LIMITS OF DIFFERENTIABLE FUNCTIONS

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Abstract. Suppose that \( \{f_n\} \) is a sequence of differentiable functions defined on \([0,1]\) which converges uniformly to some differentiable function \( f \), and \( \{f'_n\} \) converges pointwise to some function \( g \). Let \( M = \{x : f'(x) \neq g(x)\} \). In this paper we characterize such sets \( M \) under various hypotheses. It follows from one of our characterizations that \( M \) can be the entire interval \([0,1]\).

1. Introduction

We say that a sequence of differentiable functions \( \{f_n\} \) defined on the interval \([0,1]\) is proper if \( \{f_n\} \) converges uniformly to some differentiable function \( f \) and \( \{f'_n\} \) converges pointwise to some function \( g \). For such proper \( \{f_n\} \), we let \( \Delta(\{f_n\}) = \{x : f'(x) \neq g(x)\} \). It is a standard theorem in elementary analysis texts [6] that if \( \{f_n\} \) is proper and \( \{f'_n\} \) converges uniformly to some function \( g \), then \( \Delta(\{f_n\}) = \emptyset \). It is rather easy to construct an example of a proper \( \{f_n\} \) where \( \Delta(\{f_n\}) \neq \emptyset \). In this paper we investigate the following questions:

Question 1. Is there a proper \( \{f_n\} \) such that \( \Delta(\{f_n\}) = [0,1] \)?

Question 2. Can \( \{\Delta(\{f_n\}) : \{f_n\} \text{ is proper}\} \) be characterized?

Theorems 1, 2, 3 and 5 answer Question 2 under various hypotheses. It will follow from Theorem 2 that Question 1 has an affirmative answer. However, Theorem 4 implies that in order to make \( \Delta(\{f_n\}) = [0,1] \), the derivatives have to be complicated in some sense.

We now state some definitions and background theorems. Recall that the density topology \( D \) on \( \mathbb{R} \) is

\[ \{M \subset \mathbb{R} : M \text{ is Lebesgue measurable and has density 1 at each of its points}\} \]

Sets in \( D \) are said to be open in density topology. Whenever we say that a set is open, closed, \( G_\delta \), \( F_\sigma \), etc., we mean that it is open, closed, \( G_\delta \), \( F_\sigma \), etc., in the ordinary topology on \( \mathbb{R} \). Whenever we want a set to be open or closed in the density topology, we will specifically so state. A function \( f : [0,1] \to \mathbb{R} \) is approximately dense.

Received by the editors April 20, 1994 and, in revised form, July 6, 1994; this paper was presented at the Special Session on Real Analysis at the 1992 AMS meeting in Baltimore.

1991 Mathematics Subject Classification. Primary 26A24, 26A21; Secondary 40A30.

Key words and phrases. \( F_\sigma \), \( G_\delta \), density topology, approximate continuity, nowhere measure dense.

This is the core part of the author’s dissertation which was directed by Professor Jack B. Brown.

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Thus the preimage of every open set under \( f \) is open in the density topology. A set \( M \subset [0,1] \) is nowhere measure dense in the interval \( J \) iff the interior of \( M \cap J \) in the density topology is nowhere dense in \( J \) in the ordinary topology. We will freely use the following facts about density topology, approximate continuity and Lebesgue integration theory throughout the paper. Their proofs may be found in [1], [3], [2].

**Fact 1.** If \( M \subset \mathbb{R} \) is measurable, then there exists an \( F_\sigma \) set \( N \subset M \) such that \( N \) is open in the density topology and \( \lambda(M \setminus N) \), the Lebesgue measure of \( M \setminus N \), is zero.

**Fact 2.** Every bounded approximately continuous function is a derivative.

**Fact 3** (Zahorski’s Theorem [7]). If \( G_0 \) and \( G_1 \) are two disjoint \( G_\delta \) sets which are closed in the density topology, then there exists an approximately continuous function \( f : [0,1] \to [0,1] \) such that \( f^{-1}(i) = G_i \) for \( i = 0, 1 \).

