

## ON FUNCTIONS THAT ARE TRIVIAL COCYCLES FOR A SET OF IRRATIONALS. II

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(Communicated by J. Marshall Ash)

ABSTRACT. Two results are obtained about the topological size of the set of irrationals for which a given function is a trivial cocycle. An example of a continuous function which is a coboundary with non- $L^1$  cobounding function is constructed.

A function  $v : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  is called an (additive) coboundary for an irrational  $\alpha$  if there is a measurable function  $w : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  such that  $v(x) = w(x) - w(x + \alpha)$  a.e. (where we parameterize  $\mathbb{R}/\mathbb{Z}$  by the interval  $[0,1)$  with addition mod 1). It is called trivial if  $v(x) - c$  is a coboundary for some  $c \in \mathbb{R}$ . In either case the function  $w$ , which is unique up to an additive constant, is called the cobounding function. The question of whether particular functions or classes of functions are coboundaries for a given  $\alpha$  has applications in ergodic theory and the representation theory of non-Type I groups (see, for example, [BM1],[ILR]). Recent research has revealed an interesting interplay between classes of functions and the types of irrationals for which they can be coboundaries (e.g., [BM2], [M]). Thus it is natural to look at the coboundary question from an opposite point of view, fixing a function  $v$ , and asking for exactly which irrationals  $v$  is a coboundary. A simple Fourier series argument shows that a trigonometric polynomial must be a coboundary for every irrational  $\alpha$ . For other types of functions, this question is much harder to answer.

In a 1988 paper in this journal [B], L. Baggett presented a proof that the set of irrationals for which a given continuous function is a coboundary must be of the first category, unless the function is a trigonometric polynomial. Shortly after this paper appeared, P. Liardet, A. Iwanik, and P. Hellekalek pointed out a gap in the proof. This gap remains unfilled. In this paper, we present an altered version of that proof in the case that the original function is real-analytic. We also present a parallel result for  $L^1$  functions in which the cobounding function is also required to be  $L^1$ . Finally, we display a counterexample which shows that the requirement of an  $L^1$  cobounding function can be a genuine restriction.

**Theorem 1.** *Let  $v$  be an integrable, real-analytic function on the open interval  $(0,1)$ , which is not a trigonometric polynomial. Then the set  $S$  of all irrationals for which  $v$  is a trivial cocycle is of the first category.*

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Received by the editors November 15, 1993 and, in revised form, June 21, 1994.

1991 *Mathematics Subject Classification.* Primary 28D05, 11K38.

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*Proof.* Suppose  $S$  is not of the first category. Let  $f(x) = e^{2\pi iv(x)}$ . For each integer  $k$  and each positive integer  $j$ , let  $A_{k,j}$  be the set of numbers (rational or irrational)  $\alpha$  for which there exists a constant  $\lambda$  and a measurable function  $g$  such that

- (1)  $\|g\|_2 \leq 1$ .
- (2)  $|\lambda| = 1$ .
- (3)  $|\int_0^1 g(x)e^{2\pi ikx} dx| \geq \frac{1}{j}$ .
- (4)  $f(x)g(x+\alpha) = \lambda g(x)$  for almost all  $x \in [0, 1)$ .

We claim that each set  $A_{k,j}$  is a closed set. Let  $\{\alpha_n\}$  be a sequence in  $A_{k,j}$  converging to  $\alpha$ . For each  $\alpha_n$ , there exists a constant  $\lambda_n$  and a measurable function  $g_n$  satisfying (1)–(4) above. By passing to a subsequence twice, we may assume that  $\lambda_n \rightarrow \lambda$  and  $g_n \rightarrow g$  weakly in  $L^2$ .  $\lambda$  and  $g$  satisfy conditions (1)–(3) above, and a computation shows that  $f(x)g(x+\alpha) = \lambda g(x)$  for almost all  $x \in [0, 1)$ . Thus  $\alpha \in A_{k,j}$ .

Clearly  $S \subseteq \bigcup_{k,j} A_{k,j}$ . By the Baire Theorem, some set  $A_{k_0,j_0}$  must contain an open interval. Therefore, there exists a positive integer  $Q$  such that for every  $q \geq Q$  there is a rational number  $p/q \in A_{k_0,j_0}$ , and thus a constant  $\lambda_q$  and a function  $g_q$  satisfying

- (1)  $\|g_q\|_2 \leq 1$ .
- (2)  $|\lambda_q| = 1$ .
- (3)  $|\int_0^1 g_q(x)e^{2\pi ik_0 x} dx| \geq \frac{1}{j_0}$ .
- (4)  $f(x)g_q(x + \frac{p}{q}) = \lambda_q g_q(x)$  for almost all  $x \in [0, 1)$ .

