

A BANACH SUBSPACE OF $L_{1/2}$ WHICH DOES NOT EMBED IN L_1 (ISOMETRIC VERSION)

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(Communicated by Dale Alspach)

ABSTRACT. For every $n \geq 3$, we construct an n -dimensional Banach space which is isometric to a subspace of $L_{1/2}$ but is not isometric to a subspace of L_1 . The isomorphic version of this problem (posed by S. Kwapien in 1969) is still open. Another example gives a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$. Note that, from the isomorphic point of view, all the spaces L_q with $q < 1$ have the same Banach subspaces.

1. INTRODUCTION

A well-known fact is that the space L_1 is isometric to a subspace of L_q for every $q < 1$. It is natural to ask whether the spaces L_q with $q < 1$ contain any Banach space structure not generated by L_1 . This question was first formulated in 1969 by Kwapien [6] in the following form: Need every Banach subspace of L_0 be also a subspace of L_1 ? Later the question was mentioned by Maurey [8, Question 124].

In 1970, Nikishin [9] proved that every Banach subspace of L_0 is isomorphic to a subspace of L_q for every $q < 1$. Therefore, if we replace the space L_0 in Kwapien's question by any of the spaces L_q with $q < 1$ we get an equivalent question.

Since all the spaces L_q with $q < 1$ embed in L_0 , Nikishin's result also shows that these spaces are all the same from the isomorphic Banach space point of view. Namely, every Banach space which is isomorphic to a subspace of L_q with $q < 1$ is also isomorphic to a subspace of L_p for every other $p < 1$.

In this paper we show that the answer to the isometric version of Kwapien's question is negative. For every $n \in \mathbb{N}$, $n \geq 3$, there exists an n -dimensional Banach space which is isometric to a subspace of $L_{1/2}$ but is not isometric to a subspace of L_1 . Using this example it is easy to see that the spaces L_q with $q < 1$ may be different from the isometric Banach space point of view. We give, however, a direct example illustrating the difference by constructing a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$.

The isomorphic version of Kwapien's question is still open. The most recent related result seems to be a theorem of Kalton [2], who proved that a Banach space X embeds in L_1 if and only if $\ell_1(X)$ embeds in L_0 .

The isometric version of Kwapien's question can be reformulated in the language of positive definite functions. In fact, a Banach space $(X, \|\cdot\|)$ is isometric to a subspace of L_p with $0 < p \leq 2$ if and only if the function $\exp(-\|x\|^p)$ is positive

Received by the editors April 28, 1994 and, in revised form, July 13, 1994.

1991 *Mathematics Subject Classification*. Primary 46B04; Secondary 46E30, 60E10.

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definite [1]. The main example of this paper gives a norm such that the function $\exp(-\|x\|^{1/2})$ is positive definite but the function $\exp(-\|x\|)$ is not positive definite. This result is close to problems of Schoenberg’s type (see [4]).

In this article, we consider real Banach spaces only.

2. THE IDEA OF THE CONSTRUCTION

Let f be an infinitely differentiable even function on the unit sphere S_n in \mathbb{R}^n . We spoil the Euclidean norm $\|x\|_2$ in \mathbb{R}^n by means of the function f . Namely, for $\lambda > 0$ consider the function

$$(1) \quad \mathcal{N}_\lambda(x) = \|x\|_2 \left(1 + \lambda f\left(\frac{x}{\|x\|_2}\right) \right), \quad x \in \mathbb{R}^n.$$

One can choose λ small enough so that \mathcal{N}_λ is a norm in \mathbb{R}^n . This follows from a simple one-dimensional consideration: if $a, b \in \mathbb{R}$, g is a convex function on $[a, b]$ with $g'' > \delta > 0$ on $[a, b]$ and $h \in C^2[a, b]$, then the functions $g + \lambda h$ have positive second derivatives on $[a, b]$ for sufficiently small λ 's and, hence, are convex on $[a, b]$.

Let $\lambda_f = \sup\{\lambda > 0 : \mathcal{N}_t \text{ is a norm in } \mathbb{R}^n \text{ for every } t \leq \lambda\}$. For each $\lambda \leq \lambda_f$, we denote by X_λ the Banach space with the norm $\|x\|_\lambda = \mathcal{N}_\lambda(x)$.

Theorem 2 from the paper [5] shows that, for every $q > 0$ which is not an even integer, there exists a small enough number λ such that the space X_t is isometric to a subspace of L_q for every $t \leq \lambda$. This fact was used in [5] to prove that for every compact subset Q of $(0, \infty) \setminus \{2k, k \in \mathbb{N}\}$ there exists a Banach space different from Hilbert spaces which is isometric to a subspace of L_q for every $q \in Q$.

