PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 1, January 1996

# REAL HYPERSURFACES OF $CP^n$ WITH NON-NEGATIVE RICCI CURVATURE

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(Communicated by Christopher B. Croke)

ABSTRACT. We prove the non-existence of Levi flat compact real hypersurfaces without boundary in $CP^n, n > 1$, with non-negative totally real Ricci curvature.

## 1. INTRODUCTION

Let $CP^n$ be the $n$-dimensional complex projective space with complex structure $J$ and Kaehler metric $g$. It is well known (cf. A. Bejancu [1], p. 21) that a real hypersurface $M$ of $CP^n$ is a CR-submanifold of $CP^n$. More precisely, the CR-structure on $M$ is defined as follows. Denote by $TM^{\perp}$ the normal bundle of $M$ and consider the *totally real distribution* $RM = J(TM^{\perp})$ on $M$. Then the complementary orthogonal distribution $HM$ to $RM$ in $TM$ is of rank $2(n-1)$ and it is invariant under $J$, that is, $J(HM) = HM$. That is why $HM$ is called the *holomorphic distribution* on $M$. For any $x \in M$ denote by $(HM)_x$ the fibre of $HM$ over $x$ and define the complex vector space

$$H_x^{1,0}(M) = \{W_x = X_x - iJ_x(X_x) \colon X_x \in (HM)_x\}.$$

Consider the complexified tangent bundle $T^cM = TM \otimes C$ of $M$ and note that

$$H^{1,0}(M) = \bigcup_{x \in M} H_x^{1,0}(M)$$

is an involutive complex vector subbundle of $T^cM$ such that

$$H^{1,0}(M) \cap \overline{H^{1,0}(M)} = \{0\}.$$

Hence $H^{1,0}(M)$ defines a CR-structure on $M$ (cf. A. Boggess [3], p. 121). Besides, we have $HM = H^{1,0}(M) \oplus \overline{H^{1,0}(M)}$.

Though $H^{1,0}(M)$ is involutive, it is not necessary that $HM$ be integrable. Thus M. Okumura [6] and Y. Maeda [5] found different geometric characterizations of a class of real hypersurfaces of $CP^n$ with non-integrable $HM$.

Received by the editors October 12, 1993 and, in revised form, July 19, 1994.
1991 *Mathematics Subject Classification.* Primary 53C40, 53C55; Secondary 53C12, 53C42, 53C16.
*Key words and phrases.* Real hypersurface, CR-submanifold, complex projective space, Levi flat, totally real Ricci curvature, holomorphic distribution.

For any $x \in M$ consider a unit vector $N_x \in T_xM^{\perp}$ and let $\xi_x = -J_x(N_x)$. Then following A. Boggess [3], define the Levi form at the point $x$ as the map

$$L_x\colon H_x^{1,0}(M) \to \mathfrak{R}, \qquad L_x(W_x) = -g_x([X, JX]_x, \xi_x),$$

where $X$ is the $HM$-vector field extension of $X_x = \frac{1}{2}(W_x + \overline{W}_x)$. We say that $M$ is a *Levi flat CR-submanifold* if $L_x$ vanishes for any $x \in M$. Taking into account that $\xi_x \in (RM)_x$, from the definition of $L_x$ it follows that in case $HM$ is integrable, $M$ is Levi flat. The converse of this assertion is a consequence of Theorem 1 in A. Boggess [3], p. 158. Hence *$M$ is Levi flat iff $HM$ is integrable.*

Next, suppose $M$ is an orientable real hypersurface in $CP^n$. Then there exists a globally defined unit normal vector field $N$ on $M$. Thus the totally real distribution $RM$ is globally spanned by $\xi = -JN$. The Ricci curvature of $M$ in the direction $\xi$ is called the *totally real Ricci curvature* and it is denoted by $\mathrm{Ric}(\xi, \xi)$. Our paper has the origin in the remark that the totally real Ricci curvature of $K$-contact manifold is a positive number (cf. D. Blair [2], p. 65) and that $HM$ is not integrable. So it is natural to ask whether $HM$ is not integrable in general for a real hypersurface of $CP^n$ with non-negative totally real Ricci curvature. In this respect we prove the following result.

**Theorem.** *Let $M$ be a compact orientable real hypersurface without boundary of $CP^n, n > 1$, such that $\mathrm{Ric}(\xi, \xi) \geq 0$ everywhere on $M$. Then $HM$ is not integrable on $M$.*

The authors would like to express their sincere gratitude to the referee for many valuable suggestions to improve the paper.