We also may assume that  $p$  is relatively prime to  $q$ . It follows by condition (4) that for each  $q \geq Q$  there exists a  $p$  relatively prime to  $q$  such that

$$(*) \quad f(x)f(x + \frac{p}{q})f(x + \frac{2p}{q}) \dots f(x + \frac{(q-1)p}{q}) = \lambda_q^q$$

for every  $x$  for which  $g_q(x) \neq 0$ , and, by condition (3), this is certainly a set of positive measure.

Now the function

$$f(x)f(x + \frac{p}{q}) \dots f(x + \frac{(q-1)p}{q})$$

has discontinuities at most at the multiples of  $\frac{p}{q}$ , and on each subinterval  $(\frac{j}{q}, \frac{(j+1)}{q})$  it is real-analytic. By the identity theorem for real-analytic functions, it follows that

$$f(x)f(x + \frac{p}{q}) \dots f(x + \frac{p(q-1)}{q})$$

is identically  $\lambda_q^q$  on some one of these subintervals. By the invariance of  $(*)$  under translation by  $\frac{p}{q}$ , it follows that

$$f(x)f(x + \frac{p}{q}) \dots f(x + \frac{p(q-1)}{q}) \equiv \lambda_q^q$$

for all  $x$  not of the form  $\frac{pj}{q}$ .

Now  $f(x) = e^{2\pi iv(x)}$ , so we have that

$$(**) \quad v(x) + v(x + \frac{p}{q}) + \dots + v(x + \frac{p(q-1)}{q}) = c_q + N_q(x)$$

where  $c_q$  is a constant and  $N_q$  is an integer-valued function. Because  $v$  is continuous, we have that  $N_q$  is constant on the subintervals  $(\frac{j}{q}, \frac{(j+1)}{q})$ .

Using (\*\*), we compute the  $n$ th Fourier coefficient of  $v$ ,  $c_{nq}(v)$ , and obtain

$$qc_{nq}(v) = 0$$

for every nonzero integer  $n$ . Since this computation holds for every  $q \geq Q$ , it follows immediately that  $v$  is a trigonometric polynomial.  $\square$

*Remark.* Michael Herman [H, Theorem 4.11] proved a similar result under the additional hypothesis that for all  $n \neq 0$ ,  $c_n(v) \neq 0$ .

By requiring the cobounding functions to be integrable, we obtain the following stronger result.

**Theorem 2.** *Let  $v$  be a real-valued  $L^1$  function on  $\mathbb{R}/\mathbb{Z}$ , which is not a trigonometric polynomial. Then the set of all irrationals for which  $v$  is a trivial cocycle with  $L^1$  cobounding function is of the first category.*

*Proof.* By the Riemann-Lebesgue Lemma, if  $v$  is a coboundary for  $\alpha$  with  $L^1$  cobounding function  $w$ , then  $|c_n(w)| = |c_n(v)|/|1 - e^{2\pi i n \alpha}| \rightarrow 0$  as  $|n| \rightarrow \infty$ . Thus it will suffice to find a dense  $G_\delta$  set  $E$  of irrationals such that for  $\alpha \in E$ ,  $|c_n(v)|/|1 - e^{2\pi i n \alpha}| \not\rightarrow 0$ . Since  $v$  is not a trigonometric polynomial,  $\exists \{m_k\}_{k=1}^\infty$ ,  $m_k \rightarrow \infty$ , such that  $|c_{m_k}(v)| = \epsilon_k \neq 0$ . Choose  $a_k$  so that  $a_k > \frac{1}{m_k \epsilon_k}$ . Let

$$A_k = \bigcup_{j=1}^{m_k-1} \left( \frac{j}{m_k} - \frac{1}{a_k m_k^2}, \frac{j}{m_k} + \frac{1}{a_k m_k^2} \right).$$

If  $\alpha \in A_k$ , then  $\exists j$  such that  $\left| \alpha - \frac{j}{m_k} \right| < \frac{1}{a_k m_k^2}$ , which implies  $|m_k \alpha - j| < \frac{1}{a_k m_k}$  and hence  $|1 - e^{2\pi i m_k \alpha}| < \frac{1}{a_k m_k}$ . Thus we see that for  $\alpha \in A_k$ ,  $|c_{m_k}(v)|/|1 - e^{2\pi i m_k \alpha}| > 1$ . Let  $E_n = \bigcup_{k=n}^\infty A_k$ .  $E_n$  is open and dense for each  $n$ , and by the above we have that if  $\alpha \in E_n$ ,  $\exists m_k$ ,  $k \geq n$ , such that  $|c_{m_k}(v)|/|1 - e^{2\pi i m_k \alpha}| > 1$ . Set  $E = \bigcap_{n=1}^\infty E_n$ .  $\square$

The apparent advantage of the second theorem over the first raises the natural question of whether an  $L^1$  coboundary, or even an analytic coboundary, must have an  $L^1$  cobounding function.

