PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 1, January 1996

A NOTE ON THE KERNEL OF A LOCALLY NILPOTENT DERIVATION

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. This note concerns locally nilpotent derivations D of the polynomial ring $\mathbf{C}[X_1, \ldots, X_n]$. It is shown that if D annihilates a polynomial in two variables, then D annihilates a variable.

Let k be a field of characteristic 0, and let A be a commutative k-algebra. A k-derivation D on A is said to be *locally nilpotent* if, for every $f \in A$, there is an integer $s \geq 0$ such that $D^s f = 0$. This paper is motivated by the following open question.

Question. For $n \geq 2$, does every locally nilpotent derivation on the polynomial ring $k[X_1, \ldots, X_n]$ annihilate a variable?

(Recall that $f \in k[X_1, \ldots, X_n]$ is a *variable* if there exist

$$f_2, \ldots, f_n \in k[X_1, \ldots, X_n]$$

such that $k[f, f_2, \ldots, f_n] = k[X_1, \ldots, X_n]$.) Geometrically, this question is equivalent to asking whether, with respect to some choice of coordinates, a given algebraic action of the additive group of k on $\mathbf{A}^n$ (affine n-space over k) fixes a hyperplane (cf. [3]). When $n = 2$, an affirmative answer to the question is given by the following result, due to Rentschler [1].

Theorem 1. *If D is a locally nilpotent derivation on $k[X, Y]$, then there exists a tame k-algebra automorphism γ of $k[X, Y]$ and a polynomial $f \in k[X]$ such that $\gamma D \gamma^{-1} = f(X) \cdot \frac{\partial}{\partial Y}$.*

Using Rentschler's Theorem, it is shown below that an affirmative answer can also be given in the following case.

Theorem 2. *Let D be a locally nilpotent derivation on $R_n(\mathbf{C}) = \mathbf{C}[X_1, \ldots, X_n]$, and suppose that the set*

$$\mathcal{T} = (\mathbf{C}[X_1, X_2] \cap \ker(D)) - \mathbf{C}$$

is non-empty (where $\ker(D)$ denotes the kernel of D). Then there exists a variable $\rho \in R_n(\mathbf{C})$ such that $\rho \in \mathbf{C}[X_1, X_2]$ and $D\rho = 0$.

In other words, if D annihilates a polynomial in two variables over $\mathbf{C}$, then D annihilates a variable. Before giving the proof, some preliminary results are needed.

Received by the editors June 6, 1994 and, in revised form, July 15, 1994.
1991 *Mathematics Subject Classification.* Primary 13B25; Secondary 14L30.

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Let $R_n(k)$ denote $k[X_1, \ldots, X_n]$. Given any locally nilpotent derivation D on $R_n(k)$, and given $f \in R_n(k), f \neq 0$, define

$$\nu_D(f) = \max\{s \mid D^s f \neq 0\}.$$

Also, set $\nu_D(0) = -\infty$. (Note that $\nu_D(f) = \deg_t \exp(tD)(f).$) The following properties are immediate (the first as a corollary to the Leibniz rule):

(i) $\nu_D(fg) = \nu_D(f) + \nu_D(g)$.
(ii) $\nu_D(f+g) \leq \max\{\nu_D(f), \nu_D(g)\}$.
(iii) $\nu_D(Df) = \nu_D(f) - 1$ if $Df \neq 0$.

Next, a multiplicatively closed subset $S \subset R_n(k)$ is said to be *saturated* if the following property holds:

$$fg \in S \Leftrightarrow f \in S \text{ and } g \in S.$$

It is well known that, for any locally nilpotent derivation D on $R_n(k)$, the set $(\ker(D) - 0)$ is saturated, since

$$\begin{aligned} fg \in (\ker D - 0) &\Leftrightarrow D(fg) = 0 \qquad (f, g \neq 0) \\ &\Leftrightarrow 0 = \nu_D(fg) = \nu_D(f) + \nu_D(g) \\ &\Leftrightarrow \nu_D(f) = \nu_D(g) = 0 \\ &\Leftrightarrow f, g \in (\ker D - 0). \end{aligned}$$

A noteworthy consequence of saturation is the following.

Proposition. *Let D be a locally nilpotent derivation of $R_n(k)$, let $S \subset R_n(k)$ be a multiplicative subset, and let $\widetilde{D}$ be the extension of D to the localization $S^{-1}R_n(k)$ (via the quotient rule). The following are equivalent:*

(i) *$\widetilde{D}$ is locally nilpotent.*
(ii) *$S \subset (\ker(D) - 0)$.*

Proof. If $\widetilde{D}$ is locally nilpotent and $f \in S$ is given, then $(\ker \widetilde{D} - 0)$ is saturated, and

$$0 = \widetilde{D}(1) = \widetilde{D}\left(\frac{1}{f} \cdot f\right) \Rightarrow f \in \ker \widetilde{D} \Rightarrow 0 = \widetilde{D}f = Df.$$

Conversely, if $S \subset (\ker(D) - 0)$, let $h = (f/g)$ for $f \in R_n(k)$ and $g \in S$; then $\widetilde{D}^s h = \widetilde{D}^s(f/g) = g^{-1}\widetilde{D}^s f = g^{-1} D^s f = 0$ for $s \gg 0$. $\square$