## 2. Preliminaries

Let $M$ be an orientable real hypersurface of $CP^n$. Denote by $\nabla$ and $\overline{\nabla}$ the Levi Civita connection on $M$ and $CP^n$ respectively. Then we have the well-known formulae

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{2.1}$$

$$\overline{\nabla}_X N = -AX, \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on $M, N$ is the unit normal vector field to $M$ and $A$ is the shape operator of $M$. Let $\eta$ be the 1-form dual to $\xi$, that is, $\eta(X) = g(X, \xi)$ for any $X \in \mathfrak{X}(M)$. Then we have $JX = JPX + \eta(X)N$, where $P$ is the projection morphism of $TM$ on $HM$. On using (2.1) and (2.2) and taking into account that $J$ is parallel with respect to $\overline{\nabla}$ one obtains

$$\nabla_X \xi = JPAX, \qquad \forall X \in \mathfrak{X}(M). \tag{2.3}$$

Consider $CP^n$ as a complex space form of the constant holomorphic sectional curvature $c = 4$. Then using the formulae of curvature tensor field of $CP^n$ (cf. Kobayashi-Nomizu [5], p. 167), the equations of Gauss and Codazzi for the immersion of $M$ in $CP^n$ become

$$\begin{aligned} g(R(X,Y)Z,W) &\\ = g(Y,Z)g(X,W) &- g(X,Z)g(Y,W) + g(Z,JY)g(JX,W) \\ &- g(Z,JX)g(JY,W) + 2g(X,JY)g(JZ,W) \\ &+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) \end{aligned} \tag{2.4}$$

and

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = g(X,\xi)JY - g(Y,\xi)JX + 2g(X,JY)\xi, \tag{2.5}$$

respectively, for any $X, Y, Z, W \in \mathfrak{X}(M)$, where $R$ is the curvature tensor corresponding to $\nabla$.

Finally denote by $\mathcal{F}(M)$ the algebra of smooth functions on $M$ and by $\Gamma(HM)$ and $\Gamma(RM)$ the $\mathcal{F}(M)$-modules of smooth vector fields on $M$ which belong to $HM$ and $RM$ respectively. We may always consider a local orthonormal field of frames $\{E_1, \ldots, E_{n-1}, JE_1, \ldots, JE_{n-1}, \xi\}$ on a coordinate neighborhood $U \subset M$, where $\{E_i, JE_i\} \subset \Gamma(HM), i \in \{1, \ldots, n-1\}$. Such a frame field is said to be a *local CR-frame field* on $M$.

## 3. Proof of the Theorem

First we prove

**Lemma.** *Let $M$ be a compact orientable real hypersurface without boundary of $CP^n$. Suppose* $\mathrm{Ric}(\xi,\xi) \geq 0$ *everywhere on $M$ and $HM$ is an integrable distribution on $M$. Then we have*

(i) $\mathrm{Ric}(\xi,\xi) = 0$,
(ii) $\nabla_X \xi = 0, \forall X \in \Gamma(HM)$,
(iii) $\nabla_X Y \in \Gamma(HM), \forall X, Y \in \Gamma(HM)$,
(iv) $AX \in \Gamma(RM), \forall X \in \Gamma(HM)$,
(v) $\|A\xi\|^2 = 2(n-1) + g(A\xi,\xi)^2$.

*Proof.* Choose a local CR-frame field $\{E_i, JE_i, \xi\}$ on $M$. Then taking into account that $A$ and $J$ are symmetric and antisymmetric with respect to $g$, that is, $g(AX,Y) = g(X,AY)$ and $g(JX,Y) = -g(X,JY)$ holds for $X, Y \in \mathfrak{X}(M)$, and using (2.3) obtain

$$\delta\eta = \operatorname{div}\xi = 0. \tag{3.1}$$

On the other hand, by direct calculations, using the integrability of $HM$ obtain $d\eta(X,Y) = 0$ and $d\eta(\xi,X) = g(\nabla_\xi \xi, X)$, for any $X, Y \in \Gamma(HM)$. Hence locally on $U$ we have

$$\|d\eta\|^2 = 2\sum_{i=1}^{n-1}\{g(\nabla_\xi\xi, E_i)^2 + g(\nabla_\xi\xi, JE_i)^2\} = 2\|\nabla_\xi\xi\|^2. \tag{3.2}$$

