

**REAL HYPERSURFACES OF  $CP^n$   
WITH NON-NEGATIVE RICCI CURVATURE**

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ABSTRACT. We prove the non-existence of Levi flat compact real hypersurfaces without boundary in  $CP^n$ ,  $n > 1$ , with non-negative totally real Ricci curvature.

1. INTRODUCTION

Let  $CP^n$  be the  $n$ -dimensional complex projective space with complex structure  $J$  and Kaehler metric  $g$ . It is well known (cf. A. Bejancu [1], p. 21) that a real hypersurface  $M$  of  $CP^n$  is a CR-submanifold of  $CP^n$ . More precisely, the CR-structure on  $M$  is defined as follows. Denote by  $TM^\perp$  the normal bundle of  $M$  and consider the *totally real distribution*  $RM = J(TM^\perp)$  on  $M$ . Then the complementary orthogonal distribution  $HM$  to  $RM$  in  $TM$  is of rank  $2(n - 1)$  and it is invariant under  $J$ , that is,  $J(HM) = HM$ . That is why  $HM$  is called the *holomorphic distribution* on  $M$ . For any  $x \in M$  denote by  $(HM)_x$  the fibre of  $HM$  over  $x$  and define the complex vector space

$$H_x^{1,0}(M) = \{W_x = X_x - iJ_x(X_x) : X_x \in (HM)_x\}.$$

Consider the complexified tangent bundle  $T^cM = TM \otimes C$  of  $M$  and note that

$$H^{1,0}(M) = \bigcup_{x \in M} H_x^{1,0}(M)$$

is an involutive complex vector subbundle of  $T^cM$  such that

$$H^{1,0}(M) \cap \overline{H^{1,0}(M)} = \{0\}.$$

Hence  $H^{1,0}(M)$  defines a CR-structure on  $M$  (cf. A. Boggess [3], p. 121). Besides, we have  $HM = H^{1,0}(M) \oplus \overline{H^{1,0}(M)}$ .

Though  $H^{1,0}(M)$  is involutive, it is not necessary that  $HM$  be integrable. Thus M. Okumura [6] and Y. Maeda [5] found different geometric characterizations of a class of real hypersurfaces of  $CP^n$  with non-integrable  $HM$ .

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For any  $x \in M$  consider a unit vector  $N_x \in T_x M^\perp$  and let  $\xi_x = -J_x(N_x)$ . Then following A. Boggess [3], define the Levi form at the point  $x$  as the map

$$L_x : H_x^{1,0}(M) \rightarrow \mathfrak{R}, \quad L_x(W_x) = -g_x([X, JX]_x, \xi_x),$$

where  $X$  is the  $HM$ -vector field extension of  $X_x = \frac{1}{2}(W_x + \overline{W}_x)$ . We say that  $M$  is a *Levi flat CR-submanifold* if  $L_x$  vanishes for any  $x \in M$ . Taking into account that  $\xi_x \in (RM)_x$ , from the definition of  $L_x$  it follows that in case  $HM$  is integrable,  $M$  is Levi flat. The converse of this assertion is a consequence of Theorem 1 in A. Boggess [3], p. 158. Hence  $M$  is Levi flat iff  $HM$  is integrable.

Next, suppose  $M$  is an orientable real hypersurface in  $CP^n$ . Then there exists a globally defined unit normal vector field  $N$  on  $M$ . Thus the totally real distribution  $RM$  is globally spanned by  $\xi = -JN$ . The Ricci curvature of  $M$  in the direction  $\xi$  is called the *totally real Ricci curvature* and it is denoted by  $\text{Ric}(\xi, \xi)$ . Our paper has the origin in the remark that the totally real Ricci curvature of  $K$ -contact manifold is a positive number (cf. D. Blair [2], p. 65) and that  $HM$  is not integrable. So it is natural to ask whether  $HM$  is not integrable in general for a real hypersurface of  $CP^n$  with non-negative totally real Ricci curvature. In this respect we prove the following result.

