PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 12, December 1995

GENERALIZATIONS OF SEMI-FREDHOLM OPERATORS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. On nonseparable Hilbert spaces there are multiple sets of operators that are analogous to the semi-Fredholm operators on a separable space. We develop the properties of these sets and relate those properties to some recent research. We conclude with a theorem that indicates precisely how far one can go from a given generalized semi-Fredholm operator (or generalized Fredholm operator) and retain the property of generalized semi-Fredholmness (or generalized Fredholmness).

1. INTRODUCTION

By "operator" we mean a bounded linear operator T defined on a Hilbert space H; we do not assume H to be separable. Let $\mathscr{B}(H)$ denote the operators on H. When H is not separable there are multiple sets of operators that might be used as natural generalizations of the Fredholm operators on a separable Hilbert space. See Definition 2.7 and Theorem 2.8 of [11], and the references listed there. Much less attention has been given to the semi-Fredholm operators. This paper continues our study of nonseparable Hilbert spaces ([5], [6], [7]) by determining properties of several classes of operators analogous to the semi-Fredholm operators on a separable space.

We shall use some concepts from earlier papers. Let $U|T|$ be the usual polar factorization of T, and let $E(\cdot)$ be the spectral measure for the nonnegative operator $|T|$. In [5] we defined ess nul T by the equation

$$\text{ess null } T = \inf\{(\dim E([0, \varepsilon))H : \varepsilon > 0\},$$

and by definition ess def T is ess nul T^*. In [5] we showed that the closure of the invertible operators is the set of operators T such that ess nul $T =$ ess def T.

In [7] we defined the modulus of invertibility, denoted $\rho(T)$, by the equation

$$\rho(T) = \inf\{\lambda: \ \dim E((\lambda - \varepsilon, \lambda + \varepsilon))H = \dim H \text{ for } \varepsilon > 0\}.$$

This quantity is related to the distance between T and the invertible operators. It follows from Theorem 4 in [8] that $\rho(T)$ is precisely the distance from T

Received by the editors May 27, 1994.
1991 *Mathematics Subject Classification*. Primary 47A58, 47A53; Secondary 47A05.
Key words and phrases. Semi-Fredholm operator, nonseparable Hilbert space, essential nullity, modulus of invertibility, stability theorem.

to the set $\{A\colon \text{ess nul } A = 0\}$. In this paper we use a natural generalization of $\rho(T)$ defined in §2.

Recall that the minimum modulus of T, denote $m(T)$, is defined to be $\inf\{\|Tf\|\colon \|f\| = 1\}$. The essential spectrum of T, denoted $\sigma_e(T)$, is the set $\{z\colon (T - zI)$ is not Fredholm$\}$, and the essential minimum modulus $m_e(T)$ is defined to be $\inf\{\lambda\colon \lambda \in \sigma_e(|T|)\}$. In [2] we showed that $m_e(T)$ was the minimum modulus of the projection of T into the Calkin algebra, which is defined in the next section. In [2] we computed the distance from T to the invertible operators in terms of $m_e(T)$. In this paper we shall use a generalization of $m_e(T)$.

We shall use the following standard notation. The closure of a set $\mathscr{S}$ is denoted $\mathscr{S}^-$. The restriction of T to a subspace H_0 is denoted $T|H_0$.

2. Preliminary results

We begin by defining a set of operators that form a closed two-sided ideal for the ring of operators $\mathscr{B}(H)$. We say that A belongs to $\mathscr{I}_\beta$ provided that any (closed) subspace K contained in the range of A, denoted AH, has the property that $\dim K < \beta$. The next theorem will relate the preceding definition to other definitions that have appeared in the literature.

Theorem 1. *The following are equivalent.*

(i) $A \in \mathscr{I}_\beta$.

(ii) *There is a sequence* $\{A_k\} \subset \mathscr{B}(H)$ *such that* $\|A - A_k\| \to 0$ *as* $k \to \infty$ *and* $\dim(A_k H)^- < \beta$.

(iii) *For each positive* ε *there is a subspace* H_ε *such that* $\|A|H_\varepsilon\| < \varepsilon$ *and* $\dim(H_\varepsilon)^\perp < \beta$.

