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CONJUGATE BUT NON INNER CONJUGATE SUBFACTORS

MARIE CHODA

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ABSTRACT. It is shown that for each $r \in \{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$, there exist at least infinitely many subfactors of the hyperfinite II_1 factor R with index r, which are pairwise conjugate but non inner conjugate. In the case that r is an integer, we have uncountably many such subfactors of R.

1. INTRODUCTION

Two von Neumann subalgebras B and C of a von Neumann algebra A are said to be conjugate if there exists a *-automorphism θ of A with $\theta(B) = C$. If we can take the θ as an inner automorphism of A, then B and C are said to be inner conjugate.

Kosaki and Yamagami showed in [12], among other things, that there are infinitely many inner conjugacy classes of subfactors in the hyperfinite II_1 factor R with the Jones index 2 ([10]). Otherwise, the subfactors of R with index 2 are given as the fixed point algebras outer automorphisms on R with the period 2 ([10]) so that they are all conjugate by [4]. Bearing this fact in mind, we generalize their result in two directions.

One is that, for a given $r \in \{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$, there exists a countable infinity of non inner conjugate but conjugate subfactors of R with index r. More precisely,

Theorem 1. *Let P be a subfactor of the hyperfinite II_1 factor R with $[R : P] < \infty$. Assume that either $[R : P] \leq 4$ (except one of type $A_1^{(1)}$) or $P \subset R$ is Jones' inclusion given in [10]. Then there exists a countable infinity of subfactors of R which are pairwise non inner conjugate but conjugate to P.*

A similar result holds for the subfactors of Wenzl ([21]), of Wassermann ([20]) and of Pimsner-Popa ([15, 16]). We construct such subfactors by an easy method (i.e. tensor product). To distinguish the inner conjugacy classes of these subfactors, we use basic (in the sense of [2]) *-endomorphisms of R instead of bimodules as in [12].

Another generalization is that for each integer n, there exist uncountably many non inner conjugate subfactors of R which are pairwise conjugate. More precisely,

Theorem 2. *Let G be a finite group and α the outer action of G on the hyperfinite II_1 factor R. Then there exist uncountably many subfactors of R which are pairwise non inner conjugate but conjugate to the fixed point algebra R^α of G under α.*

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2. Basic *-endomorphisms

Throughout this paper, M is a II_1 factor with separable predual and τ is the unique faithful normal trace of M, $\tau(1) = 1$. We denote by $\mathrm{End}(M, \tau)$ (resp. $\mathrm{Aut}(M)$) the set of trace-preserving *-endomorphisms (resp. automorphisms) of M. If $\sigma \in \mathrm{End}(M, \tau)$, then σ is unit-preserving, faithful and σ-weakly continuous. Hence $\sigma(M)$ is a subfactor of M. Assume that the index $[M : \sigma(M)] < \infty$. By iterating the Jones basic construction ([10]) from the inclusion $\sigma(M) \subset M$, the tower of II_1 factors $\{M_n\}_{n\geq 1}$ is obtained:

$$\sigma(M) \subset M \subset M_1 = \langle M, e_1\rangle \subset \cdots \subset M_n = \langle M_{n-1}, e_n\rangle \cdots,$$

where e_n is the Jones projection for the inclusion $M_{n-2} \subset M_{n-1}$. The trace τ is extended to the unique trace τ of M_n $(n \geq 1)$. Let M_∞ be the II_1 factor obtained from the GNS construction of $\bigcup_n M_n$ by τ.

A $\sigma \in \mathrm{End}(M, \tau)$ is called basic ([2]) if $[M : \sigma(M)] < \infty$ and σ satisfies one of the following equivalent conditions:

(1) σ is extended to a *-isomorphism from M_1 onto M.

(2) There exists a projection $e \in M$ such that

$$\sigma^2(M) = \{e\}' \cap \sigma(M), \qquad E_{\sigma(M)}(e) = [M : \sigma(M)]^{-1},$$

where E_B is the τ-preserving conditional expectation of M onto the von Neumann subalgebra B of M. Such a projection $e \in M$ is called a basic projection for σ.

Two *-endomorphisms ρ and σ of M are said to be unitarily equivalent if there exists a unitary $u \in M$ with $u\rho(a)u^* = \sigma(a)$ $(a \in M)$. The unitary equivalence class of ρ is denoted by $[\rho]$. Then

(i) if $[\rho] = [\sigma]$, then $[\rho^n] = [\rho^{n-1}\sigma]$, for all n;

(ii) if $[\rho] = [\sigma \cdot \theta]$ for some $\theta \in \mathrm{Aut}(M)$, then $\rho(A)' \cap A$ is inner conjugate to $\sigma(A)' \cap A$.

