PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 1, January 1996

A BANACH SUBSPACE OF $L_{1/2}$ WHICH DOES NOT EMBED IN L_1 (ISOMETRIC VERSION)

ALEXANDER KOLDOBSKY

(Communicated by Dale Alspach)

ABSTRACT. For every $n \geq 3$, we construct an n-dimensional Banach space which is isometric to a subspace of $L_{1/2}$ but is not isometric to a subspace of L_1. The isomorphic version of this problem (posed by S. Kwapien in 1969) is still open. Another example gives a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$. Note that, from the isomorphic point of view, all the spaces L_q with $q < 1$ have the same Banach subspaces.

1. INTRODUCTION

A well-known fact is that the space L_1 is isometric to a subspace of L_q for every $q < 1$. It is natural to ask whether the spaces L_q with $q < 1$ contain any Banach space structure not generated by L_1. This question was first formulated in 1969 by Kwapien [6] in the following form: Need every Banach subspace of L_0 be also a subspace of L_1? Later the question was mentioned by Maurey [8, Question 124].

In 1970, Nikishin [9] proved that every Banach subspace of L_0 is isomorphic to a subspace of L_q for every $q < 1$. Therefore, if we replace the space L_0 in Kwapien's question by any of the spaces L_q with $q < 1$ we get an equivalent question.

Since all the spaces L_q with $q < 1$ embed in L_0, Nikishin's result also shows that these spaces are all the same from the isomorphic Banach space point of view. Namely, every Banach space which is isomorphic to a subspace of L_q with $q < 1$ is also isomorphic to a subspace of L_p for every other $p < 1$.

In this paper we show that the answer to the isometric version of Kwapien's question is negative. For every $n \in N$, $n \geq 3$, there exists an n-dimensional Banach space which is isometric to a subspace of $L_{1/2}$ but is not isometric to a subspace of L_1. Using this example it is easy to see that the spaces L_q with $q < 1$ may be different from the isometric Banach space point of view. We give, however, a direct example illustrating the difference by constructing a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$.

The isomorphic version of Kwapien's question is still open. The most recent related result seems to be a theorem of Kalton [2], who proved that a Banach space X embeds in L_1 if and only if $\ell_1(X)$ embeds in L_0.

The isometric version of Kwapien's question can be reformulated in the language of positive definite functions. In fact, a Banach space $(X, \|\cdot\|)$ is isometric to a subspace of L_p with $0 < p \leq 2$ if and only if the function $\exp(-\|x\|^p)$ is positive

Received by the editors April 28, 1994 and, in revised form, July 13, 1994.
1991 *Mathematics Subject Classification*. Primary 46B04; Secondary 46E30, 60E10.

definite [1]. The main example of this paper gives a norm such that the function $\exp(-\|x\|^{1/2})$ is positive definite but the function $\exp(-\|x\|)$ is not positive definite. This result is close to problems of Schoenberg's type (see [4]).

In this article, we consider real Banach spaces only.

2. The idea of the construction

Let f be an infinitely differentiable even function on the unit sphere S_n in $\mathbb{R}^n$. We spoil the Euclidean norm $\|x\|_2$ in $\mathbb{R}^n$ by means of the function f. Namely, for $\lambda > 0$ consider the function

$$\mathcal{N}_\lambda(x) = \|x\|_2\big(1+\lambda f(\frac{x}{\|x\|_2})\big),\ x \in \mathbb{R}^n. \tag{1}$$

One can choose λ small enough so that $\mathcal{N}_\lambda$ is a norm in $\mathbb{R}^n$. This follows from a simple one-dimensional consideration: if $a, b \in \mathbb{R}$, g is a convex function on $[a,b]$ with $g'' > \delta > 0$ on $[a,b]$ and $h \in C^2[a,b]$, then the functions $g + \lambda h$ have positive second derivatives on $[a,b]$ for sufficiently small λ's and, hence, are convex on $[a,b]$.

Let $\lambda_f = \sup\{\lambda > 0 :\ \mathcal{N}_t$ is a norm in $\mathbb{R}^n$ for every $t \leq \lambda\}$. For each $\lambda \leq \lambda_f$, we denote by X_λ the Banach space with the norm $\|x\|_\lambda = \mathcal{N}_\lambda(x)$.

Theorem 2 from the paper [5] shows that, for every $q > 0$ which is not an even integer, there exists a small enough number λ such that the space X_t is isometric to a subspace of L_q for every $t \leq \lambda$. This fact was used in [5] to prove that for every compact subset Q of $(0,\infty)\setminus\{2k, k \in N\}$ there exists a Banach space different from Hilbert spaces which is isometric to a subspace of L_q for every $q \in Q$.

For $q \in (0,1]$, let $\lambda_q = \sup\{\lambda > 0 :\ X_t$ is isometric to a subspace of L_q for every $t \leq \lambda\}$. If $0 < q < p \leq 2$, then the space L_p is isometric to a subspace of L_q; therefore, $\lambda_p \leq \lambda_q$. In particular, $\lambda_1 \leq \lambda_q$ for every $q < 1$. Clearly, $\lambda_1 \leq \lambda_f$.