**Fact 4.** If \( g \) is an integrable derivative defined on \([0,1]\), then \( f(x) = \int_0^x g \) is differentiable everywhere and \( f'(x) = g(x) \) for all \( x \in [0,1] \).

**Fact 5.** Suppose \( \{g_n\} \) is a sequence of integrable functions defined on the interval \([0,1]\) such that \( \{g_n\} \) is dominated by an \( L^1 \) function and \( \{g_n\} \) converges pointwise to \( g \). Let \( f_n(x) = \int_0^x g_n \) and \( f(x) = \int_0^x g \) for each \( x \in [0,1] \). Then \( \{f_n\} \) converges uniformly to \( f \).

2. Main results

**Lemma 1.** If \( \{f_n\} \) is proper and \( \{f'_n\} \) is dominated by an \( L^1 \) function, then \( \Delta(\{f_n\}) \) has measure zero.

**Proof.** Without loss of generality assume that \( f_n(0) = 0 \) for all \( n \). Let \( g \) be the pointwise limit of \( \{f'_n\} \) and \( h(x) = \int_0^x g \). Note that \( h'(x) = g(x) \) for almost all \( x \in [0,1] \). By the Lebesgue Dominated Convergence Theorem we have that

\[
 f(x) = \lim_{n \to \infty} f_n(x) = h(x) \quad \text{for all} \quad x \in [0,1].
\]

Thus \( f' = g \) a.e. and \( \Delta(\{f_n\}) \) has measure zero. \( \square \)

**Lemma 2.** Suppose \( \{f_n\} \) is proper. Then \( \Delta(\{f_n\}) \) is \( G_{\delta\sigma} \).

**Proof.** Let \( g \) be the pointwise limit of \( \{f'_n\} \). Since derivatives are of Baire class 1, \( g \) is in Baire class 2 and hence \( \Delta(\{f_n\}) = (f' - g)^{-1}(R \setminus \{0\}) \) is \( G_{\delta\sigma} \) [4]. \( \square \)

**Theorem 1** (General Dominated Case). A set \( M \subset [0,1] \) is \( G_{\delta\sigma} \) and of measure zero iff \( M = \Delta(\{f_n\}) \) for some proper \( \{f_n\} \) where \( \{f'_n\} \) is dominated by an \( L^1 \) function.

**Proof.** (\( \Rightarrow \)) This direction follows from Lemmas 1 and 2.

(\( \Leftarrow \)) The proof of this direction has a flavor similar to a result of Preiss [5]. Let \( M = \bigcup \{G_k \} \) be a \( G_{\delta\sigma} \) set of measure zero where each \( G_k \) is \( G_\delta \). Now for each positive integer \( k \), let \( \{U_{k,n}\}_{n=1}^\infty \) be a decreasing sequence of open sets such that \( G_k = \bigcap_{n=1}^\infty U_{k,n} \). For each \( n \) and \( k \) we may obtain by Fact 3 an approximately continuous function \( h_{k,n} : [0,1] \to [0,1] \) such that \( h_{k,n}(1) = G_k \) and \( h_{k,n}^{-1}(0) = (U_{k,n})^c \), the complement of \( U_{k,n} \). Note that for each \( k \), \( \{h_{k,n}\}_{n=1}^\infty \) converges pointwise to \( \chi_{G_k} \),
the characteristic function of $G_k$. Now set

$$g_n = \sum_{k=1}^{\infty} \frac{1}{2^{k}} h_{k,n} \quad \text{and} \quad g = \sum_{k=1}^{\infty} \frac{1}{2^{k}} \chi_{G_k}.$$ 

It follows that $g_n$ is a bounded derivative because it is the uniform sum of a sequence of bounded approximately continuous functions. Also note that $\{g_n\}$ converges pointwise to $g$.

Now let $f_n(x) = \int_{0}^{x} g_n$. That $f'_{n}(x) = g_n(x)$ for all $x \in [0,1]$ follows from Fact 4. We also have that $\int_{0}^{x} g = 0$ for all $x \in [0,1]$ as $g$ is nonzero only on the measure zero set $M$. Since $\{g_n\}$ is a uniformly bounded sequence of integrable functions which converges pointwise to $g$, by Fact 5 it follows that $\{f_n\}$ converges uniformly to the zero function. But $\{f'_{n}\}$ converges pointwise to the function $g$ which is nonzero precisely on set $M$. Therefore, $\Delta(\{f_n\}) = M$. 