Example 1. Let A_0 be the $m \times m$ matrices and $M = \bigotimes_{i=1}^{\infty}(A_i, \tau_i)$, where $A_i = A_0$, and τ_i is the normalized unique trace of A_i $(i \geq 1)$. Let η_i be the canonical embedding of A_0 onto A_i (under the identification with the subfactor of M) with $\eta_i(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots$. Then the *-endomorphism σ of M defined by $\sigma(\eta_i(x)) = \eta_{i+1}(x)$ $(x \in A_0)$ is basic.

Example 2 (Jones' inclusion). Let $r \in \{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$ and put $\lambda = r^{-1}$. Let P be the factor generated by the sequence of projections $\{e_i; i \geq 1\}$ with

$$e_ie_je_i = \begin{cases} \lambda e_i & (|i-j| = 1), \\ e_ie_j & (|i-j| \neq 1), \end{cases}$$

and $Q = \{e_i : i \geq 2\}''$. The inclusion $Q \subset P$ of hyperfinite II_1 factors is called Jones' inclusion and satisfies that $[P : Q] = r$, Q is irreducible (i.e. $Q' \cap P = \mathbf{C}1$) if $r \leq 4$ ([10]) and $\dim(Q' \cap P) = 2$ if $r > 4$ ([15]). A basic *-endomorphism σ of P with $\sigma(P) = Q$ is defined by $\sigma(e_i) = e_{i+1}$.

Example 3 (Wenzl's inclusion). The method in Example 2 is generalized by Wenzl ([21]), considering certain unitary representations π of the Braid group on infinitely many generators $g_0, g_1, \cdots$ that admit positive Markov traces. The irreducible inclusion $N \subset M$ is given as $N = \{\pi(g_j); j \geq 2\}'' \subset M = \{\pi(g_j); j \geq 1\}''$. A basic *-endomorphism σ of M with $\sigma(M) = N$ is defined by $\sigma(\pi(g_i)) = \pi(g_{i+1})$ $(i \geq 1)$.

Example 4 (Ocneanu's inclusion). The above examples are generalized using bipartite graphs by Ocneanu ([13, 14]). Let $\mathcal{G}$ be a finite bipartite graph with a biunitary connection W with respect to the distinguished point * among the even vertices of $\mathcal{G}$. Considering the algebra $String^{(n)}\mathcal{G}$ of n-strings on $\mathcal{G}$, we give a trace τ and an embedding i_n^{n+k} of $String^{(n)}\mathcal{G}$ into $String^{(n+k)}\mathcal{G}$, which is compatible with τ. The trace on $String^{(n)}\mathcal{G}$ which is compatible with the embedding is unique. The hyperfinite II_1 factor $M(\mathcal{G})$ is obtained by the GNS construction of $(\bigcup_n String^{(n)}\mathcal{G})$ with respect to τ. Another embedding φ of $Strings^{(n)}(\mathcal{G})$ into $Strings^{(n+1)}(\mathcal{G})$ different with i_n^{n+1} is induced by the unitary matrix in the biunitary axiom of W. The φ is a *-isomorphism compatible with the inclusion i_n^{n+k} and preserves τ so that it is extended to $M(\mathcal{G})$. Ocneanu's subfactor $N(\mathcal{G})$ is nothing else but $\varphi(M(\mathcal{G}))$. The φ is a basic *-endomorphism of $M(\mathcal{G})$.

Example 5 (Pimsner-Popa's inclusion). Let σ be the basic *-endomorphism of M in Example 1 for $m = 2$ and η_i be the same as in Example 1. Let θ be an outer automorphism of M. Put

$$\rho(x) = \eta_1(e_{11})\sigma(x) + \eta_1(e_{22})\sigma(\theta(x)) \quad (x \in M),$$

where $\{e_{ij}\}_{i,j=1,2}$ is the canonical matrix unit of $M_2(\mathbf{C})$. Then $\rho \in \mathrm{End}(M, \tau)$. The subfactor $N = \rho(M)$ is Pimsner-Popa's subfactor ([15, 17]) associated with θ. Since $[\sigma] = [\theta^{-1} \cdot \sigma \cdot \theta]$, ρ is a basic *-endomorphism of M with $\rho(M) = N$.