Proof. (i) $\Rightarrow$ (ii) Let $U|A|$ be the usual polar factorization of A, and let $E(\cdot)$ be the spectral measure for $|A|$. Let a positive ε be given, and let H_ε denote $E([\varepsilon, \infty))H$. Since $|A|$ is obviously bounded below on H_ε and U is isometric on $(|A|H)^-$, we see that $|A|H_\varepsilon$ and $AH_\varepsilon = U|A|H_\varepsilon$ are both closed subspaces. Clearly AH_ε is contained in AH, and, consequently, $\dim AH_\varepsilon < \beta$.

Define the operator A_ε to agree with A on H_ε and to agree with 0 on $(H_\varepsilon)^\perp$. Clearly $\dim A_\varepsilon H < \beta$, and

$$\begin{aligned}\|A - A_\varepsilon\| &= \|(A - A_\varepsilon)|(H_\varepsilon)^\perp\| \\ &= \|A|(H_\varepsilon)^\perp\| = \||A|\,|(H_\varepsilon)^\perp\| \le \varepsilon.\end{aligned}$$

It is clear that (ii) holds.

(ii) $\Rightarrow$ (iii) Let positive ε be given, and choose k such that $\|A - A_k\| < \varepsilon$ with $\dim(A_k H)^- < \beta$. Let $K_\varepsilon = ((A_k)^* H)^-$, and note that A_k is zero on $(K_\varepsilon)^\perp$. Clearly, $\|A|(K_\varepsilon)^\perp\| < \varepsilon$ holds. Let $H_\varepsilon = (K_\varepsilon)^\perp$, and note that

$$\dim(H_\varepsilon)^\perp = \dim K_\varepsilon = \dim(A_k H) < \beta.$$

(iii) $\Rightarrow$ (i) Let K be a (closed) subspace contained in AH, and let A^+ be the inverse of $A|(A^* H)^-$. Let $H_0 = A^+ K$, and note that $A|H_0$ maps H_0 onto K in a one-to-one way. Thus, H_0 and K have the same dimension, and there is a positive ε such that

$$\|Af\| \ge 2\varepsilon\|f\| \quad \text{for } f \in H_0.$$

By hypothesis there is a subspace H_ε such that $\|A|H_\varepsilon\| < \varepsilon$ and $\dim(H_\varepsilon)^\perp < \beta$. Let P be the projection onto $(H_\varepsilon)^\perp$. If f is a nonzero vector belonging to $\ker P|H_0$ then $f = (I-P)f \in H_\varepsilon$ and $\|Af\| \le \varepsilon\|f\|$. This contradicts the construction of H_0, and so we conclude that $P|H_0$ is one-to-one. It follows that

$$\dim K = \dim H_0 = \dim PH_0 \le \dim(H_\varepsilon)^\perp < \beta,$$

which verifies (i).

We need a modest generalization of the modules of invertibility, which was defined in [7]. Let β be an infinite cardinal number that does not exceed $\dim H$. Define $\rho_\beta(T)$ by the equation

$$\rho_\beta(T) = \inf\{\lambda\colon \dim E((\lambda-\delta, \lambda+\delta))H \ge \beta \quad \text{for } \delta > 0\}$$

where $E(\cdot)$ is the spectral measure for $|T| = (T^*T)^{1/2}$. Consideration of the function $u(x) = \dim E([0, x))H$ shows that the set of λ above cannot be empty. It is clear that $\gamma \ge \beta$ implies $\rho_\gamma(T) \ge \rho_\beta(T)$.

The first part of the next result is well known (see [11]); the rest of it is new.