**Theorem.** *Let  $M$  be a compact orientable real hypersurface without boundary of  $CP^n, n > 1$ , such that  $\text{Ric}(\xi, \xi) \geq 0$  everywhere on  $M$ . Then  $HM$  is not integrable on  $M$ .*

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## 2. PRELIMINARIES

Let  $M$  be an orientable real hypersurface of  $CP^n$ . Denote by  $\nabla$  and  $\overline{\nabla}$  the Levi Civita connection on  $M$  and  $CP^n$  respectively. Then we have the well-known formulae

$$(2.1) \quad \overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

$$(2.2) \quad \overline{\nabla}_X N = -AX,$$

for any  $X, Y \in \mathfrak{X}(M)$ , where  $\mathfrak{X}(M)$  is the Lie algebra of smooth vector fields on  $M$ ,  $N$  is the unit normal vector field to  $M$  and  $A$  is the shape operator of  $M$ . Let  $\eta$  be the 1-form dual to  $\xi$ , that is,  $\eta(X) = g(X, \xi)$  for any  $X \in \mathfrak{X}(M)$ . Then we have  $JX = JPX + \eta(X)N$ , where  $P$  is the projection morphism of  $TM$  on  $HM$ . On using (2.1) and (2.2) and taking into account that  $J$  is parallel with respect to  $\overline{\nabla}$  one obtains

$$(2.3) \quad \nabla_X \xi = JPAX, \quad \forall X \in \mathfrak{X}(M).$$

Consider  $CP^n$  as a complex space form of the constant holomorphic sectional curvature  $c = 4$ . Then using the formulae of curvature tensor field of  $CP^n$  (cf. Kobayashi-Nomizu [5], p. 167), the equations of Gauss and Codazzi for the immersion of  $M$  in  $CP^n$  become

$$(2.4) \quad \begin{aligned} g(R(X, Y)Z, W) &= g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(Z, JY)g(JX, W) \\ &\quad - g(Z, JX)g(JY, W) + 2g(X, JY)g(JZ, W) \\ &\quad + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W) \end{aligned}$$

and

$$(2.5) \quad (\nabla_X A)(Y) - (\nabla_Y A)(X) = g(X, \xi)JY - g(Y, \xi)JX + 2g(X, JY)\xi,$$

respectively, for any  $X, Y, Z, W \in \mathfrak{X}(M)$ , where  $R$  is the curvature tensor corresponding to  $\nabla$ .

Finally denote by  $\mathcal{F}(M)$  the algebra of smooth functions on  $M$  and by  $\Gamma(HM)$  and  $\Gamma(RM)$  the  $\mathcal{F}(M)$ -modules of smooth vector fields on  $M$  which belong to  $HM$  and  $RM$  respectively. We may always consider a local orthonormal field of frames  $\{E_1, \dots, E_{n-1}, JE_1, \dots, JE_{n-1}, \xi\}$  on a coordinate neighborhood  $U \subset M$ , where  $\{E_i, JE_i\} \subset \Gamma(HM), i \in \{1, \dots, n-1\}$ . Such a frame field is said to be a *local CR-frame field* on  $M$ .

### 3. PROOF OF THE THEOREM

First we prove

**Lemma.** *Let  $M$  be a compact orientable real hypersurface without boundary of  $CP^n$ . Suppose  $\text{Ric}(\xi, \xi) \geq 0$  everywhere on  $M$  and  $HM$  is an integrable distribution on  $M$ . Then we have*

- (i)  $\text{Ric}(\xi, \xi) = 0$ ,
- (ii)  $\nabla_X \xi = 0, \forall X \in \Gamma(HM)$ ,
- (iii)  $\nabla_X Y \in \Gamma(HM), \forall X, Y \in \Gamma(HM)$ ,
- (iv)  $AX \in \Gamma(RM), \forall X \in \Gamma(HM)$ ,
- (v)  $\|A\xi\|^2 = 2(n-1) + g(A\xi, \xi)^2$ .

*Proof.* Choose a local CR-frame field  $\{E_i, JE_i, \xi\}$  on  $M$ . Then taking into account that  $A$  and  $J$  are symmetric and antisymmetric with respect to  $g$ , that is,  $g(AX, Y) = g(X, AY)$  and  $g(JX, Y) = -g(X, JY)$  holds for  $X, Y \in \mathfrak{X}(M)$ , and using (2.3) obtain

$$(3.1) \quad \delta\eta = \text{div } \xi = 0.$$

On the other hand, by direct calculations, using the integrability of  $HM$  obtain  $d\eta(X, Y) = 0$  and  $d\eta(\xi, X) = g(\nabla_\xi \xi, X)$ , for any  $X, Y \in \Gamma(HM)$ . Hence locally on  $U$  we have

$$(3.2) \quad \|d\eta\|^2 = 2 \sum_{i=1}^{n-1} \{g(\nabla_\xi \xi, E_i)^2 + g(\nabla_\xi \xi, JE_i)^2\} = 2\|\nabla_\xi \xi\|^2.$$

