REAL HYPERSURFACES OF $\mathbb{CP}^n$
WITH NON-NEGATIVE RICCI CURVATURE

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Abstract. We prove the non-existence of Levi flat compact real hypersurfaces without boundary in $\mathbb{CP}^n, n > 1$, with non-negative totally real Ricci curvature.

1. Introduction

Let $\mathbb{CP}^n$ be the $n$-dimensional complex projective space with complex structure $J$ and Kähler metric $g$. It is well known (cf. A. Bejancu [1], p. 21) that a real hypersurface $M$ of $\mathbb{CP}^n$ is a CR-submanifold of $\mathbb{CP}^n$. More precisely, the CR-structure on $M$ is defined as follows. Denote by $TM^\perp$ the normal bundle of $M$ and consider the totally real distribution $RM = J(TM^\perp)$ on $M$. Then the complementary orthogonal distribution $HM$ to $RM$ in $TM$ is of rank $2(n-1)$ and it is invariant under $J$, that is, $J(HM) = HM$. That is why $HM$ is called the holomorphic distribution on $M$. For any $x \in M$ denote by $(HM)_x$ the fibre of $HM$ over $x$ and define the complex vector space

$$H^{1,0}_x(M) = \{W_x = X_x - iJ_x(X_x) : X_x \in (HM)_x\}.$$ 

Consider the complexified tangent bundle $T^cM = TM \otimes C$ of $M$ and note that

$$H^{1,0}(M) = \bigcup_{x \in M} H^{1,0}_x(M)$$

is an involutive complex vector subbundle of $T^cM$ such that

$$H^{1,0}(M) \cap \overline{H^{1,0}(M)} = \{0\}.$$ 

Hence $H^{1,0}(M)$ defines a CR-structure on $M$ (cf. A. Boggess [3], p. 121). Besides, we have $HM = H^{1,0}(M) \oplus \overline{H^{1,0}(M)}$.

Though $H^{1,0}(M)$ is involutive, it is not necessary that $HM$ be integrable. Thus M. Okumura [6] and Y. Maeda [5] found different geometric characterizations of a class of real hypersurfaces of $\mathbb{CP}^n$ with non-integrable $HM$.

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For any $x \in M$ consider a unit vector $N_x \in T_x M \perp$ and let $\xi_x = -J_x(N_x)$. Then following A. Boggess [3], define the Levi form at the point $x$ as the map

$$L_x : H^{1,0}_x(M) \to \mathfrak{g}, \quad L_x(W_x) = -g_x([X,JX],\xi_x),$$

where $X$ is the $HM$-vector field extension of $X_x = \frac{1}{2}(W_x + \overline{W}_x)$. We say that $M$ is a Levi flat CR-submanifold if $L_x$ vanishes for any $x \in M$. Taking into account that $\xi_x \in (RM)_x$, from the definition of $L_x$ it follows that in case $HM$ is integrable, $M$ is Levi flat. The converse of this assertion is a consequence of Theorem 1 in A. Boggess [3], p. 158. Hence $M$ is Levi flat iff $HM$ is integrable.

Next, suppose $M$ is an orientable real hypersurface in $CP^n$. Then there exists a globally defined unit normal vector field $N$ on $M$. Thus the totally real distribution $RM$ is globally spanned by $\xi = -JN$. The Ricci curvature of $M$ in the direction $\xi$ is called the totally real Ricci curvature and it is denoted by $\text{Ric}(\xi, \xi)$. Our paper has the origin in the remark that the totally real Ricci curvature of $K$-contact manifold is a positive number (cf. D. Blair [2], p. 65) and that $HM$ is not integrable. So it is natural to ask whether $HM$ is not integrable in general for a real hypersurface of $CP^n$ with non-negative totally real Ricci curvature. In this respect we prove the following result.

**Theorem.** Let $M$ be a compact orientable real hypersurface without boundary of $CP^n$, $n \geq 1$, such that $\text{Ric}(\xi, \xi) \geq 0$ everywhere on $M$. Then $HM$ is not integrable on $M$.