**Lemma 3.** Let $M \subset [0,1]$ be a nonempty $F_\sigma$ set which is open in the density topology, and let $A > 0$. Then, there exists a bounded approximately continuous function $f : [0,1] \rightarrow \mathbb{R}$ such that

1. $\int_{0}^{1} f = A$,
2. $f^{-1}(0) = M^c$, and
3. $f \geq 0$.

**Proof.** By Fact 3 there is a bounded approximately continuous function $h$ which satisfies conditions (2) and (3). Let $f = c \cdot h$, where $c = \frac{A}{\int_{0}^{1} h}$. This $f$ satisfies all three required conditions. 

**Lemma 4.** If $M$ is $F_\sigma$, open in the density topology and $M^c$ is dense in $[0,1]$, then there exists proper $\{f_n\}$ such that $\Delta(\{f_n\})$.

**Proof.** Suppose $M$ is $F_\sigma$, open in the density topology, and $M^c$ is dense in $[0,1]$. Let $g$ be an approximately continuous function such that $0 \leq g(x) \leq 1$ for all $x$ and $g^{-1}(0) = M^c$. Write $M = \bigcup_{i=1}^{\infty} F_i$ where each $F_i$ is a nowhere dense closed subset of $[0,1]$. Since each $F_i$ is a nowhere dense closed subset of $[0,1]$ and $M$ is open in the density topology, it follows that for each interval $J$ for which $M \cap J \neq \emptyset$ we have $(M \cap J) \setminus (\bigcup_{i=1}^{n} F_i) \neq \emptyset$ for all $n$.

We now construct a sequence of derivatives $\{g_n\}$ in the following manner. Fix $n$. Let $N_n = M^c \cup F_1 \cup \ldots \cup F_n$. Observe that $N_n$ is a $G_\delta$ set and is also closed in the density topology. Let $\{x(n,k)\}_{k=1}^{m(n)+1}$ be a partition of $[0,1]$ such that $x(n,i) \in M^c$ and $|x(n,i) - x(n,i+1)| < \frac{1}{2^n}$ for each $1 \leq i \leq m(n)$. Fix $1 \leq i \leq m(n)$. Observe that $[x(n,i), x(n,i+1)) \setminus N_n$ is either a nonempty $F_\sigma$ set which is open in the density topology or $[x(n,i), x(n,i+1)] \subset M^c$. If the latter is the case, let $h_{n,i}$ be the zero function on $[x(n,i), x(n,i+1)]$ and we have that $\int_{x(n,i)}^{x(n,i+1)} h_{n,i} = f_{x(n,i)}^{x(n,i+1)}g$. Otherwise using Lemma 3, obtain a bounded, nonnegative approximately continuous function $h_{n,i}$ defined on $[x(n,i), x(n,i+1)]$ such that $h_{n,i}^{-1}(0) = N_n \cap [x(i), x(i+1)]$ and $\int_{x(n,i)}^{x(n,i+1)} h_{n,i} = f_{x(n,i)}^{x(n,i+1)}g$. Now, let $h_n$ be the union of $h_{n,1}, \ldots, h_{n,m(n)}$. Then, $h_n$ is bounded and approximately continuous. Let $g_n = g - h_n$. As $\{h_n\}$ converges pointwise to the zero function, $\{g_n\}$ converges pointwise to $g$.

Let $f_n(x) = \int_{0}^{x} g_n$. Since $g_n$ is bounded and approximately continuous, $f'_n = g_n$. Let us next show that $\|f'_n\|$, the sup norm of $f'_n$, is less than $2^{-n+1}$. Let $x \in [0,1]$. 