Theorem 2. *$\mathcal{J}_\beta$ is a closed two-sided ideal of $\mathcal{B}(H)$, and $\mathcal{B}(H)/\mathcal{J}_\beta$ is a C^*-algebra. If $m_\beta(T)$ denotes the minimum modulus of the image of T in $\mathcal{B}(H)/\mathcal{J}_\beta$, denoted $[T]$, then $m_\beta(T) > 0$ if and only if $[T]$ is left-invertible. Furthermore, $m_\beta(T) = \rho_\beta(T)$.*

Proof. It is routine to show that $\mathcal{J}_\beta$ is a closed two-sided ideal; see Theorem 0 of [11]. It follows from Theorem 5.38 of [10] that $\beta(H)/\mathcal{J}_\beta$ is a C^*-algebra provided the involution $[T]^*$ is defined to be $[T^*]$. Let this C^*-algebra be represented as an algebra of operators. Thus, $m_\beta(T)$ is meaningful. Since the square root is the limit of polynomials, we have

$$\|[T]\| = ([T]^*[T])^{1/2} = ([T^*T])^{1/2} = [(T^*T)^{1/2}] = [|T|].$$

If $\sigma(A)$ denotes the spectrum of A, then it is elementary to see that

$$m(A) = \inf\{\lambda\colon \lambda \in \sigma(|A|)\}$$

and, thus,

$$m_\beta(T) = \inf\{\lambda\colon \lambda \in \sigma([|T|]).$$

Suppose $[B]$ is a left inverse for $[T]$. For the sake of a contradiction assume that $m_\beta(T) = 0$ and choose a sequence of unit vectors $\{g_1, g_2, \dots\}$ from the space K on which $\mathcal{B}(H)/\mathcal{J}_\beta$ is represented such that $\|[T]g_k\| \to 0$. Since

$$1 = \|g_k\| = \|[B][T]g_k\| \le \|[B]\|\,\|[T]g_k\|$$

holds, we have a contradiction. Thus, $m_\beta(T)$ is positive provided $[T]$ is left-invertible.

In order to prove the converse we assume that $m_\beta(T) > 0$. Choose ε_0 such that $0 < \varepsilon < 2\varepsilon_0$ implies that $\dim E([0, \varepsilon))H = \operatorname{ess\,nul} T$. Since $[E([0, \varepsilon))]$ is a nonnegative idempotent, it is a projection. If g is a unit vector in the range of $[E([0, \varepsilon))]$, then

$$\varepsilon > \|[T][E([0, \varepsilon))]g\| \ge m_\beta(T),$$

which implies that no such unit vector exists. Thus, $\dim E([0, \varepsilon))H = \operatorname{ess\,nul} T$ is less than β.

Define R on $E((\varepsilon_0, \infty))H$ to be the inverse of $|T|\,|E((\varepsilon_0, \infty))H$ and on $E([0, \varepsilon_0])H$ let it be zero. The preceding paragraph implies that $[E([0, \varepsilon_0])] = [0]$ or $[E((\varepsilon_0, \infty))] = [I]$. Since

$$(RU^*)T = RU^*U|T| = R|T| = E((\varepsilon_0, \infty)),$$

we see that $[T]$ is left-invertible.

Now we direct our attention to proving that $m_\beta(T) = \rho_\beta(T)$. First, we recall a useful fact from Theorem 2.6 of [11]. The coset $[T]$ is left-invertible if and only if T is bounded below on some subspace with codimension less than β.

Consider the case that $m_\beta(T) = 0$. So $[T]$ is not left-invertible. If $\varepsilon > 0$ and $E(\cdot)$ is the spectral measure for $|T|$, then $|T|$ is bounded below on $E((\varepsilon, \infty))H$. According to the preceding paragraph, the codimension of $E((\varepsilon, \infty))H$, which is $\dim E([0, \varepsilon])H$, must not be less than β. It follows from

$$\dim E([0, \varepsilon])H \geq \beta$$

and the arbitrariness of ε that $\rho_\beta(T) = 0$.

Conversely, assume that $\rho_\beta(|T|) = 0$ and note that $\dim E([0, \varepsilon])H \geq \beta$ for all positive ε sufficiently small. If $m_\beta(T)$ were positive, then there would be a subspace H_0 such that $T|H_0$ is bounded below and $\operatorname{codim} H_0 < \beta$. If Q is the orthogonal projection onto $(H_0)^\perp$ and ε is a positive number such that $\|Tf\| \geq 2\varepsilon\|f\|$ for $f \in H_0$, then the restriction of Q to $E([0, \varepsilon])H$ must have nontrivial kernel. Suppose $g \in E([0, \varepsilon])H$ and $Qg = 0$. Since $g \in H_0 \cap E([0, \varepsilon])H$, we have a contradiction; this proves that no subspace such as H_0 exists. Hence, $m_\beta(T) = 0$, and we have proved that $m_\beta(T) = 0$ if and only if $\rho_\beta(T) = 0$.