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## 2. Preliminaries

Let $M$ be an orientable real hypersurface of $CP^n$. Denote by $\nabla$ and $\overline{\nabla}$ the Levi Civita connection on $M$ and $CP^n$ respectively. Then we have the well-known formulae

\begin{alignat}{2}
\nabla_X Y &= \nabla_X Y + g(AX,Y)N, \\
\nabla_X N &= -AX,
\end{alignat}

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on $M$, $N$ is the unit normal vector field to $M$ and $A$ is the shape operator of $M$. Let $\eta$ be the 1-form dual to $\xi$, that is, $\eta(X) = g(X, \xi)$ for any $X \in \mathfrak{X}(M)$. Then we have $JX = JPX + \eta(X)N$, where $P$ is the projection morphism of $TM$ on $HM$. On using (2.1) and (2.2) and taking into account that $J$ is parallel with respect to $\nabla$ one obtains

\begin{alignat}{2}
\nabla_X \xi &= JPAX, \quad \forall X \in \mathfrak{X}(M).
\end{alignat}

Consider $CP^n$ as a complex space form of the constant holomorphic sectional curvature $c = 4$. Then using the formulae of curvature tensor field of $CP^n$ (cf. Kobayashi-Nomizu [5], p. 167), the equations of Gauss and Codazzi for the immersion of $M$ in $CP^n$ become

\begin{alignat}{2}
g(R(X,Y)Z,W) &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(Z,JY)g(JX,W) \\
&\quad - g(Z,JX)g(JY,W) + 2g(X,JY)g(JZ,W) \\
&\quad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W).
\end{alignat}
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and

\[(\nabla_X A)(Y) - (\nabla_Y A)(X) = g(X,\xi)JY - g(Y,\xi)JX + 2g(X,JY)\xi,\]

respectively, for any $X,Y,Z,W \in \mathfrak{X}(M)$, where $R$ is the curvature tensor corresponding to $\nabla$.

Finally denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(HM)$ and $\Gamma(RM)$ the $F(M)$-modules of smooth vector fields on $M$ which belong to $HM$ and $RM$ respectively. We may always consider a local orthonormal field of frames \{ $E_1, \ldots, E_{n-1}, JE_1, \ldots, JE_{n-1}, \xi$ \} on a coordinate neighborhood $U \subset M$, where \{ $E_i, JE_i$ \} $\subset \Gamma(HM), i \in \{1, \ldots, n-1\}$. Such a frame field is said to be a local CR-frame field on $M$.

3. Proof of the Theorem

First we prove

**Lemma.** Let $M$ be a compact orientable real hypersurface without boundary of $CP^n$. Suppose $\text{Ric}(\xi, \xi) \geq 0$ everywhere on $M$ and $HM$ is an integrable distribution on $M$. Then we have

(i) $\text{Ric}(\xi, \xi) = 0$,

(ii) $\nabla_X \xi = 0, \forall X \in \Gamma(HM)$,

(iii) $\nabla_X Y \in \Gamma(HM), \forall X,Y \in \Gamma(HM)$,

(iv) $AX \in \Gamma(RM), \forall X \in \Gamma(HM)$,

(v) $\|A\xi\|^2 = 2(n-1) + g(A\xi, \xi)^2$.

Proof. Choose a local CR-frame field \{ $E_i, JE_i, \xi$ \} on $M$. Then taking into account that $A$ and $J$ are symmetric and antisymmetric with respect to $g$, that is, $g(AX, Y) = g(X, AY)$ and $g(JX, Y) = -g(X, JY)$ holds for $X,Y \in \mathfrak{X}(M)$, and using (2.3) obtain

\[(3.1) \quad \delta\eta = \text{div} \xi = 0.\]

On the other hand, by direct calculations, using the integrability of $HM$ obtain $d\eta(X,Y) = 0$ and $d\eta(\xi, X) = g(\nabla_X \xi, X)$, for any $X,Y \in \Gamma(HM)$. Hence locally on $U$ we have

\[(3.2) \quad \|d\eta\|^2 = 2 \sum_{i=1}^{n-1} \{ g(\nabla_\xi \xi, E_i)^2 + g(\nabla_\xi \xi, JE_i)^2 \} = 2\|\nabla_\xi \xi\|^2.\]