For $q \in (0, 1]$, let $\lambda_q = \sup\{\lambda > 0 : X_t \text{ is isometric to a subspace of } L_q \text{ for every } t \leq \lambda\}$. If $0 < q < p \leq 2$, then the space L_p is isometric to a subspace of L_q ; therefore, $\lambda_p \leq \lambda_q$. In particular, $\lambda_1 \leq \lambda_q$ for every $q < 1$. Clearly, $\lambda_1 \leq \lambda_f$.

Now we can explain the idea of getting a Banach subspace of L_q with $q < 1$ which is not isometric to a subspace of L_1 . Suppose we can find a function f so that λ_1 is strictly less than λ_f , and also λ_1 is strictly less than λ_q . Then, for every $\lambda \in (\lambda_1, \min(\lambda_f, \lambda_q)]$, the space X_λ is a Banach space with the desired property.

Similarly, for $q < p < 1$, if we manage to find a function f so that $\lambda_p < \lambda_q$ and $\lambda_p < \lambda_f$ we get an example of a Banach space which embeds isometrically in L_q but does not embed in L_p .

The construction in [5] is based on the use of spherical harmonics and, in general, does not give a chance to calculate the numbers λ_q exactly. We are, however, able to choose a function f for which it is possible to calculate the numbers λ_q for certain values of q . Our calculations do not depend on the results from [5] mentioned above, so the paper [5] only shows a direction for constructing examples.

We shall use one simple characterization of finite-dimensional subspaces of L_q .

Proposition 1. *Let q be a positive number which is not an even integer, let $(X, \|\cdot\|)$ be an n -dimensional Banach space, and suppose there exists a continuous function b on the sphere S_n in \mathbb{R}^n such that, for every $x \in \mathbb{R}^n$,*

$$(2) \quad \|x\|^q = \int_{S_n} |(x, \xi)|^q b(\xi) \, d\xi$$

where (x, ξ) stands for the scalar product in \mathbb{R}^n .

Then X is isometric to a subspace of L_q if and only if b is a non-negative (not identically zero) function.

Proof. If b is a non-negative function we can assume without loss of generality that $\int_{S_n} b(\xi) d\xi = 1$. Choose any measurable (with respect to Lebesgue measure) functions f_1, \dots, f_n on $[0, 1]$ having the joint distribution $b(\xi)d\xi$. Then, by (2), the operator $x \mapsto \sum x_i f_i$, $x \in \mathbb{R}^n$, is an isometry from X to $L_q([0, 1])$.

Conversely, if X is a subspace of $L_q([0, 1])$ choose any functions $f_1, \dots, f_n \in L_q$ which form a basis in X , and let μ be the joint distribution of the functions f_1, \dots, f_n with respect to Lebesgue measure. Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \|x\|^q &= \left\| \sum_{k=1}^n x_k f_k \right\|^q = \int_0^1 \left| \sum_{k=1}^n x_k f_k(t) \right|^q dt \\ &= \int_{\mathbb{R}^n} |(x, \xi)|^q d\mu(\xi) = \int_{S_n} |(x, \xi)|^q d\nu(\xi) \end{aligned}$$

where ν is the projection of μ to the sphere. (For every Borel subset A of S_n , $\nu(A) = \int_{\{tA, t \in \mathbb{R}\}} \|x\|_2^q d\mu(x)$.) It follows from (2) that

$$\int_{S_n} |(x, \xi)|^q b(\xi) d\xi = \int_{S_n} |(x, \xi)|^q d\nu(\xi)$$

for every $x \in \mathbb{R}^n$. Since q is not an even integer, we can apply the uniqueness theorem for measures on the sphere from [3] to show that $d\nu(\xi) = b(\xi) d\xi$, which means that $b(\xi) d\xi$ is a measure and the function b is non-negative. \square

The representation (2) exists for every smooth enough function on the sphere (see, for example, Theorem 1 from [5]). We are going to choose special smooth norms for which it is possible to calculate the function b exactly and then check if b is non-negative. In this way we calculate the numbers λ_q for these norms.

We need the representation (2) for some simple functions on the sphere.