For the sake of a contradiction suppose $m_\beta(T)$ and $\rho_\beta(T)$ are positive and

$$m_\beta(T) = \lambda < \rho_\beta(T).$$

Then $m_\beta(|T| - \lambda) = 0$ while $\rho_\beta(|T| - \lambda) > 0$, which is a contradiction. A similar argument dismisses the opposite inequality, and we have proved that $m_\beta(T) = \rho_\beta(T)$.

If $\beta = \aleph_0$, then it is well known that $\mathscr{I}_\beta$ is the set of compact operators, and $\mathscr{B}(H)/\mathscr{I}_\beta$ is the usual Calkin algebra. In that case the notation $m_\beta(T)$ is replaced with the notation $m_e(T)$ for the essential minimum modulus, which was studied in [2]. Of course, in that case $\rho_\beta(T)$ is the modulus of invertibility $\rho(T)$.

3. Basic results

Let β be an infinite cardinal number that does not exceed $\dim H$. We say that A is *β-left-invertible* provided there is an operator B such that $(I - BA)$ belongs to $\mathscr{I}_\beta$. Define *β-right-invertible* analogously. We say that A is *β-semi-invertible* if it has at least one of the two preceding properties and that it is *β-Fredholm* if it has both properties.

If H is a separable Hilbert space and $\beta = \dim H$, then the preceding concepts reduce to “left-semi-Fredholm”, and “right-semi-Fredholm”, “semi-Fredholm”, and “Fredholm”, respectively. A strong argument can be made that the generalization that is most analogous to the original concept is obtained by setting $\beta = \dim H$ in the case that H is non-separable.

We can give some enlightening characterizations of the previously defined concepts.

Theorem 3. *The following are equivalent.*

(i) T *is* β*-left-ivnertible.*
(ii) $\rho_\beta(T) > 0$.
(iii) $\operatorname{ess\,nul} T < \beta$.
(iv) *For all positive* ε *sufficiently small the inequality* $\|T|H_\varepsilon\| < \varepsilon$ *implies that* $\dim H_\varepsilon < \beta$.

Proof. (i) $\Rightarrow$ (ii) This follows from Theorem 2.

(ii) $\Rightarrow$ (iii) For brevity let $\tau = \rho_\beta(T)$ and choose λ to satisfy the inequalities $0 \le \lambda < \tau$. For each λ there is a positive number δ_λ such that

$$\dim E((\lambda - \delta_\lambda, \lambda + \delta_\lambda))H < \beta.$$

Choose positive ε sufficiently small that $\dim E([0, \varepsilon))H = \operatorname{ess\,nul} T$ and $\varepsilon < \tau$; note that $\{(\lambda - \delta_\lambda, \lambda + \delta_\lambda) : 0 \le \lambda < \tau\}$ is an open cover of $[0, \varepsilon]$. Let $\{(\lambda_1 - \delta_1, \lambda_1 + \delta_1), \dots, (\lambda_n - \delta_n, \lambda_n + \delta_n)\}$ be a finite subcover. We note that

$$\begin{aligned}\dim E([0, \varepsilon))H &\le \sum_{j=1}^{n} \dim E((\lambda_j - \delta_j, \lambda_j + \delta_j))H \\ &< n\beta = \beta.\end{aligned}$$

Since ε was arbitrarily small, it follows that $\operatorname{ess\,nul} T < \beta$.

(iii) $\Rightarrow$ (iv) Choose positive ε sufficiently small that $\dim E([0, \varepsilon))H = \operatorname{ess\,nul} T$. By Lemma 4 of [5] it follows that $\|T|H_\varepsilon\| < \varepsilon$ implies that $\dim H_\varepsilon \le \operatorname{ess\,nul} T < \beta$.