Since for each $x \in M$, we have a coordinate neighborhood $U$ and a local CR-frame field $\{E_i, JE_i, \xi\}$ on $U$, (3.2) holds for each $x \in M$ and hence globally on $M$. Next, we recall that on any compact orientable Riemannian manifold $M$ without boundary we have (cf. K. Yano [7], p. 41)

$$\int_M \left\{\mathrm{Ric}(X,X) - \frac{1}{2}\|d\alpha\|^2 + \|\nabla X\|^2 - (\delta\alpha)^2\right\} dv = 0,$$

where $\alpha$ is a 1-form dual to $X$ on $M$. Replace $X$ by $\xi$ and $\alpha$ by $\eta$ in the above integral formula and use (3.1) to obtain

$$\int_M \left\{\mathrm{Ric}(\xi,\xi) - \frac{1}{2}\|d\eta\|^2 + \|\nabla\xi\|^2\right\} dv = 0. \tag{3.3}$$

Since on each coordinate neighborhood $U$, we have

$$\|\nabla\xi\|^2 = \sum_{i=1}^{n-1}\{\|\nabla_{E_i}\xi\|^2 + \|\nabla_{JE_i}\xi\|^2\} + \|\nabla_\xi\xi\|^2,$$

on account of (3.2) we thus find

$$\|\nabla\xi\|^2 - \frac{1}{2}\|d\eta\|^2 = \sum_{i=1}^{n-1}\{\|\nabla_{E_i}\xi\|^2 + \|\nabla_{JE_i}\xi\|^2\} \geq 0 \tag{3.4}$$

on each $U$ and consequently on $M$. As $\mathrm{Ric}(\xi,\xi) \geq 0$, it follows that the integrand in (3.3) is non-negative. Hence we must have $\mathrm{Ric}(\xi,\xi) = 0$ and $\|\nabla\xi\|^2 - \frac{1}{2}\|d\eta\|^2 = 0$. The second equation together with (3.4) gives $\nabla_{E_i}\xi = \nabla_{JE_i}\xi = 0$ for each local CR-frame field $\{E_i, JE_i, \xi\}$ and consequently we obtain (i) and (ii). Taking into account that $\nabla$ is a Riemannian connection from (ii) we infer (iii). Moreover (iv) follows from (ii) on using (2.3). Finally using (iv), from (2.4) obtain

$$g(R(E_i,\xi)\xi, E_i) = 1 - g(A\xi, E_i)^2$$

and

$$g(R(JE_i,\xi)\xi, JE_i) = 1 - g(A\xi, JE_i)^2.$$

Hence (i) gives

$$\begin{aligned}
0 = \mathrm{Ric}(\xi,\xi) &= \sum_{i=1}^{n-1}\{g(R(E_i,\xi)\xi,E_i) + g(R(JE_i,\xi)\xi,JE_i)\}\\
&= 2(n-1) - \sum_{i=1}^{n-1}\{g(A\xi,E_i)^2 + g(A\xi,JE_i)^2\}\\
&= 2(n-1) - \{\|A\xi\|^2 - g(A\xi,\xi)^2\},
\end{aligned}$$

on any coordinate neighborhood $U$, which gives (v). This completes the proof of the Lemma. □

*Remark.* The assertions (i)–(iv) of the Lemma hold in a more general setting, namely, in case the complex projective space is replaced by an arbitrary Kaehler manifold.

Now we proceed with the proof of the Theorem. Suppose $HM$ is integrable. Then using (iv) and (2.1) and taking into account that $J$ is parallel with respect to $\overline{\nabla}$ obtain

$$\nabla_X JY = J\nabla_X Y, \quad \forall X, Y \in \Gamma(HM). \tag{3.5}$$

Thus on a coordinate neighborhood $U$, (iii) and (3.5) imply

$$\nabla_{E_i}E_j = \sum_{k=1}^{n-1}\{a_{ijk}E_k + b_{ijk}JE_k\}, \qquad \nabla_{E_i}JE_j = \sum_{k=1}^{n-1}\{a_{ijk}JE_k - b_{ijk}E_k\}, \tag{3.6}$$

and

(3.7)

$$\nabla_{JE_i}E_j = \sum_{k=1}^{n-1}\{c_{ijk}E_k + d_{ijk}JE_k\}, \qquad \nabla_{JE_i}JE_j = \sum_{k=1}^{n-1}\{c_{ijk}JE_k - d_{ijk}E_k\},$$

where $\{a_{ijk}, b_{ijk}, c_{ijk}, d_{ijk}\}$ are smooth functions on $U$ satisfying

$$a_{ijk} + a_{ikj} = 0, \quad c_{ijk} + c_{ikj} = 0, \quad b_{ijk} = b_{ikj}, \quad d_{ijk} = d_{ikj}. \tag{3.8}$$