Lemma 1. For every $x = (x_1, \dots, x_n)$ from the unit sphere S_n in \mathbb{R}^n and every $q > 0$ we have

$$(3) \quad x_n^2 = \frac{\Gamma((n+q)/2)}{2\pi^{(n-1)/2}\Gamma((q+1)/2)} \int_{S_n} |(x, \xi)|^q \left(\frac{n+q}{q} \xi_n^2 - \frac{1}{q} \right) d\xi.$$

Therefore,

$$(4) \quad x_n^2 = \frac{\Gamma((2n+1)/4)}{2\pi^{(n-1)/2}\Gamma(3/4)} \int_{S_n} |(x, \xi)|^{1/2} ((2n+1)\xi_n^2 - 2) d\xi$$

and

$$(5) \quad x_n^2 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)|((n+1)\xi_n^2 - 1) d\xi.$$

Besides,

$$(6) \quad x_n^4 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)|(- (n+3)(n+1)\xi_n^4 + 6(n+1)\xi_n^2 - 3) d\xi.$$

Proof. It is a well-known simple fact (see, for example, [7]) that for every $x \in \mathbb{R}^n$ and every $k > 0$,

$$(7) \quad (x_1^2 + \dots + x_n^2)^k = \frac{\Gamma((n + 2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k + 1)/2)} \int_{S_n} |(x, \xi)|^{2k} d\xi.$$

Differentiate both sides of (7) by x_n twice, and then use the fact that $x \in S_n$ to get

$$1 + (2k - 2)x_n^2 = (2k - 1) \frac{\Gamma((n + 2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k + 1)/2)} \int_{S_n} |(x, \xi)|^{2k-2} \xi_n^2 d\xi.$$

Use (7) with the exponent $2k - 2$ instead of $2k$ to get

$$(2k - 2)x_n^2 = \frac{1}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)|^{2k-2} \left((2k - 1) \frac{\Gamma((n + 2k)/2)}{\Gamma((2k + 1)/2)} \xi_n^2 - \frac{\Gamma((n + 2k - 2)/2)}{\Gamma((2k - 1)/2)} \right) d\xi.$$

Now use the fact that $\Gamma(x + 1) = x\Gamma(x)$ and put $2k - 2 = q$ to get (3).

To prove (6), differentiate both sides of (7) four times by x_n (remember that $x_1^2 + \dots + x_n^2 = 1$; do not factor the second and the third derivatives!), and then put $k = 5/2$:

$$-x_n^4 + 6x_n^2 + 3 = \frac{4\Gamma((n + 5)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)| \xi_n^4 d\xi.$$

Now use (5), (7) with $k = 1/2$, and the fact that $\Gamma(x + 1) = x\Gamma(x)$ to get (6). \square

3. EXAMPLES

For every $\lambda > 0$ define a function \mathcal{N}_λ on \mathbb{R}^n by

$$\mathcal{N}_\lambda(x) = (x_1^2 + \dots + x_n^2)^{1/2} \left(1 + \lambda \frac{x_1^2 + \dots + x_{n-1}^2 - 2x_n^2}{x_1^2 + \dots + x_n^2} \right)^2, \quad x \in \mathbb{R}^n.$$

Lemma 2. \mathcal{N}_λ is a convex function if and only if $\lambda \leq \frac{1}{11}$.

Proof. The function \mathcal{N}_λ is convex if and only if the following function of two variables is convex:

$$g(x, y) = (x^2 + y^2)^{1/2} \left(1 + \lambda \frac{x^2 - 2y^2}{x^2 + y^2} \right)^2.$$

Calculating the second derivatives of the function g we get

$$\begin{aligned} & a^2 \frac{\partial^2 g}{\partial x^2} + 2ab \frac{\partial^2 g}{\partial x \partial y} + b^2 \frac{\partial^2 g}{\partial y^2} \\ &= (x^2 + y^2)^{-7/2} (ay - bx)^2 \\ & \cdot (x^4(1 - 10\lambda - 11\lambda^2) + x^2y^2(2 - 2\lambda + 104\lambda^2) + y^4(1 + 8\lambda - 20\lambda^2)). \end{aligned}$$

The function g is convex if and only if the latter expression is non-negative for every choice of a, b, x, y . Clearly, it happens if and only if $\frac{-1}{10} \leq \lambda \leq \frac{1}{11}$. \square

For $\lambda \leq \frac{1}{11}$ denote by X_λ the Banach space with the norm \mathcal{N}_λ .

Theorem 1. *Let $n \geq 3$. If*

$$\alpha_n = \frac{(18n^2 - 18n)^{1/2} - 3n + 1}{9n^2 - 12n - 1} < \lambda \leq \frac{1}{6n - 4},$$

then the Banach space X_λ is isometric to a subspace of $L_{1/2}$ and, at the same time, X_λ is not isometric to a subspace of L_1 .

