable. We claim that if $\mu_H(X) \geq 0$, then we have $m = \infty$. Indeed, assume that $m < \infty$ and let M' be the maximal destabilizing subsheaf of $(F^m)^*E$. If we define the $\mathbb{Q}$-line bundle $M := M'/p^m$, then

$$\frac{c_1(E).H^{d-1}}{2} < M.H^{d-1}.$$

Applying Lemma 1 to $(F^{m-1})^*E$, we obtain

$$M.H^{d-1} \leq \frac{1}{2}\left\{c_1(E).H^{d-1} - \frac{\mu_H(X)}{p^m}\right\},$$

which is a contradiction if $\mu_H(X) \geq 0$. Hence in case (1) we are done by Proposition 1.

Assume that $\mu_H(X) < 0$. If we set

$$\alpha = \frac{(2M - c_1(E)).H^{d-1}}{H^d},$$

then we have

$$0 < \alpha \leq -\frac{\mu_H(X)}{p^m H^d}.$$

Since $(2M - c_1(E) - \alpha H).H^{d-1} = 0$, the Hodge index theorem yields

$$(2M - c_1(E) - \alpha H)^2.H^{d-2} \leq 0.$$

Therefore we have

$$\begin{aligned}\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} &\leq (2M - c_1(E))^2.H^{d-2} \\ &\leq \{2\alpha(2M - c_1(E)).H - \alpha^2 H^2\}.H^{d-2} \\ &= \alpha^2 H^d \\ &\leq \frac{\mu_H(X)^2}{p^{2m}H^d} \leq \frac{\mu_H(X)^2}{p^2 H^d}.\end{aligned}$$

This completes the proof. □

The above theorem implies vanishing results for the cohomology of line bundles as in [M2].

Corollary 1. *Let X be a smooth projective variety of dimension $d \geq 2$ with an ample line bundle H. Let L be a nef line bundle on X. Assume that either*

(1) $\mu_H(X) \geq 0$ *and* $L^2.H^{d-2} > 0$, *or*
(2) $\mu_H(X) < 0$ *and*

$$L^2.H^{d-2} > \frac{\mu_H(X)^2}{p^2 H^d}.$$

Then we have $H^1(X, L^{-1}) = 0$.

Proof. If $H^1(X, L^{-1}) \neq 0$, then we obtain a non-split extension

$$0 \to \mathscr{O}_X \to E \to L \to 0.$$

If we show that E is μ-semistable with respect to H, then we obtain a contradiction by Theorem 1, since we have $\{c_1(E)^2 - 4c_2(E)\}.H^{d-2} = L^2.H^{d-2}$. To show the μ-semistability of E, assume that there exist a subline bundle $M \hookrightarrow E$ with
$(2M - L).H^{d-1} > 0$ and an exact sequence

$$0 \to M \to E \to \mathscr{I}_Z(L - M) \to 0$$

where Z is a codimension-two subscheme. The composition map $M \hookrightarrow E \to L$ is not zero, since otherwise we would obtain a nontrivial map $M \to \mathscr{O}_X$, which is impossible. Hence there exists an effective divisor D which is linearly equivalent to $L - M$ and satisfies $(L - 2D).H^{d-1} > 0$. Then, by the Hodge index theorem, we have $(L - 2D).D.H^{d-2} > 0$. On the other hand, we have $Z.H^{d-2} = c_2(E(-M)).H^{d-2} = (D - L).D.H^{d-2} \geq 0$. It follows that $D = 0$ or, equivalently, $L = M$, hence the original sequence must split. This contradiction proves that E is μ-semistable. □

We say that T_X is *nef* if the tautological line bundle $\mathscr{O}(1)$ on $\mathbb{P}(T_X)$ is nef. For example, T_X is nef if it is globally generated.