Since for each  $x \in M$ , we have a coordinate neighborhood  $U$  and a local CR-frame field  $\{E_i, JE_i, \xi\}$  on  $U$ , (3.2) holds for each  $x \in M$  and hence globally on  $M$ . Next, we recall that on any compact orientable Riemannian manifold  $M$  without boundary we have (cf. K. Yano [7], p. 41)

$$\int_M \left\{ \text{Ric}(X, X) - \frac{1}{2}\|d\alpha\|^2 + \|\nabla X\|^2 - (\delta\alpha)^2 \right\} dv = 0,$$

where  $\alpha$  is a 1-form dual to  $X$  on  $M$ . Replace  $X$  by  $\xi$  and  $\alpha$  by  $\eta$  in the above integral formula and use (3.1) to obtain

$$(3.3) \quad \int_M \left\{ \text{Ric}(\xi, \xi) - \frac{1}{2}\|d\eta\|^2 + \|\nabla \xi\|^2 \right\} dv = 0.$$

Since on each coordinate neighborhood  $U$ , we have

$$\|\nabla\xi\|^2 = \sum_{i=1}^{n-1} \{\|\nabla_{E_i}\xi\|^2 + \|\nabla_{JE_i}\xi\|^2\} + \|\nabla_\xi\xi\|^2,$$

on account of (3.2) we thus find

$$(3.4) \quad \|\nabla\xi\|^2 - \frac{1}{2}\|d\eta\|^2 = \sum_{i=1}^{n-1} \{\|\nabla_{E_i}\xi\|^2 + \|\nabla_{JE_i}\xi\|^2\} \geq 0$$

on each  $U$  and consequently on  $M$ . As  $\text{Ric}(\xi, \xi) \geq 0$ , it follows that the integrand in (3.3) is non-negative. Hence we must have  $\text{Ric}(\xi, \xi) = 0$  and  $\|\nabla\xi\|^2 - \frac{1}{2}\|d\eta\|^2 = 0$ . The second equation together with (3.4) gives  $\nabla_{E_i}\xi = \nabla_{JE_i}\xi = 0$  for each local CR-frame field  $\{E_i, JE_i, \xi\}$  and consequently we obtain (i) and (ii). Taking into account that  $\nabla$  is a Riemannian connection from (ii) we infer (iii). Moreover (iv) follows from (ii) on using (2.3). Finally using (iv), from (2.4) obtain

$$g(R(E_i, \xi)\xi, E_i) = 1 - g(A\xi, E_i)^2$$

and

$$g(R(JE_i, \xi)\xi, JE_i) = 1 - g(A\xi, JE_i)^2.$$

Hence (i) gives

$$\begin{aligned} 0 = \text{Ric}(\xi, \xi) &= \sum_{i=1}^{n-1} \{g(R(E_i, \xi)\xi, E_i) + g(R(JE_i, \xi)\xi, JE_i)\} \\ &= 2(n-1) - \sum_{i=1}^{n-1} \{g(A\xi, E_i)^2 + g(A\xi, JE_i)^2\} \\ &= 2(n-1) - \{\|A\xi\|^2 - g(A\xi, \xi)^2\}, \end{aligned}$$

on any coordinate neighborhood  $U$ , which gives (v). This completes the proof of the Lemma.  $\square$

*Remark.* The assertions (i)–(iv) of the Lemma hold in a more general setting, namely, in case the complex projective space is replaced by an arbitrary Kaehler manifold.

Now we proceed with the proof of the Theorem. Suppose  $HM$  is integrable. Then using (iv) and (2.1) and taking into account that  $J$  is parallel with respect to  $\bar{\nabla}$  obtain

$$(3.5) \quad \nabla_X JY = J\nabla_X Y, \quad \forall X, Y \in \Gamma(HM).$$

Thus on a coordinate neighborhood  $U$ , (iii) and (3.5) imply

$$(3.6) \quad \nabla_{E_i} E_j = \sum_{k=1}^{n-1} \{a_{ijk} E_k + b_{ijk} JE_k\}, \quad \nabla_{E_i} JE_j = \sum_{k=1}^{n-1} \{a_{ijk} JE_k - b_{ijk} E_k\},$$

and

$$(3.7) \quad \nabla_{JE_i} E_j = \sum_{k=1}^{n-1} \{c_{ijk} E_k + d_{ijk} JE_k\}, \quad \nabla_{JE_i} JE_j = \sum_{k=1}^{n-1} \{c_{ijk} JE_k - d_{ijk} E_k\},$$

where  $\{a_{ijk}, b_{ijk}, c_{ijk}, d_{ijk}\}$  are smooth functions on  $U$  satisfying

$$(3.8) \quad a_{ijk} + a_{ikj} = 0, \quad c_{ijk} + c_{ikj} = 0, \quad b_{ijk} = b_{ikj}, \quad d_{ijk} = d_{ikj}.$$