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and \( i \) be such that \( x \in [x(n,i), x(n,i + 1)] \). Then

\[
|f_n(x)| = \left| \int_0^x g_n \right| \leq \left| \int_0^{x(n,i)} g - h_n \right| + \left| \int_{x(n,i)}^x g - h_n \right| \\
\leq 0 + \left| \int_0^{x(n,i)} g \right| + \left| \int_{x(n,i)}^x h_n \right| < 2^{-n} + 2^{-n} = 2^{-n+1}.
\]

The above estimate on \( \|f_n\| \) implies that \( \{f_n\} \) converges uniformly to the zero function. We also know that \( \{g_n\} \) converges pointwise to \( g \) and \( g^{-1}(0) = M^c \). Therefore, \( M = \Delta(\{f_n\}) \).

**Theorem 2** (General Nondominated Case). A set \( M \subset [0,1] \) is \( G_{\delta,\sigma} \) iff there exists proper \( \{f_n\} \) such that \( M = \Delta(\{f_n\}) \).

**Proof.** \((\Rightarrow)\) This direction follows from Lemma 2.

\((\Rightarrow)\) Let \( M \) be \( G_{\delta,\sigma} \). From Fact 1 obtain two disjoint sets \( M_1 \) and \( M_2 \) such that \( M_1 \cup M_2 = M \), \( M_1 \) is \( G_{\delta,\sigma} \) set of measure zero and \( M_2 \) is an \( F_\sigma \) set which is open in the density topology and \( M_2 \) is dense in \([0,1]\). By Theorem 1 and Lemma 4, obtain proper sequences \( \{f_n\} \) and \( \{h_n\} \) such that \( M_1 = \Delta(\{f_n\}) \) and \( M_2 = \Delta(\{h_n\}) \). Then \( \{f_n + h_n\} \) is proper and \( M = \Delta(\{f_n + h_n\}) \).

**Lemma 5.** Suppose \( \{f_n\} \) is proper and for all \( n \), \( f_n \in C^1 \), i.e. \( f_n' \) is continuous. Then \( \Delta(\{f_n\}) \) is \( F_\sigma \).

**Proof.** Let \( g \) be the pointwise limit of \( \{f_n'\} \). Then \( g \) and \( f' \) are of Baire class 1. Therefore, \( \Delta(\{f_n\}) = (f' - g)^{-1}(\mathbb{R} \setminus \{0\}) \) is \( F_\sigma \).

**Theorem 3** (Dominated \( C^1 \) Case). A set \( M \subset [0,1] \) is \( F_\sigma \) and of measure zero iff \( M = \Delta(\{f_n\}) \) for some proper \( \{f_n\} \) where \( f_n \in C^1 \) for all \( n \) and \( \{f_n'\} \) is dominated by an \( L^1 \) function.

**Proof.** \((\Rightarrow)\) This direction follows from Lemmas 1 and 5.

\((\Rightarrow)\) Suppose \( M \) is \( F_\sigma \) and of measure zero. Let \( M = \bigcup_{k=1}^\infty M_k \) where each \( M_k \) is closed. Let \( \{G_k,n\} \) be such that each \( G_k,n \) is a finite collection of closed intervals and

\[
\begin{align*}
(1) & \quad \bigcup_{k=1}^\infty G_{k,n} \subset \bigcup_{k=1}^\infty G_{k,n+1}, \\
(2) & \quad \bigcup_{n=1}^\infty G_{k,n} = M_k.
\end{align*}
\]

Now let \( h_{k,n} \) be a continuous function defined on \([0,1]\) such that \( 0 \leq h_{k,n}(x) \leq 1 \), \( h_{k,n}(M_k) = 1 \), and \( h_{k,n}(\bigcup G_{k,n}) = 0 \). Note that \( \{h_{k,n}\}_{n=1}^\infty \) converges pointwise to \( \chi_{M_k} \), the characteristic function of \( M_k \). Now set \( g_n = \sum_{k=1}^\infty 2^{-k} \cdot h_{k,n} \) and \( g = \sum_{k=1}^\infty 2^{-k} \cdot \chi_{M_k} \). Observe that \( g_n \) is a continuous function for all \( n \), \( 0 \leq g_n(x) \leq 1 \) and \( \{g_n\} \) converges pointwise to \( g \).