Now we can explain the idea of getting a Banach subspace of L_q with $q < 1$ which is not isometric to a subspace of L_1. Suppose we can find a function f so that λ_1 is strictly less than λ_f, and also λ_1 is strictly less than λ_q. Then, for every $\lambda \in (\lambda_1,\ \min(\lambda_f, \lambda_q)]$, the space X_λ is a Banach space with the desired property.

Similarly, for $q < p < 1$, if we manage to find a function f so that $\lambda_p < \lambda_q$ and $\lambda_p < \lambda_f$ we get an example of a Banach space which embeds isometrically in L_q but does not embed in L_p.

The construction in [5] is based on the use of spherical harmonics and, in general, does not give a chance to calculate the numbers λ_q exactly. We are, however, able to choose a function f for which it is possible to calculate the numbers λ_q for certain values of q. Our calculations do not depend on the results from [5] mentioned above, so the paper [5] only shows a direction for constructing examples.

We shall use one simple characterization of finite-dimensional subspaces of L_q.

Proposition 1. *Let q be a positive number which is not an even integer, let $(X, \|\cdot\|)$ be an n-dimensional Banach space, and suppose there exists a continuous function b on the sphere S_n in $\mathbb{R}^n$ such that, for every $x \in \mathbb{R}^n$,*

$$\|x\|^q = \int_{S_n} |(x,\xi)|^q\ b(\xi)\ d\xi \tag{2}$$

where (x,ξ) stands for the scalar product in $\mathbb{R}^n$.

Then X is isometric to a subspace of L_q if and only if b is a non-negative (not identically zero) function.

Proof. If b is a non-negative function we can assume without loss of generality that $\int_{S_n} b(\xi)\, d\xi = 1$. Choose any measurable (with respect to Lebesgue measure) functions $f_1, \dots, f_n$ on $[0,1]$ having the joint distribution $b(\xi)d\xi$. Then, by (2), the operator $x \mapsto \sum x_i f_i,\ x \in \mathbb{R}^n$, is an isometry from X to $L_q([0,1])$.

Conversely, if X is a subspace of $L_q([0,1])$ choose any functions $f_1, ..., f_n \in L_q$ which form a basis in X, and let μ be the joint distribution of the functions $f_1, ..., f_n$ with respect to Lebesgue measure. Then, for every $x \in R^n$,

$$\|x\|^q = \|\sum_{k=1}^n x_k f_k\|^q = \int_0^1 |\sum_{k=1}^n x_k f_k(t)|^q dt$$
$$= \int_{R^n} |(x,\xi)|^q \ d\mu(\xi) = \int_{S_n} |(x,\xi)|^q \ d\nu(\xi)$$

where ν is the projection of μ to the sphere. (For every Borel subset A of S_n, $\nu(A) = \int_{\{tA, t\in R\}} \|x\|_2^q d\mu(x)$.) It follows from (2) that

$$\int_{S_n} |(x,\xi)|^q \ b(\xi)\ d\xi = \int_{S_n} |(x,\xi)|^q \ d\nu(\xi)$$

for every $x \in \mathbb{R}^n$. Since q is not an even integer, we can apply the uniqueness theorem for measures on the sphere from [3] to show that $d\nu(\xi) = b(\xi)\ d\xi$, which means that $b(\xi)\ d\xi$ is a measure and the function b is non-negative. □

The representation (2) exists for every smooth enough function on the sphere (see, for example, Theorem 1 from [5]). We are going to choose special smooth norms for which it is possible to calculate the function b exactly and then check if b is non-negative. In this way we calculate the numbers λ_q for these norms.

We need the representation (2) for some simple functions on the sphere.

Lemma 1. *For every $x = (x_1, \dots, x_n)$ from the unit sphere S_n in $\mathbb{R}^n$ and every $q > 0$ we have*

$$x_n^2 = \frac{\Gamma((n+q)/2)}{2\pi^{(n-1)/2}\Gamma((q+1)/2)} \int_{S_n} |(x,\xi)|^q \big(\frac{n+q}{q}\xi_n^2 - \frac{1}{q}\big)\ d\xi. \tag{3}$$

Therefore,

$$x_n^2 = \frac{\Gamma((2n+1)/4)}{2\pi^{(n-1)/2}\Gamma(3/4)} \int_{S_n} |(x,\xi)|^{1/2}\big((2n+1)\xi_n^2 - 2\big)\ d\xi \tag{4}$$

and

$$x_n^2 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x,\xi)|\big((n+1)\xi_n^2 - 1\big)\ d\xi. \tag{5}$$

Besides,

$$x_n^4 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x,\xi)|\big(-(n+3)(n+1)\xi_n^4 + 6(n+1)\xi_n^2 - 3\big)\ d\xi. \tag{6}$$

Proof. It is a well-known simple fact (see, for example, [7]) that for every $x \in \mathbb{R}^n$ and every $k > 0$,

$$(x_1^2 + \cdots + x_n^2)^k = \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k+1)/2)} \int_{S_n} |(x,\xi)|^{2k} \, d\xi. \tag{7}$$