Lemma. *Given a polynomial $q \in \mathbf{C}[X, Y]$, suppose $f \in \mathbf{C}[X, Y]$ is an irreducible non-constant divisor of both $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$. Then there exists $c \in \mathbf{C}$ such that f divides $(q + c)$.*

Proof. Let $Z \subset \mathbf{C}^2$ be the curve defined by f; by hypothesis, $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$ evaluated along Z are zero. Given a non-singular point $P \in Z$, let $\alpha(t) = (x(t), y(t))$ be a local parametrization of Z at P. Define $Q\colon \mathbf{C} \to \mathbf{C}$ to be the evaluation of q along α, i.e., $Q = q \circ \alpha$. Then

$$\frac{dQ}{dt} = \frac{\partial q}{\partial X}(\alpha(t)) \cdot \frac{dx}{dt} + \frac{\partial q}{\partial Y}(\alpha(t)) \cdot \frac{dy}{dt} = 0.$$

Therefore $Q \equiv c$ for some $c \in \mathbf{C}$, which implies that $q \equiv c$ along the entire connected component of Z containing P. But f being irreducible implies Z is connected (in the complex topology); see, for example, [2, Chapter 7, §2] . Hence, $(q - c)$ vanishes along Z, and f divides $(q - c)$. $\square$

Proof of Theorem 2. To simplify notation, let $X = X_1$ and $Y = X_2$. If $q \in \mathcal{T}$ is of minimal degree, then q has the following property:

$$(q + c) \text{ is irreducible for all } c \in \mathbf{C}. \tag{1}$$

To see this, note first that $(q + c) \in \ker(D)$ for all $c \in \mathbf{C}$. If $(q + c)$ is reducible for some c, let q_0 be one of its irreducible factors. Since $\ker(D)$ is saturated, $q_0 \in \ker(D)$; hence $q_0 \in \mathcal{T}$ as well. But q_0 would then be of smaller degree than q, which is impossible. So $(q + c)$ must be irreducible, as claimed.

Next, since $q \in \mathbf{C}[X, Y]$, we may write

$$0 = Dq = \frac{\partial q}{\partial X} DX + \frac{\partial q}{\partial Y} DY. \tag{2}$$

By the preceding lemma, condition (1) above implies that $\frac{\partial q}{\partial X}$ and $\frac{\partial q}{\partial Y}$ have no common factor. It follows from (2) that $\frac{\partial q}{\partial Y}$ divides DX and $\frac{\partial q}{\partial X}$ divides DY. So there exist $l, m \in R_n(\mathbf{C})$ such that $DX = l \cdot \frac{\partial q}{\partial Y}$ and $DY = m \cdot \frac{\partial q}{\partial X}$. Substitution in (2) then shows that $m = -l$. We thus have

$$DX = l \cdot \frac{\partial q}{\partial Y} \quad \text{and} \quad DY = -l \cdot \frac{\partial q}{\partial X}.$$

If $l = 0$, the theorem is proved, so assume $l \neq 0$.

Define a derivation Δ on $\mathbf{C}[X, Y]$, by setting $\Delta X = \frac{\partial q}{\partial Y}$ and $\Delta Y = -\frac{\partial q}{\partial X}$. Observe the following:

$$Df = l \cdot \Delta f \quad \text{for all } f \in \mathbf{C}[X, Y]. \tag{3}$$

Claim. Δ *is locally nilpotent.*

Proof of Claim. Suppose, to the contrary, that there is an element $p \in \mathbf{C}[X, Y]$ such that, for all $n \geq 0$, $\Delta^n p \neq 0$. Then $\nu_D(\Delta^n p) \geq 0$ for all $n \geq 0$. Set $L = \nu_D(l)$; since $l \neq 0$, $L \geq 0$. Using (3) above, we see that, for all $n \geq 1$,

$$D(\Delta^{(n-1)} p) = l \cdot \Delta(\Delta^{(n-1)} p) = l \cdot \Delta^n p.$$

Now apply ν_D to each side of this equation:

$$\nu_D(\Delta^{(n-1)} p) - 1 = L + \nu_D(\Delta^n p) \Rightarrow \nu_D(\Delta^n p) = \nu_D(\Delta^{(n-1)} p) - (L + 1).$$

Recursive application of this rule yields $\nu_D(\Delta^n p) = \nu_D(p) - n(L + 1)$. But this implies $-\infty < \nu_D(\Delta^n p) < 0$ for n sufficiently large, a contradiction. Therefore Δ is locally nilpotent, and the Claim is proved.

We can now apply Rentschler's Theorem to Δ: there exist X' and Y' in $\mathbf{C}[X, Y]$ and $f \in \mathbf{C}[X']$ such that $\mathbf{C}[X', Y'] = \mathbf{C}[X, Y]$, and $\Delta = f(X') \frac{\partial}{\partial Y'}$. Consequently, $DX' = l \cdot \Delta X' = 0$. Since X' is clearly a variable in $R_n(\mathbf{C})$, we may take $\rho = X'$. ☐

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