As $\nabla_\xi \xi$ is orthogonal to $\xi$, there exist smooth functions $\{a_i, b_i\}, i \in \{1, \ldots, n-1\}$, such that

$$\nabla_\xi \xi = \sum_{k=1}^{n-1} \{a_k E_k + b_k JE_k\}. \tag{3.9}$$

Finally using (ii) and (3.9) and taking into account that $\nabla$ is a Riemannian connection, obtain

$$\nabla_\xi E_i = \sum_{k=1}^{n-1} \{a_{ik} E_k + b_{ik} JE_k\} - a_i \xi, \qquad \nabla_{E_i} \xi = 0, \tag{3.10}$$

and

$$\nabla_\xi JE_i = \sum_{k=1}^{n-1} \{c_{ik} E_k + d_{ik} JE_k\} - b_i \xi, \qquad \nabla_{JE_i} \xi = 0, \tag{3.11}$$

where $\{a_{ik}, b_{ik}, c_{ik}, d_{ik}\}$ are smooth functions on $U$. Now using (2.1), (2.2) and (3.9) and taking into account that $\overline{\nabla}$ is a Riemannian connection we infer that

$$a_i = -g(A\xi, JE_i) \quad \text{and} \quad b_i = g(A\xi, E_i).$$

Thus on account of (iv), we have

$$AE_i = b_i \xi, \quad AJE_i = -a_i \xi, \quad A\xi = \sum_{k=1}^{n-1} \{b_k E_k - a_k JE_k\} + c\xi, \tag{3.12}$$

where $c$ is a smooth function on $U$. Then (v) in the Lemma and the last equation in (3.12) imply

$$\sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} = 2(n-1). \tag{3.13}$$

Further, take $X = E_i$ and $Y = \xi$ in (2.5), use (3.6), (3.9), (3.10) and (3.12) and equate the components with respect to the holomorphic frame $\{E_k, JE_k\}$ of $\Gamma(HM)$ to obtain

(3.14)
$$\begin{cases} E_i(a_k) = a_i a_k - b_i b_k + \delta_{ik} + \sum_{j=1}^{n-1} \{b_j b_{ijk} - a_j a_{ijk}\}, \\ E_i(b_k) = a_i b_k + b_i a_k - \sum_{j=1}^{n-1} \{b_j a_{ijk} + a_j b_{ijk}\}, \qquad i, k \in \{1, \ldots, n-1\}. \end{cases}$$

In a similar way, take $X = JE_i$ and $Y = \xi$ in (2.5) and use (3.7), (3.9), (3.11) and (3.12) to infer

$$\begin{cases} JE_i(b_k) = b_i b_k - a_i a_k + \delta_{ik} - \sum_{j=1}^{n-1} \{b_j c_{ijk} + a_j\, d_{ijk}\}, \\ JE_i(a_k) = a_i b_k + b_i a_k + \sum_{j=1}^{n-1} \{b_j\, d_{ijk} - a_j c_{ijk}\}. \end{cases} \tag{3.15}$$

From (3.14) and (3.15) on using (3.8) it is easy to obtain

$$\frac{1}{2} E_i \left( \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right) = a_i \left( 1 + \sum_{k=1}^{n-1} \{(a_k)^2 + (b_k)^2\} \right) \tag{3.16}$$

and

$$\frac{1}{2}JE_i\left(\sum_{k=1}^{n-1}\{(a_k)^2+(b_k)^2\}\right)=b_i\left(1+\sum_{k=1}^{n-1}\{(a_k)^2+(b_k)^2\}\right), \tag{3.17}$$

respectively. On using (3.13) in (3.16) and (3.17) obtain $a_i = 0$ and $b_i = 0$ for any $i \in \{1, \ldots, n-1\}$. Thus we get a contradiction to (3.13) and this completes the proof of the theorem. As a direct consequence of the Theorem we have

**Corollary.** *Let $M$ be a Levi flat compact real hypersurface without boundary of $CP^n, n > 1$. Then there exists an open subset $U$ of $M$ such that the totally real Ricci curvature of $M$ is negative everywhere on $U$.*

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