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ON WEAK* CONVERGENCE IN H^1

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ABSTRACT. A bounded sequence of functions in H^1 which converges in measure on a set of positive measure of the unit circle converges weak*. An example is given to show that weak* convergence cannot be replaced by weak convergence.

The following result appears in the paper [2] of Jones and Journé.

Theorem. *Suppose $\{f_n\}$ is a sequence of $H^1(\mathbf{R}^d)$ functions such that $\|f_n\|_{H^1} \leq 1$ for all n and such that $f_n(x) \to f(x)$ for almost every $x \in \mathbf{R}^d$. Then $f \in H^1(\mathbf{R}^d)$, $\|f\|_{H^1} \leq 1$, and*

$$\int_{\mathbf{R}^d} f_n \phi \, dx \to \int_{\mathbf{R}^d} f \phi \, dx$$

for all $\phi \in \mathrm{VMO}(\mathbf{R}^d)$.

Since $H^1(\mathbf{R}^d)$ is isomorphic to the dual space of $\mathrm{VMO}(\mathbf{R}^d)$, this result can be paraphrased as saying that norm boundedness and pointwise almost everywhere convergence imply weak* convergence in real H^1. It is not hard to see that convergence a.e. can be replaced by convergence in measure. It is the purpose of this note to demonstrate that for the Hardy space H^1 of analytic functions in the unit disk it is enough to assume that this convergence occurs only on some set of positive measure.

Theorem. *Let $\{f_n\}$ be a sequence in H^1 such that $\|f_n\|_1 \leq 1$ and let E be a set of positive measure on the unit circle such that $\{f_n|_E\}$ converges in measure on E to some function ϕ. Then $\{f_n\}$ converges weak* to a function f in H^1 and the boundary values of f coincide almost everywhere on E with the values of ϕ.*

This theorem follows immediately from the following simple proposition and a deep theorem of Khintchin and Ostrowski which will be discussed below. It should be noted that $(\mathfrak{B}, \text{weak}^*)$ is metrizable, since the predual VMOA of H^1 is separable, and thus it is enough to consider sequential convergence.

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Proposition. *If $\mathfrak{B}$ denotes the unit ball in H^1 and if κ is the compact-open topology on $\mathfrak{B}$, then the identity map $\mathfrak{I}\colon(\mathfrak{B},\text{weak}^*)\to(\mathfrak{B},\kappa)$ is continuous and hence is a homeomorphism. Thus weak* sequential convergence on $\mathfrak{B}$ is characterized by uniform convergence on compact subsets of $\mathbf{D}$.*

Proof. Since $(\mathfrak{B},\text{weak}^*)$ is compact and $(\mathfrak{B},\kappa)$ is Hausdorff, it is enough to show that $\mathfrak{I}\colon(\mathfrak{B},\text{weak}^*)\to(\mathfrak{B},\kappa)$ is continuous. To this end let $\{f_n\}$ be a sequence in $\mathfrak{B}$ which converges weak* to f. Because $\mathfrak{B}$ is a normal family, every subsequence of $\{f_n\}$ has a further subsequence which converges uniformly on compact sets to some function. Since point evaluations inside the unit disk are weak* continuous, this function must coincide with f. That completes the proof.

The following theorem of Khintchin and Ostrowski is used to complete the proof of the theorem above. An exposition of this theorem can be found in [5, pp. 118–131], or in a slightly weaker form in [7, p. 206].

Theorem. *Let $\{f_n\}$ be a sequence of functions analytic in the unit disk satisfying the following conditions:*

(i) *There exists a constant $C>0$ such that*

$$\int_0^{2\pi}\log^+|f_n(re^{i\theta})|\,d\theta\le C.$$

(ii) *On some set E of positive measure on the unit circle the sequence $f_n(e^{i\theta})$ of boundary values of the functions $f_n(z)$ converges in measure to a function ϕ.*

Then the sequence $f_n(z)$ converges uniformly on compact subsets of the unit disk to a function $f(z)$, and the sequence $f_n(e^{i\theta})$ converges in measure on E to $f(e^{i\theta})$, the boundary values of the function $f(z)$.

The classical theory of Hardy spaces [1] shows that a bounded sequence in H^1 satisfies condition (i), and Montel's theorem will extract from any subsequence a further subsequence which converges uniformly on compact subsets to a function which clearly belongs to H^1. The significance of the Khintchin-Ostrowski theorem is that the limit function has boundary values coinciding a.e. with ϕ on the set E. Since E, being a set of postive measure, is a set of uniqueness, every convergent subsequence has the same limit function and hence the original sequence converges to this limit uniformly on compact sets.

In closing it is important to note that weak* convergence cannot be replaced by weak convergence, as the following example shows. Let E_n be a disjoint sequence of measurable sets on the unit circle. Then clearly $|E_n|\to 0$, where $|E|$ denotes the (normalized) measure of the set E. Let $\{\epsilon_n\}$ be a sequence of numbers between 0 and 1 which converges to 0. For each n choose $C_n>0$ so that $C_n|E_n|+\epsilon_n(1-|E_n|)=1$, and let F_n be the outer function given by

$$|F_n(e^{i\theta})|=\begin{cases} C_n & \text{if } e^{i\theta}\in E_n,\\ \epsilon_n & \text{otherwise.}\end{cases}$$

Then $\|F_n\|_1=C_n|E_n|+\epsilon_n(1-|E_n|)=1$. Evidently, F_n converges pointwise to 0 on the unit circle and so also weak* to zero. On the other hand let $h\in L^\infty$ be a unimodular function arranged so that $hF_n=|F_n|$ on E_n for each n. Then

$$\begin{aligned}\left|\int_0^{2\pi}hF_n\,dm\right| &\ge \int_{E_n}|F_n|\,dm-\int_{E_n'}|F_n|\,dm\\ &\ge 1-\epsilon_n\to 1,\end{aligned}$$

as $n \to \infty$. Hence $\{F_n\}$ does not converge weakly to 0. With a bit more effort such a sequence $\{F_n\}$ can be produced which is equivalent to the usual unit vector basis of l^1. It is known [6] that the existence of such a sequence characterizes nonreflexive subspaces of L^1.

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