**Theorem 3.** *Given any irrational  $\alpha$ , there exists a continuous coboundary  $v$  for  $\alpha$ , whose cobounding function is not  $L^1$ .*

*Proof.* Choose a sequence of rationals  $\{\frac{p_n}{q_n}\}$  satisfying

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{n^3 2^{2n+1} q_n}.$$

(This can be done by choosing a subsequence of the convergents to  $\alpha$  so that each element,  $\frac{p_n}{q_n}$ , of this subsequence has the property that  $q_n \geq n^3 2^{2n+1}$ .) For each  $n \geq 1$ , we define the function  $u_n$  by

$$u_n(x) = \begin{cases} 2^{n+1} + n2^{2n+1}q_n x & \text{if } x \in (-\frac{1}{n2^n q_n}, 0), \\ 2^{n+1} - n2^{2n+1}q_n x & \text{if } x \in (0, \frac{1}{n2^n q_n}), \end{cases}$$

and then define

$$w_n(x) = \sum_{p=0}^{q_n-1} u_n\left(x - \frac{p}{q_n}\right).$$

(The function  $w_n$  is triangular on  $(\frac{p}{q_n} - \frac{1}{n2^n q_n}, \frac{p}{q_n} + \frac{1}{n2^n q_n})$  with  $w_n(\frac{p}{q_n}) = 2^{n+1}$  for  $p = 0, 1, \dots, q_n - 1$ , and 0 everywhere else.) Finally, let

$$w(x) = \sum_{n=1}^{\infty} w_n(x).$$

To show that  $w$  is finite a.e., we show that  $S_N = \sum_{n=1}^N w_n$  is Cauchy in measure. Indeed, for any  $N > M$ ,  $S_N - S_M = \sum_{n=M+1}^N w_n$  is supported on a set of measure  $\sum_{n=M+1}^N (q_n)(\frac{1}{n2^n q_n})$ , which goes to zero as  $N$  and  $M$  go to infinity. We see that  $w$  is not in  $L^1$  by noting that  $\int |w_n(x)| dx = \frac{2}{n}$  so that by the monotone convergence theorem we have

$$\int |w(x)| dx = \sum_{n=1}^{\infty} \int |w_n(x)| dx = \sum_{n=1}^{\infty} \frac{2}{n} = \infty.$$

Now we define

$$v(x) = w(x) - w(x + \alpha) = \sum_{n=1}^{\infty} w_n(x) - w_n(x + \alpha).$$

Since the  $w_n(x) - w_n(x + \alpha)$  are continuous, it will follow from the M-test that  $v$  is continuous, if we can show that  $|w_n(x) - w_n(x + \alpha)| < \frac{1}{n^2}$ . By the periodicity of  $w_n$ , we have that

$$|w_n(x) - w_n(x + \alpha)| = \left| w_n\left(x + \frac{p_n}{q_n}\right) - w_n(x + \alpha) \right| \leq n2^{2n+1} q_n \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{n^2},$$

since  $n2^{2n+1} q_n$  is the maximum slope of a secant line of  $w_n$ . □

*Remark.* For certain  $\alpha$ , we can modify the above construction to give  $C^r$  coboundaries with non- $L^1$  cobounding functions. In particular, if there is a sequence of rational approximations to  $\alpha$ ,  $\{\frac{p_n}{q_n}\}$ , such that  $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^{r+2}}$ , we can replace the continuous, piecewise linear functions  $w_n$  with  $C^r$ , piecewise  $(r + 1)$ st degree polynomials, with the same integral as before, and with the property that  $\sum w_n^{(r)}(x) - w_n^{(r)}(x + \alpha)$  converges uniformly, thus giving  $v$  a continuous  $r$ th derivative. Y. Meyer [H, p. 187] has a related result in the  $r = 1$  case, which implies that if  $\alpha$  has bounded partial quotients in its continued fraction expansion, then there exists a  $C^1$  function which is a coboundary for  $\alpha$  with noncontinuous cobounding function. The question of whether there are analytic coboundaries with non- $L^1$  cobounding functions remains unanswered.

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