Example 6 (Wasserman's inclusion). Let M and σ be the same as in Example 5. Let G be a subgroup of $SU(2)$. Then G acts on M by the infinite tensor product of its action by conjugation on $M_2(\mathbf{C})$. Let $A = (M_2(\mathbf{C}) \otimes M)^G \supset B = (1 \otimes M)^G$. Then N is a subfactor of M with index 4 and the principal graph is obtained as one of the extended Coxeter graph corresponding G's ([5, 20]). Since the action of G commutes with σ, σ becomes a basic *-endomorphism of M with $\sigma(M) = N$.

Proposition 1. *Let $N \subset M$ be an inclusion of hyperfinite II_1 factors with $[M : N] \leq 4$. Then there exists a basic *-endomorphism σ of M with $\sigma(M) = N$.*

Proof. The complete classification of subfactors with index ≤ 4 of the hyperfinite II_1 factor is established in [18]. The principal graph of $N \subset M$ is one of the Coxeter graphs A_n $(n \geq 3), D_{2n}$ $(n \geq 2), E_6, E_8$ and the extended Coxeter graphs $A_n^{(1)}$ $(2 \leq n \leq \infty), D_n^{(1)}$ $(n \geq 4), A_\infty, D_\infty, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ ([1, 5, 6, 7, 8, 10, 11, 13, 14, 16, 17, 19]).

The subfactor of type $A_n, 3 \leq n < \infty$, is unique ([10, 11, 13, 14, 16]) and it is a Jones' subfactor so that a basic *-endomorphism is given in Example 2. The subfactors of types $D_{2n}, E_6, E_8, D_n^{(1)}$ are constructed by the method of string algebras via flat connections due to Ocneanu ([7, 8, 11, 13, 14]) and they respectively have basic *-endomorphisms in Example 4. Each inclusion of type $A_{2n-1}^{(1)}$ $(1 \leq n < \infty)$ is given as Pimpner-Popa's inclusion $M \supset N$ associated with the $\theta = s_n^\gamma$ by Connes ([4]) for some γ with $\gamma^n = 1$ ([14, 17]). For $A_\infty^{(1)}$, the inclusion is unique and the unique (up to the outer conjugacy) aperiodic automorphism ([3]) corresponds to θ ([17]). Hence a basic *-endomorphism of M of type $A_{2n-1}^{(1)}$ $(1 \leq n \leq \infty)$ with N as its range is given as ρ in Example 5. The subfactor with one of $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$

as the principal graph is unique ([11]), so it is given by Wasserman's method ([5]). Hence it has a basic *-endomorphism in Example 6.

Thus each subfactor is given as the range of a basic *-endomorphism. □

3. Non inner conjugate subfactors

In this section, we distinguish subfactors of the hyperfinite II_1 factor which are given through the method of the tensor product by using unitary equivalence relations of basic *-endomorphisms.

Lemma 2. *Let $\rho, \sigma \in \mathrm{End}(M, \tau)$. Then $\rho(M)$ is inner conjugate to $\sigma(M)$ if and only if there exists a $\theta \in \mathrm{Aut}(M)$ with*

$$[\sigma \cdot \theta] = [\rho].$$

Proof. Assume $u\rho(M)u^* = \sigma(M)$ for some unitary $u \in M$. Then $\sigma^{-1}(u\rho(M)u^*) = M$. Putting $\theta = \sigma^{-1} \cdot \mathrm{Ad}(u) \cdot \rho$, we have $\theta \in \mathrm{Aut}(M)$ and $[\sigma \cdot \theta] = [\rho]$. □

Lemma 3. *Let R be the hyperfinite II_1 factor with the canonical trace τ and a basic σ have $[R : \sigma(R)] = r < \infty$. Put*

$$M = \bigotimes_{m \in \mathbf{Z}} (R_m, \tau_m), \quad M(i) = \bigotimes_{j<i} (R_j, \tau_j) \otimes (\sigma_i(R), \tau_i) \otimes \bigotimes_{i<k} (R_k, \tau_k),$$

where $R_m = R, \tau_m = \tau, \sigma_m = \sigma$ $(m \in \mathbf{Z})$. Assume that σ satisfies the condition $(*)$:

(*) $\quad \exists n \in \mathbf{N} : \sigma^n(R)' \cap R$ *is not isomorphic to* $(\sigma^{n-1}(R)' \cap R) \otimes (\sigma(R)' \cap R)$.

Then the subfactors $\{M(i) : i \in \mathbf{Z}\}$ of M are pairwise conjugate but non inner conjugate.