(iv) $\Rightarrow$ (i) Theorem 2 showed the equivalence "T is β-left-invertible" and "$m_\beta(T) > 0$"; the final conclusion of Theorem 2 shows that this is equivalent to "$\rho_\beta(T) > 0$". Thus, it suffices to show that (iv) implies $\rho_\beta(T) > 0$.

Let ε be a positive number such as promised in (iv), and let $E(\cdot)$ be the spectral measure for $|T|$. Let $H_\varepsilon = E([0, \varepsilon/2])H$, and note that $\|T|H_\varepsilon\| < \varepsilon$. Thus, we know that $\dim H_\varepsilon < \beta$. It follows that we can choose positive δ for any λ satisfying $0 \le \lambda < \varepsilon/2$ such that $\dim E((\lambda - \delta, \lambda + \delta))H \le \dim H_\varepsilon < \beta$. It is now clear that $\rho_\beta(T) \ge \varepsilon/2$, which concludes the proof.

We omit the proof of the next theorem since it can be proved by analogy to the proof of Theorem 3 or deduced as a consequence of that theorem.

Theorem 4. *The following are equivalent.*

(i) T *is* β*-right-invertible.*
(ii) $\rho_\beta(T^*) > 0$.
(iii) $\operatorname{ess\,def} T < \beta$.
(iv) *For all positive* ε *sufficiently small the inequality* $\|T^*|H_\varepsilon\| < \varepsilon$ *implies that* $\dim H_\varepsilon < \beta$.

The next theorem follows from Theorems 3 and 4. Part (iii) of the next theorem gives a particularly simple characterization of "β-Fredholm".

Theorem 5. *The following are equivalent.*

(i) T *is* β*-Fredholm.*
(ii) $\min\{\rho_\beta(T), \rho_\beta(T^*)\} > 0$.

(iii) $\max\{\text{ess nul}\, T, \text{ess def}\, T\} < \beta$.

(iv) *For all positive* ε *sufficiently small either of the inequalities* $\|T|H_\varepsilon\| < \varepsilon$ *or* $\|T^*|H_\varepsilon\| < \varepsilon$ *implies that* $\dim H_\varepsilon < \beta$.

If $\rho_\beta(T)$ and $\rho_\beta(T^*)$ are both positive, then they are equal. This follows from the fact that $\rho_\beta(A) = m_\beta(A)$, proved in Theorem 2, and the argument used to prove part (vii) of Theorem 2 of [2].

Let $\mathscr{S}_\beta$ denote the β-semi-invertible operators in $\mathscr{B}(H)$. The next theorem establishes some properties analogous to familiar properties of the semi-Fredholm operators.

Theorem 6. (i) $\mathscr{S}_\beta$ *is open in operator norm.*

(ii) $\mathscr{S}_\beta$ *is dense in* $\mathscr{B}(H)$.

(iii) *If every subspace of* KH *has dimension less than* β *and* $A \in \mathscr{S}_\beta$, *then* $(A+K) \in \mathscr{S}_\beta$.

(iv) *If* A *and* B *are* β*-left-invertible* (β*-right-invertible), then* AB *is the same.*

(v) *If* $A \in \mathscr{S}_\beta$ *and* B *is* β*-Fredholm, then* $AB, BA \in \mathscr{S}_\beta$.

(vi) *If* B *is invertible and* $A \in \mathscr{S}_\beta$, *then* $BAB^{-1} \in \mathscr{S}_\beta$.

Proof. (i) In view of Theorems 3 and 4, for each $T \in \mathscr{S}_\beta$ we have either $\rho_\beta(T) > 0$ or $\rho_\beta(T^*) > 0$. For a representative case we assume the former. Recalling the notation and conventions in Theorem 2 and its proof, we note that $m_\beta(T) > 0$. Let $\delta = m_\beta(T)$, and choose A such that $\|T - A\| < \delta$.