As  $\nabla_\xi \xi$  is orthogonal to  $\xi$ , there exist smooth functions  $\{a_i, b_i\}, i \in \{1, \dots, n-1\}$ , such that

$$(3.9) \quad \nabla_\xi \xi = \sum_{k=1}^{n-1} \{a_k E_k + b_k J E_k\}.$$

Finally using (ii) and (3.9) and taking into account that  $\nabla$  is a Riemannian connection, obtain

$$(3.10) \quad \nabla_\xi E_i = \sum_{k=1}^{n-1} \{a_{ik} E_k + b_{ik} J E_k\} - a_i \xi, \quad \nabla_{E_i} \xi = 0,$$

and

$$(3.11) \quad \nabla_\xi J E_i = \sum_{k=1}^{n-1} \{c_{ik} E_k + d_{ik} J E_k\} - b_i \xi, \quad \nabla_{J E_i} \xi = 0,$$

where  $\{a_{ik}, b_{ik}, c_{ik}, d_{ik}\}$  are smooth functions on  $U$ . Now using (2.1), (2.2) and (3.9) and taking into account that  $\bar{\nabla}$  is a Riemannian connection we infer that

$$a_i = -g(A\xi, J E_i) \quad \text{and} \quad b_i = g(A\xi, E_i).$$

Thus on account of (iv), we have

$$(3.12) \quad A E_i = b_i \xi, \quad A J E_i = -a_i \xi, \quad A \xi = \sum_{k=1}^{n-1} \{b_k E_k - a_k J E_k\} + c \xi,$$

where  $c$  is a smooth function on  $U$ . Then (v) in the Lemma and the last equation in (3.12) imply

$$(3.13) \quad \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} = 2(n-1).$$

Further, take  $X = E_i$  and  $Y = \xi$  in (2.5), use (3.6), (3.9), (3.10) and (3.12) and equate the components with respect to the holomorphic frame  $\{E_k, J E_k\}$  of  $\Gamma(HM)$  to obtain

$$(3.14) \quad \begin{cases} E_i(a_k) = a_i a_k - b_i b_k + \delta_{ik} + \sum_{j=1}^{n-1} \{b_j b_{ijk} - a_j a_{ijk}\}, \\ E_i(b_k) = a_i b_k + b_i a_k - \sum_{j=1}^{n-1} \{b_j a_{ijk} + a_j b_{ijk}\}, \quad i, k \in \{1, \dots, n-1\}. \end{cases}$$

In a similar way, take  $X = J E_i$  and  $Y = \xi$  in (2.5) and use (3.7), (3.9), (3.11) and (3.12) to infer

$$(3.15) \quad \begin{cases} J E_i(b_k) = b_i b_k - a_i a_k + \delta_{ik} - \sum_{j=1}^{n-1} \{b_j c_{ijk} + a_j d_{ijk}\}, \\ J E_i(a_k) = a_i b_k + b_i a_k + \sum_{j=1}^{n-1} \{b_j d_{ijk} - a_j c_{ijk}\}. \end{cases}$$

From (3.14) and (3.15) on using (3.8) it is easy to obtain

$$(3.16) \quad \frac{1}{2} E_i \left( \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right) = a_i \left( 1 + \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right)$$

and

$$(3.17) \quad \frac{1}{2}JE_i \left( \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right) = b_i \left( 1 + \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right),$$

respectively. On using (3.13) in (3.16) and (3.17) obtain  $a_i = 0$  and  $b_i = 0$  for any  $i \in \{1, \dots, n-1\}$ . Thus we get a contradiction to (3.13) and this completes the proof of the theorem. As a direct consequence of the Theorem we have

**Corollary.** *Let  $M$  be a Levi flat compact real hypersurface without boundary of  $CP^n$ ,  $n > 1$ . Then there exists an open subset  $U$  of  $M$  such that the totally real Ricci curvature of  $M$  is negative everywhere on  $U$ .*

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