Since for each $x \in M$, we have a coordinate neighborhood $U$ and a local CR-frame field \{ $E_i, JE_i, \xi$ \} on $U$, (3.2) holds for each $x \in M$ and hence globally on $M$. Next, we recall that on any compact orientable Riemannian manifold $M$ without boundary we have (cf. K. Yano [7], p. 41)

\[\int_M \left\{ \text{Ric}(X,X) - \frac{1}{2}\|d\alpha\|^2 + \|\nabla X\|^2 - (\delta\alpha)^2 \right\} dv = 0,\]

where $\alpha$ is a 1-form dual to $X$. Replace $X$ by $\xi$ and $\alpha$ by $\eta$ in the above integral formula and use (3.1) to obtain

\[(3.3) \quad \int_M \left\{ \text{Ric}(\xi, \xi) - \frac{1}{2}\|d\eta\|^2 + \|\nabla \xi\|^2 \right\} dv = 0.\]
Since on each coordinate neighborhood $U$, we have
\[ \|\nabla \xi\|^2 = \sum_{i=1}^{n-1} \left\{ \|\nabla E_i \xi\|^2 + \|\nabla J_E \xi\|^2 \right\} + \|\nabla \xi\|^2, \]
on account of (3.2) we thus find
\[ \|\nabla \xi\|^2 - \frac{1}{2}\|d\eta\|^2 = \sum_{i=1}^{n-1} \left\{ \|\nabla E_i \xi\|^2 + \|\nabla J_E \xi\|^2 \right\} \geq 0 \]
on each $U$ and consequently on $M$. As $\text{Ric}(\xi, \xi) \geq 0$, it follows that the integrand in (3.3) is non-negative. Hence we must have $\text{Ric}(\xi, \xi) = 0$ and $\|\nabla \xi\|^2 - \frac{1}{2}\|d\eta\|^2 = 0$. The second equation together with (3.4) gives $\nabla E_i \xi = \nabla J_E \xi = 0$ for each local CR-frame field $\{E_i, J_E \xi\}$ and consequently we obtain (i) and (ii). Taking into account that $\nabla$ is a Riemannian connection from (ii) we infer (iii). Moreover (iv) follows from (ii) on using (2.3). Finally using (iv), from (2.4) obtain
\[ g(R(E_i, \xi), E_i) = 1 - g(A\xi, E_i)^2 \]
and
\[ g(R(J_E i, \xi), J_E i) = 1 - g(A\xi, J_E i)^2. \]
Hence (i) gives
\[ 0 = \text{Ric}(\xi, \xi) = \sum_{i=1}^{n-1} \left\{ g(R(E_i, \xi), E_i) + g(R(J_E i, \xi), J_E i) \right\} \]
\[ = 2(n - 1) - \sum_{i=1}^{n-1} \left\{ g(A\xi, E_i)^2 + g(A\xi, J_E i)^2 \right\} \]
\[ = 2(n - 1) - \{ \|A\xi\|^2 - g(A\xi, \xi)^2 \}, \]
on any coordinate neighborhood $U$, which gives (v). This completes the proof of the Lemma.

Remark. The assertions (i)–(iv) of the Lemma hold in a more general setting, namely, in case the complex projective space is replaced by an arbitrary Kaehler manifold.