Proof. Let us first prove that the space X_λ is isometric to a subspace of $L_{1/2}$ if and only if $\lambda \leq \frac{1}{6n-4}$. For every $x \in S_n$, use (4) and (7) with $k = 1/4$ to get the representation (2) (with $q = 1/2$) for the norm \mathcal{N}_λ :

$$\begin{aligned} \mathcal{N}_\lambda^{1/2}(x) &= 1 + \lambda(1 - 3x_n^2) \\ &= \frac{\Gamma((2n+1)/4)}{2\pi^{(n-1)/2}\Gamma(3/4)} \int_{S_n} |(x, \xi)|^{1/2} (1 + 7\lambda - (6n+3)\lambda\xi_n^2) d\xi. \end{aligned}$$

Clearly, the function $b(\xi) = 1 + 7\lambda - (6n+3)\lambda\xi_n^2$ is non-negative on S_n if and only if $\lambda \leq \frac{1}{6n-4}$, and the fact we need follows from Proposition 1.

Let us show that X_λ is isometric to a subspace of L_1 if and only if $\lambda \leq \alpha_n$. If $x \in S_n$, then $\mathcal{N}_\lambda(x) = 1 + 2\lambda(1 - 3x_n^2) + \lambda^2(1 - 6x_n^2 + 9x_n^4)$. We use (5), (6) and (7) with $k = 1/2$ to get the representation (2) for $\mathcal{N}_\lambda(x)$ with $q = 1$:

$$\mathcal{N}_\lambda(x) = \Gamma((n+1)/2) \int_{S_n} |(x, \xi)| b(\xi_n^2) d\xi$$

where

$$b(\xi_n^2) = (1 + 8\lambda - 20\lambda^2) + \xi_n^2(n+1)(48\lambda^2 - 6\lambda) - 9\xi_n^4(n+3)(n+1)\lambda^2.$$

The function b is a quadratic function of ξ_n^2 with negative first coefficient. By Proposition 1, the space X_λ embeds in L_1 if and only if b is non-negative for every $\xi_n^2 \in [0, 1]$. Clearly, it happens if and only if both numbers $b(0) = 1 + 8\lambda - 20\lambda^2$ and $b(1) = 1 + \lambda(2 - 6n) + \lambda^2(1 + 12n - 9n^2)$ are non-negative. Since $1 + 12n - 9n^2$ is a negative number for every $n \geq 2$ and we consider only positive numbers λ , it is clear that the condition is that $\lambda \leq \alpha_n$. To prove the theorem, it suffices to note that, for every $n \geq 3$, $\frac{1}{6n-4}$ is less than $\frac{1}{11}$ and $\alpha_n < \frac{1}{6n-2} < \frac{1}{6n-4}$. \square

Remarks. 1. Since every two-dimensional Banach space is isometric to a subspace of L_1 , it is impossible to construct a two-dimensional space with the property of Theorem 1. In our example, $\alpha_2 = \frac{1}{11}$, which coincides with the bound from Lemma 2.

2. The author is unable to apply the scheme from Section 2 to every $q < 1$, although it is very likely the same idea works. For instance, getting an example for $q = 3/4$ is a matter of calculating the eighth derivative of the function $(x_1^2 + \dots + x_n^2)^k$.

Theorem 1 shows also that the spaces L_q with $q < 1$ may have different Banach subspaces. The Banach subspaces of $L_{1/2}$ constructed in Theorem 1 cannot be isometric to subspaces of L_q for all $q < 1$. In fact, if a Banach space $(X, \|\cdot\|)$ is isometric to a subspace of L_q for every $q < 1$, then, by a theorem from [1], the function $\exp(-\|x\|^q)$ is positive definite for every $q < 1$. The function $\exp(-\|x\|)$

is then positive definite as a pointwise limit of positive definite functions, and the space X (by the same result from [1]) is isometric to a subspace of L_1 .

On the other hand, we can show the difference directly. For every $\lambda > 0$ define a function \mathcal{N}_λ on \mathbb{R}^3 by

$$\mathcal{N}_\lambda(x) = (x_1^2 + x_2^2 + x_3^2)^{1/2} \left(1 + \lambda \frac{x_1^2 + x_2^2 - 2x_3^2}{x_1^2 + x_2^2 + x_3^2} \right)^4, \quad x \in \mathbb{R}^3.$$

The proof of the following theorem is similar to the proofs of Lemma 2 and Theorem 1.

Theorem 2. *The function \mathcal{N}_λ is a norm if and only if $\lambda \leq \frac{1}{23}$. The corresponding Banach space X_λ is isometric to a subspace of $L_{1/4}$ if and only if $\lambda \leq \frac{1}{26}$, and X_λ is not isometric to a subspace of $L_{1/2}$ if $\lambda > \frac{1}{28}$. Thus, for $\frac{1}{28} < \lambda \leq \frac{1}{26}$, the space X_λ is a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$.*

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