Setting \( f_n(x) = \int_0^x g_n \), we have that \( f_n'(x) = g_n(x) \) for all \( x \in [0,1] \). We also have that \( f_n \to g = 0 \) for all \( x \in [0,1] \) as \( g \) is nonzero only on the measure zero set \( M \). Since \( \{g_n\} \) is a uniformly bounded sequence of continuous functions which converges pointwise to \( g \), by Fact 5 it follows that \( \{f_n\} \) converges uniformly to the zero function. But \( \{f_n'\} \) converges pointwise to function \( g \) which is nonzero precisely on set \( M \). Therefore, \( \Delta(\{f_n\}) = M \).
Lemma 6. Suppose $M \subseteq [0,1]$ is $F_\sigma$ and nowhere measure dense. Then $M$ is the union of two disjoint $F_\sigma$ sets, one of which is of measure zero and the other nowhere dense.

Proof. Let $B_1, B_2, \ldots$ be a countable basis for $[0,1]$. Let $U = \bigcup \{B_i : \lambda(B_i \cap M) = 0\}$. Since $M$ is nowhere measure dense, $U$ is a dense open subset of $[0,1]$. Let $M_1 = M \cap U$ and $M_2 = M \setminus U$. Then $M_1$ and $M_2$ are the desired sets.

Lemma 7. Let $\{g_n\}$ be a sequence of a.e. continuous functions whose domain is interval $I$. Suppose $\{g_n\}$ converges pointwise to some function $g$ on $I$. Then there exists an interval $J \subseteq I$ and $K > 0$, such that for all $n, |g_n(x)| < K$ for a.e. $x \in J$.

Proof. We will prove this lemma by contradiction. Assume the hypothesis and that there is no $J \subseteq I$ such that $\{g_n\}$ is bounded a.e. on $J$. This implies that for every $K > 0$ and interval $J \subseteq I$, there are infinitely many integers $n$ such that $|g_n| > K$ on a positive measure set contained in $J$. Using this observation and the fact that $\{g_n\}$ is a sequence of a.e. continuous functions, we may obtain a subinterval $I_1$ of $I$ and a positive integer $n(1)$ such that $|g_{n(1)}| > 1$ on $I_1$. Proceeding in a similar fashion we may obtain a decreasing sequence of closed intervals $\{I_k\}$ and an increasing sequence of positive integers $\{n(k)\}$ such that for each $k$, $|g_{n(k)}| > k$ on $I_k$. Now let $p \in \bigcap_{k=1}^{\infty} I_k$. Then $\{g_{n(k)}(p)\}_{k=1}^{\infty}$ does not converge, contradicting the hypothesis and concluding the proof of the lemma.

Theorem 4. Suppose $\{f_n\}$ is proper, $f_n'$ is integrable and the set of discontinuities of $f_n'$ has measure zero for all $n$. Then $\Delta(\{f_n\})$ is nowhere measure dense.

Proof. Let $I$ be a subinterval of $[0,1]$. Using Lemma 7 obtain an interval $J \subseteq I$ such that $\{f_n'\}$ is bounded a.e. on $J$. By Lemma 1 we have that $\Delta(\{f_n\}) \cap I$ has measure zero. This implies that the interior of $\Delta(\{f_n\})$ in the density topology is nowhere dense in $[0,1]$.

Lemma 8. Suppose $M$ is an $F_\sigma$ nowhere dense subset of $[0,1]$. Then $M = \Delta(\{f_n\})$ for some proper $\{f_n\}$ where $f_n \in C^1$ for all $n$.

Proof. Let $U = (c[cl(M)])^c$ and $F_1, F_2, \ldots$ be a pairwise disjoint decomposition of $M$ into closed sets. Let $g$ be the function which is zero on $M^c$ and $2^{-1}$ on $F_i$. Note that $g$ is of Baire class 1. Let $\{G_n\}$ be such that each $G_n$ is a finite collection of closed intervals and

- $\bigcup G_n \subseteq \bigcup_{n=1}^{\infty} G_{n+1}$, and
- $\bigcap_{n=1}^{\infty} \bigcup G_n = U$.