Now we proceed with the proof of the Theorem. Suppose $HM$ is integrable. Then using (iv) and (2.1) and taking into account that $J$ is parallel with respect to $\nabla$ obtain
\[ \nabla_X JY = J \nabla_X Y, \quad \forall X, Y \in \Gamma(HM). \]
Thus on a coordinate neighborhood $U$, (iii) and (3.5) imply
\[ \nabla E_i E_j = \sum_{k=1}^{n-1} \{ a_{ijk} E_k + b_{ijk} J_E k \}, \quad \nabla E_i J_E j = \sum_{k=1}^{n-1} \{ a_{ijk} J_E k - b_{ijk} E_k \}, \]
and
\[ \nabla J_E E_j = \sum_{k=1}^{n-1} \{ c_{ijk} E_k + d_{ijk} J_E k \}, \quad \nabla J_E J_E j = \sum_{k=1}^{n-1} \{ c_{ijk} J_E k - d_{ijk} E_k \}, \]
where \( \{a_{ijk}, b_{ijk}, c_{ijk}, d_{ijk}\} \) are smooth functions on \( U \) satisfying
\[
(3.8) \quad a_{ijk} + a_{ikj} = 0, \quad c_{ijk} + c_{ikj} = 0, \quad b_{ijk} = b_{ikj}, \quad d_{ijk} = d_{ikj}.
\]
As \( \nabla_{\xi} \xi \) is orthogonal to \( \xi \), there exist smooth functions \( \{a_{i}, b_{i}\}, i \in \{1, \ldots, n-1\} \), such that
\[
(3.9) \quad \nabla_{\xi} \xi = \sum_{k=1}^{n-1} (a_{k}E_{k} + b_{k}JE_{k}).
\]
Finally using (ii) and (3.9) and taking into account that \( \nabla \) is a Riemannian connection, obtain
\[
(3.10) \quad \nabla_{\xi} E_{i} = \sum_{k=1}^{n-1} \{a_{ik}E_{k} + b_{ik}JE_{k}\} - a_{i} \xi, \quad \nabla_{E_{i}} \xi = 0,
\]
and
\[
(3.11) \quad \nabla_{\xi} JE_{i} = \sum_{k=1}^{n-1} \{c_{ik}E_{k} + d_{ik}JE_{k}\} - b_{i} \xi, \quad \nabla_{JE_{i}} \xi = 0,
\]
where \( \{a_{ik}, b_{ik}, c_{ik}, d_{ik}\} \) are smooth functions on \( U \). Now using (2.1), (2.2) and (3.9) and taking into account that \( \nabla \) is a Riemannian connection we infer that
\[
(3.12) \quad AE_{i} = b_{i} \xi, \quad AJE_{i} = -a_{i} \xi, \quad A^{\xi} = \sum_{k=1}^{n-1} \{b_{k}E_{k} - a_{k}JE_{k}\} + c \xi,
\]
where \( c \) is a smooth function on \( U \). Then (v) in the Lemma and the last equation in (3.12) imply
\[
(3.13) \quad \sum_{k=1}^{n-1} \{(a_{k})^{2} + (b_{k})^{2}\} = 2(n - 1).
\]
Further, take \( X = E_{i} \) and \( Y = \xi \) in (2.5), use (3.6), (3.9), (3.10) and (3.12) and equate the components with respect to the holomorphic frame \( \{E_{k}, JE_{k}\} \) of \( \Gamma(HM) \) to obtain
\[
(3.14) \quad \begin{cases}
E_{i}(a_{k}) = a_{i}a_{k} - b_{i}b_{k} + b_{i}c_{ik} + \sum_{j=1}^{n-1} \{b_{j}b_{ijk} - a_{j}a_{ijk}\}, \\
E_{i}(b_{k}) = a_{i}b_{k} + b_{i}a_{k} - \sum_{j=1}^{n-1} \{b_{j}a_{ijk} + a_{j}b_{ijk}\}, \quad i, k \in \{1, \ldots, n-1\}.
\end{cases}
\]
In a similar way, take \( X = JE_{i} \) and \( Y = \xi \) in (2.5) and use (3.7), (3.9), (3.11) and (3.12) to infer
\[
(3.15) \quad \begin{cases}
JE_{i}(b_{k}) = b_{i}b_{k} - a_{i}a_{k} + b_{i}c_{ik} - \sum_{j=1}^{n-1} \{b_{j}c_{ijk} + a_{j}d_{ijk}\}, \\
JE_{i}(a_{k}) = a_{i}b_{k} + b_{i}a_{k} + \sum_{j=1}^{n-1} \{b_{j}d_{ijk} - a_{j}c_{ijk}\}.
\end{cases}
\]
From (3.14) and (3.15) on using (3.8) it is easy to obtain
\[
(3.16) \quad \frac{1}{2} E_{i} \left( \sum_{k=1}^{n-1} \{(a_{k})^{2} + (b_{k})^{2}\} \right) = a_{i} \left( 1 + \sum_{k=1}^{n-1} \{(a_{k})^{2} + (b_{k})^{2}\} \right)
\]
and

\[
\frac{1}{2}JE_i \left( \sum_{k=1}^{n-1} \left( (a_k)^2 + (b_k)^2 \right) \right) = b_i \left( 1 + \sum_{k=1}^{n-1} \left( (a_k)^2 + (b_k)^2 \right) \right),
\]

respectively. On using (3.13) in (3.16) and (3.17) obtain \( a_i = 0 \) and \( b_i = 0 \) for any \( i \in \{1, \ldots, n-1\} \). Thus we get a contradiction to (3.13) and this completes the proof of the theorem. As a direct consequence of the Theorem we have

**Corollary.** Let \( M \) be a Levi flat compact real hypersurface without boundary of \( CP^n, n > 1 \). Then there exists an open subset \( U \) of \( M \) such that the totally real Ricci curvature of \( M \) is negative everywhere on \( U \).

**References**


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