Proof. Let

$$\rho_i = \bigotimes_{j<i} \mathrm{id}_j \otimes \sigma_i \otimes \bigotimes_{i<k} \mathrm{id}_k \,.$$

Then ρ_i is a basic *-endomorphism of M with $\rho_i(M) = M(i)$ and $[M : \rho_i(M)] = [R : P]$. It is clear that $\rho_i(M)$ is conjugate to $\rho_k(M)$ by the automorphism of M which flips the i-th factor and the k-th factor of M. Assume that $\rho_i(M)$ is inner conjugate to $\rho_k(M)$ for $i \neq k$. Then there exists a $\theta \in \mathrm{Aut}(M)$ such that $[\rho_i \cdot \theta] = [\rho_k]$ by Lemma 2. Hence $\rho_k^n(M)' \cap M$ is isomorphic to $\rho_k^{n-1}(\rho_i(M))' \cap M$ by (i) and Lemma 2. On the other hand, $\rho_k^n(M)' \cap M$ is isomorphic to $\sigma^n(R)' \cap R$, but $\rho_k^{n-1}(\rho_i(M))' \cap M$ is isomorphic to $(\sigma^{n-1}(R)' \cap R) \otimes (\sigma(R)' \cap R)$. This contradicts the assumption. □

Proof of Theorem 1. By the assumption, either $[R : P] \leq 4$ or $P \subset R$ is Jones' inclusion. Hence by Proposition 1 and Example 2, we have a basic $\sigma \in \mathrm{End}(R, \tau)$ with $\sigma(R) = P$.

For such a pair $\{R, \sigma\}$, let M and $M(i)$ be the same as in Lemma 3. It is sufficient to prove that the family $\{M(i)\}_{i \in \mathbf{Z}}$ is pairwise non inner conjugate but $M(i) \subset M$ is isomorphic to $P \subset R$, that is, there is a *-isomorphism from M onto R which transforms $M(i)$ onto P.

First we show that σ satisfies the condition (*) in Lemma 3. Since σ is basic, the n-th basic extension for $P \subset R$ is given as $\sigma^{-n}(R)$. If the inclusion $P \subset R$ is irreducible, then (*) is satisfied by $n = 2$ because $\sigma^2(R)' \cap R$ contains a basic projection for σ. Hence (*) is satisfied if $P \subset R$ is one of the cases: $[R : P] < 4$, or the type of A_∞ (Jones' inclusion of index 4), $D_n^{(1)}, D_\infty, E_n^{(1)}$ $(n = 6, 7, 8)$. If the principal graph of $P \subset R$ is $A_n^{(1)}$ $(2 \leq n \leq \infty)$, then $\sigma(R)' \cap R$ is a 2-dimensional algebra, so that $(\sigma(R)' \cap R) \otimes (\sigma(R)' \cap R)$ is abelian but $\sigma^2(R)' \cap R$ is not abelian; thus (*) is satisfied by $n = 2$. The principal graph of Jones' inclusion with index > 4 is also $A_\infty^{(1)}$ ([5]). Thus (*) is satisfied by $n = 2$ for all inclusions. Hence $M(i)$ is not inner conjugate to $M(k)$ for $i \neq k$ by Lemma 3.

Next, we show that $M(i) \subset M$ is isomorphic to $P \subset R$. Remark that the inclusion $M(i) \subset M$ has the tower of the relative commutants in the basic extensions isomorphic to that of $P \subset R$. Popa shows that if $N \subset M$ is an extremal and strongly amenable inclusion of hyperfinite II_1 factors, then $N \subset M$ is opposite isomorphic to $M_1' \cap M_\infty \subset M' \cap M_\infty$ ([18, 5.1.2 Corollary]) and that $N \subset M$ is extremal and strongly amenable if $[M : N] \leq 4$. Hence $M(i) \subset M$ is isomorphic to $P \subset R$ if $[R : P] \leq 4$. If $P \subset R$ is Jones' inclusion with $[R : P] > 4$, then

$$M(i)' \cap M = \mathbf{C}p + \mathbf{C}(1 - p)$$

for a projection $p \in M$ with $\tau(p)\tau(1-p) = [M : M(i)]^{-1}$ and $\dim(M(i)' \cap M) = 2$. Hence, $M(i) \subset M$ is isomorphic to $P \subset R$ by the characterization of locally trivial subfactors ([18, 5.2.3 Corollary]). Therefore for each given inclusion, $M(i) \subset M$ is isomorphic to $P \subset R$. □

Remark. The proof of Theorem 1 also holds for Wenzl's inclusion and the other inclusion which has the paragroup as the conjugacy invariant.