Let g be any unit vector in the Hilbert space on which $\mathscr{B}(H)/\mathscr{I}_\beta$ is represented. In the notation of the proof of Theorem 2 we have

$$\|[A]g\| \ge \|[T]g\| - \|[T-A]g\| \ge \delta - \|T-A\| > 0.$$

We may conclude that

$$\rho_\beta(A) = m_\beta(A) \ge \delta - \|T - A\| > 0.$$

It follows that $A \in \mathscr{S}_\beta$ and that $\mathscr{S}_\beta$ is open.

(ii) It suffices to approximate a given $T \notin \mathscr{S}_\beta$. Let positive ε be given, and let $T = U|T|$ be the usual polar factorization of T. Let $E(\cdot)$ be the spectral measure for $|T|$; define R to coincide with $|T|$ on $E([\varepsilon/3, \infty))H$, and let it coincide with $(\varepsilon/3)I$ on $E([0, \varepsilon/3))H$. Note that

$$\|Rf\| \ge (\varepsilon/3)\|f\| \quad \text{for } f \in H.$$

Define A to be UR on $E((0, \infty))H$ and R on $E(\{0\})H$; note that it is left-invertible since it is bounded below. Since we have

$$\|T - A\| \le \|\,|T| - R\| = \|(|T| - R)|E([0, \varepsilon/3))H\| < \varepsilon,$$

we see that $T \in (\mathscr{S}_\beta)^-$, as desired.

(iii) Since $[A+K] = [A]$, we see that $m_\beta(A+K) = m_\beta(A)$ and this follows from Theorems 2, 3, and 4.

(iv) If D and C are β-left-inverses for A and B, respectively, then CD is a β-left-inverse for AB.

(v) This follows from (iv) since B has both a β-left-inverse and a β-right-inverse.

(vi) If C is a β-left-inverse for A, then BCB^{-1} is a β-left-inverse for BAB^{-1}.

4. A STABILITY THEOREM

Now we are prepared to prove the kind of stability theorem that has been significant in perturbation theory.

Theorem 7. (i) *If $\delta_1 = \rho_\beta(T)$ and $\|T - A\| < \delta_1$, then A is β-left-inertible.*

(ii) *If $\delta_2 = \rho_\beta(T^*)$ and $\|T - A\| < \delta_2$, then A is β-right-invertible.*

(iii) *If $\delta = \min\{\delta_1, \delta_2\}$ and $\|T - A\| < \delta$, then A is β-Fredholm.*

(iv) *Each of the above statements is false if the δ constant is replaced with a larger number.*

Proof. The proof of part (i) of Theorem 6 actually proved part (i) above. To prove (ii) apply (i) to T^* and A^*. Part (iii) follows from (i) and (ii).

In order to prove (iv) we assume that a positive ε is given. Let $E(\,)$ represent the spectral measure for $|T|$ where $T = U|T|$ is the usual polar factorization. Let Q denote the projection $E([0, \varepsilon + \rho_\beta(T)))$, and let H_ε denote QH. By the definition of $\rho_\beta(T)$ we know that $\dim H_\varepsilon \geq \beta$. Define B to coincide with T on $(H_\varepsilon)^\perp$ and to be zero on H_ε; so $B = T(I - Q)$. Note that

$$\|T - B\| = \|T|H_\varepsilon\| \leq \varepsilon + \rho_\gamma(T).$$

Since

$$\operatorname{ess\,nul} B = \operatorname{nul} B = \dim H_\varepsilon \geq \beta,$$

we see that B is not β-left-invertible. This proves that δ_1 is the largest constant that makes (i) true.

Applying the construction in the preceding paragraph to T^* shows that δ_2 is the largest constant that makes (ii) true. It follows that replacing δ in part (iv) with a larger constant will result in the loss of either "β-left-invertible" or "β-right-invertible", which means that "β-Fredholm" is lost.

If one chooses β in Theorem 7 to be $\aleph_0$, so that $\mathscr{I}_\beta$ consists of the compact operators, then part (i) of the previous theorem gives the distance from a left-semi-Fredholm operator to the set of operators that are not left-semi-Fredholm. If $\beta = \aleph_0$, then part (ii) describes the distance from a right-semi-Fredholm operator to the set of operators that are not right-semi-Fredholm, and part (iii) computes the distance from a Fredholm operator to the complement of the Fredholm operators.

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