Corollary 2. *Let X be a smooth projective variety of dimension $d \geq 2$ with nef tangent bundle T_X. If L is a nef and big line bundle on X, then we have $H^1(X, L^{-1}) = 0$.*

Proof. Let Q be a quotient bundle of T_X. It can be easily seen that if T_X is nef, then $c_1(Q).H^{d-1} \geq 0$ for every ample line bundle H. In particular, we have $\mu_H(X) \geq 0$. Hence the claim follows from Corollary 1. □

Corollary 3. *Let X be a Fano variety of dimension $d \geq 2$ such that T_X is μ-semistable with respect to $-K_X$. Then we have $H^1(X, mK_X) = 0$ for all $m > 0$.*

3. Vanishing for bundles of higher rank

Let X be a smooth projective variety. A vector bundle E on X is said to be *cohomologically p-ample* if, for every coherent sheaf $\mathscr{F}$ on X, there exists an integer $m_0 = m_0(\mathscr{F})$ such that for all $m \geq m_0$ and all $i > 0$ we have $H^i(X, \mathscr{F} \otimes (F^m)^*E) = 0$. It is known that every cohomologically p-ample vector bundle is ample. A line bundle is cohomologically p-ample if and only if it is ample (cf. [K]).

Proposition 2. *Let X be a smooth projective variety of dimension $d \geq 1$ with an ample line bundle H. Let E be a vector bundle on X which is cohomologically p-ample and p-semistable with respect to H. Assume that either*

(1) $\mu_H(X) \geq 0$, *or*
(2) $\mu_H(X) < 0$ *and* $\mu_H(E) > -\mu_H(X)$.

Then we have $H^1(X, E^\vee) = 0$.

Proof. We define

$$B_X := \operatorname{Im}\left(d : F_*\mathscr{O}_X \to F_*\Omega^1_X\right)$$

where d is the differential map. Let $\mathscr{O}_X \to F_*\mathscr{O}_X$ be the natural map which sends f to f^p. Then we have the exact sequence

$$(*) \qquad 0 \to \mathscr{O}_X \to F_*\mathscr{O}_X \to B_X \to 0.$$

By assumption, it is easy to see that $\mu_H(X) > \mu_H((F^m)^*E^\vee)$ for all $m \geq 0$. It follows that $H^0(X, (F^m)^*E^\vee \otimes \Omega^1_X) = \operatorname{Hom}(T_X, (F^m)^*E^\vee) = 0$, hence we have $H^0(X, (F^m)^*E^\vee \otimes B_X) = 0$. Tensoring $(*)$ with $(F^m)^*E^\vee$ and taking cohomology, we obtain injections $H^1(X, E^\vee) \hookrightarrow H^1(X, (F^m)^*E^\vee)$ for all $m \geq 0$. On the other hand, since E is cohomologically p-ample, we have $H^1(X, (F^m)^*E^\vee) = H^{d-1}(X, (F^m)^*E \otimes \omega_X) = 0$ for sufficiently large m. Therefore, by descending induction on m, we obtain $H^1(X, E^\vee) = 0$. □

If the exact sequence $(*)$ splits, then X is called *Frobenius split*. It has been proved that Schubert varieties are Frobenius split ([M-R]). We have a stronger vanishing result for varieties which are Frobenius split.

Proposition 3. *Assume that X is Frobenius split and E is a cohomologically p-ample vector bundle on X. Then we have $H^i(X, E^\vee) = 0$ for $i < d = \dim X$.*

Proof. Since the exact sequence

$$0 \to (F^m)^*E^\vee \to (F^m)^*E^\vee \otimes F_*\mathscr{O}_X \to (F^m)^*E^\vee \otimes B_X \to 0$$

splits, we obtain injections $H^i(X, (F^m)^*E^\vee) \hookrightarrow H^i(X, (F^{m+1})^*E^\vee)$ for all m and i. Then a similar argument as in Proposition 2 completes the proof. $\square$

Acknowledgment

The author is very grateful to the referee for pointing out several mistakes in the first version of this paper.

References

[G] A. Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique* I, Séminaire Bourbaki 1959/60.

[K] S. Kleiman, *Ample vector bundles on algebraic surfaces*, Proc. Amer. Math. Soc. **21** (1969), 673–676.

[L-S] H. Lange and U. Stuhler, *Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe*, Math. Z. **156** (1977), 73–83.

[M1] A. Moriwaki, *A note on Bogomolov-Gieseker's inequality in positive characteristic*, Duke Math. J. **64** (1991), 361–375.

[M2] ——, *Frobenius pull-back of vector bundles of rank 2 over non-uniruled varieties*, Math. Ann. **296** (1993), 441–451.

[M-R] V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. (2) **122** (1985), 27–40.

Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo, 192-03, Japan

E-mail address: `nakasima@math.metro-u.ac.jp`