Using the fact that $g$ is of Baire class 1, obtain a sequence of continuous functions $\{h_n\}$ such that $h_n$ converges pointwise to $g$, and for all $n$, $h_n(\bigcup G_n) = 0$ and $0 \leq h_n(x) \leq 1$.

We now construct $g_n$ in the following manner. First, let $\{x(n,k)\}_{k=1}^{m(n)+1}$ be a partition of $[0,1]$ such that $|x(n,k) - x(n,k+1)| < 2^{-n}$ for $k = 1, 2, \ldots, m(n)$. Now for each $1 \leq k \leq m(n)$, let $a_{n,k}$ be a continuous nonnegative function defined on $[x(n,k), x(n,k+1)]$ such that

- $\int_{x(n,k)}^{x(n,k+1)} a_{n,k} = \int_{x(n,k)}^{x(n,k+1)} h_n$,
- $a_{n,k} = 0$ on $\bigcup G_n \cap [x(n,k), x(n,k+1)]$,
- $a_{n,k}^{-1}(\mathbb{R} \setminus \{0\}) \subseteq U$, and
- $a_{n,k}(x(n,k)) = a_{n,k}(x(n,k+1)) = 0$. 

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Let \( a_n \) be the union of \( a_{n,1}, a_{n,2}, \ldots, a_{n,m(n)} \) and \( g_n = h_n - a_n \). As \( \{a_n\} \) is a sequence of continuous functions which converges pointwise to the zero function, \( \{g_n\} \) is a sequence of continuous functions which converges pointwise to \( g \).

For each \( n \), let \( f_n(x) = \int_0^x g_n \). Observe that for \( 1 \leq k \leq m(n) + 1 \)

\[
f_n(x(n,k)) = \int_0^{x(n,k)} g_n = \sum_{i=1}^{k-1} \int_{x(n,i)}^{x(n,i+1)} h_n - a_n = 0.
\]

Using this observation we obtain an estimate on \( \|f_n\| \). Let \( x \in [0,1] \). Let \( k \) be a positive integer such that \( x \in [x(n,k), x(n,k+1)] \).

\[
|f_n(x)| \leq |f_n(x(n,k))| + |f_n(x) - f_n(x(n,k))|
\]

\[
= 0 + \left| \int_{x(n,k)}^{x} g_n \right| = \left| \int_{x(n,k)}^{x} h_n - a_n \right|
\]

\[
\leq \left| \int_{x(n,k)}^{x} h_n \right| + \left| \int_{x(n,k)}^{x} a_n \right|
\]

\[
< 2^{-n} + 2^{-n} = 2^{-n+1}.
\]

From above we have that \( \{f_n\} \) converges uniformly to the zero function. As \( \{f'_n\} \) converges pointwise to \( g \) and \( g^{-1}(0) = M^c \), we have that \( M = \Delta(\{f_n\}) \).

**Theorem 5** (Nondominated \( C^1 \) Case). A set \( M \subset [0,1] \) is \( F_\sigma \) and nowhere measure dense iff \( M = \Delta(\{f_n\}) \) for some proper \( \{f_n\} \) where \( f_n \in C^1 \) for all \( n \).

**Proof.** (\( \Rightarrow \)) This direction follows from Theorem 4 and Lemma 5.

(\( \Leftarrow \)) Suppose \( M \) is \( F_\sigma \) and nowhere measure dense. Then by Lemma 6, \( M = M_1 \cup M_2 \) where \( M_1 \) and \( M_2 \) are disjoint \( F_\sigma \) sets, one of which is nowhere dense and the other of measure zero. By Theorem 3 and Lemma 8, there are proper sequences \( \{f_n\} \) and \( \{h_n\} \) such that \( \Delta(\{f_n\}) = M_1 \) and \( \Delta(\{h_n\}) = M_2 \). Then \( \{f_n + h_n\} \) is proper and \( M = \Delta(\{f_n + h_n\}) \).

The author thanks the referee for making helpful suggestions which improved the exposition of the paper.

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