The most typical subfactor of a finite factor M with finite index are given as the fixed point algebra M^α under the outer action α of a finite group G. Next we discuss such subfactors.

Proposition 4. *Let G be a finite group and α an outer action of G on M. If G is nonabelian, then there are no basic *-endomorphisms of M which have the subfactor M^α as the range.*

Proof. Assume there exists a basic *-endomorphism σ of M with $\sigma(M) = M^\alpha$. Let M_∞ be the enveloping factor obtained from the tower of Jones' basic extension algebra for the inclusion $M^\alpha \subset M$. Then we have a $\beta \in \mathrm{Aut}(M_\infty)$ with $\beta(M_i) = M_{i-1}$ for all i, which is the extension of σ. Hence the principal graph and the dual principal graph for $M^\alpha \subset M$ are isomorphic. This contradicts that G is nonabelian. □

By Proposition 4, the proof of Theorem 1 does not work for the inclusion $M^\alpha \subset M$ to the outer action of a nonabelian finite group G. In the case of fixed point algebras of finite groups, we use the outerness of actions to distinguish subfactors.

Proof of Theorem 2. Let I and J be subsets of $\mathbf{Z}$ with $I \neq J$. Let

$$M = \bigotimes_{m \in Z} (R_m, \tau_m),$$

$$\beta(g) = \bigotimes_{i \in I} \alpha_i(g) \otimes \bigotimes_{i \in \mathbf{Z} \setminus I} \mathrm{id}_i, \quad \gamma(g) = \bigotimes_{i \in J} \alpha_i(g) \otimes \bigotimes_{i \in \mathbf{Z} \setminus J} \mathrm{id}_i\,.$$

Here $R_m = R, \tau_m = \tau$ (the canonical trace) of R and $\alpha_m = \alpha$ ($m \in \mathbf{Z}$).

We denote the canonical trace on M by the same notation τ. Then β and γ are outer actions of G on M. Since the outer action of G on the hyperfinite II_1 factor is unique up to conjugacy ([9]), $M^\beta \subset M$ is conjugate to $R^\alpha \subset R$ for all $I \subset \mathbf{Z}$. Assume that $uM^\beta u^* = M^\gamma$ for some $u \in M$. Let v (resp. w) be the unitary representation of G on $L^2(M, \tau)$ such that $\beta(g) = \mathrm{Ad}\, v(g)$ (resp. $\gamma(g) = \mathrm{Ad}\, w(g)$). Then $(uv(G)u^*)' \cap M = u(v(G)' \cap M)u^* = w(G)' \cap M$. Put $L = M' \cap B(L^2(M, \tau))$. Then two crossed products coincide: $L \lhd_{u\beta} G = L \lhd_\gamma G$, where $(u\beta)(g) = \mathrm{Ad}(uv(g)u^*)$ and $\gamma(g) = \mathrm{Ad}\, w(g)$ on L. Since the actions $u\beta$ and γ of G on L are outer, there exists a conditional expectation E of $L \lhd_{u\beta} G$ onto L with $E(v(g)) = E(uw(g)u^*) = 0$ if $g \neq 1$. Let $uv(g)u^* = \sum_{h \in G} w(h)x(h)$ $(x(h) \in L)$ be the Fourier expansion of $uv(g)u^*$. Then $x(1) = 0$, where 1 is the identity of G. Since $I \neq J$, $(u\beta)(g^{-1})\gamma(h)$ is an outer automorphism of L if $g \neq 1 \neq h$. Hence we have $1 = \sum_{h \neq 1} uv(g^{-1})u^* w(h)x(h) = \sum_{h \neq 1} E(uv(g^{-1})u^* w(h)x(h)) = 0$. This is a contradiction. □

Addendum

A second referee writes that since $\mathrm{End}({}_{M(i)}L^2(M)_{M(j)})$ is not isomorphic to $\mathrm{End}({}_{M(i)}L^2(M)_{M(i)})$, without any assumptions on values of indices Theorem 1 is proved by Kosaki-Yamagami's bimodule method. Theorem 1 is shown so. However, in the case that $\sigma(R) \subset R$ is of type $A_1^{(1)}$, $M(i)$ is inner conjugate to $M(j)$.

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Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara 582, Japan

E-mail address: foo527@sinet.ad.jp