Differentiate both sides of (7) by x_n twice, and then use the fact that $x \in S_n$ to get

$$1 + (2k-2)x_n^2 = (2k-1)\frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k+1)/2)} \int_{S_n} |(x,\xi)|^{2k-2} \xi_n^2 \, d\xi.$$

Use (7) with the exponent $2k-2$ instead of $2k$ to get

$$(2k-2)x_n^2 = \frac{1}{2\pi^{(n-1)/2}} \int_{S_n} |(x,\xi)|^{2k-2} \Big((2k-1)\frac{\Gamma((n+2k)/2)}{\Gamma((2k+1)/2)} \xi_n^2 - \frac{\Gamma((n+2k-2)/2)}{\Gamma((2k-1)/2)} \Big) \, d\xi.$$

Now use the fact that $\Gamma(x+1) = x\Gamma(x)$ and put $2k-2 = q$ to get (3).

To prove (6), differentiate both sides of (7) four times by x_n (remember that $x_1^2 + \cdots + x_n^2 = 1$; do not factor the second and the third derivatives!), and then put $k = 5/2$:

$$-x_n^4 + 6x_n^2 + 3 = \frac{4\Gamma((n+5)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x,\xi)| \xi_n^4 \, d\xi.$$

Now use (5), (7) with $k = 1/2$, and the fact that $\Gamma(x+1) = x\Gamma(x)$ to get (6). □

3. Examples

For every $\lambda > 0$ define a function $\mathcal{N}_\lambda$ on $\mathbb{R}^n$ by

$$\mathcal{N}_\lambda(x) = (x_1^2 + \cdots + x_n^2)^{1/2} \Big(1 + \lambda \frac{x_1^2 + \cdots + x_{n-1}^2 - 2x_n^2}{x_1^2 + \cdots + x_n^2} \Big)^2, \ x \in \mathbb{R}^n.$$

Lemma 2. *$\mathcal{N}_\lambda$ is a convex function if and only if $\lambda \le \frac{1}{11}$.*

Proof. The function $\mathcal{N}_\lambda$ is convex if and only if the following function of two variables is convex:

$$g(x,y) = (x^2 + y^2)^{1/2} \Big(1 + \lambda \frac{x^2 - 2y^2}{x^2 + y^2} \Big)^2.$$

Calculating the second derivatives of the function g we get

$$a^2 \frac{\partial^2 g}{\partial x^2} + 2ab \frac{\partial^2 g}{\partial x \partial y} + b^2 \frac{\partial^2 g}{\partial y^2} = (x^2+y^2)^{-7/2}(ay - bx)^2 \cdot \big(x^4(1 - 10\lambda - 11\lambda^2) + x^2y^2(2 - 2\lambda + 104\lambda^2) + y^4(1 + 8\lambda - 20\lambda^2) \big).$$

The function g is convex if and only if the latter expression is non-negative for every choice of a, b, x, y. Clearly, it happens if and only if $\frac{-1}{10} \le \lambda \le \frac{1}{11}$. □

For $\lambda \le \frac{1}{11}$ denote by X_λ the Banach space with the norm $\mathcal{N}_\lambda$.

Theorem 1. *Let $n \geq 3$. If*

$$\alpha_n = \frac{(18n^2 - 18n)^{1/2} - 3n + 1}{9n^2 - 12n - 1} < \lambda \leq \frac{1}{6n-4},$$

then the Banach space X_λ is isometric to a subspace of $L_{1/2}$ and, at the same time, X_λ is not isometric to a subspace of L_1.

Proof. Let us first prove that the space X_λ is isometric to a subspace of $L_{1/2}$ if and only if $\lambda \leq \frac{1}{6n-4}$. For every $x \in S_n$, use (4) and (7) with $k = 1/4$ to get the representation (2) (with $q = 1/2$) for the norm $\mathcal{N}_\lambda$:

$$\begin{aligned}\mathcal{N}_\lambda^{1/2}(x) &= 1 + \lambda(1 - 3x_n^2)\\ &= \frac{\Gamma((2n+1)/4)}{2\pi^{(n-1)/2}\Gamma(3/4)} \int_{S_n} |(x,\xi)|^{1/2}\big(1 + 7\lambda - (6n+3)\lambda\xi_n^2\big)\ d\xi.\end{aligned}$$

Clearly, the function $b(\xi) = 1 + 7\lambda - (6n+3)\lambda\xi_n^2$ is non-negative on S_n if and only if $\lambda \leq \frac{1}{6n-4}$, and the fact we need follows from Proposition 1.

Let us show that X_λ is isometric to a subspace of L_1 if and only if $\lambda \leq \alpha_n$. If $x \in S_n$, then $\mathcal{N}_\lambda(x) = 1 + 2\lambda(1 - 3x_n^2) + \lambda^2(1 - 6x_n^2 + 9x_n^4)$. We use (5), (6) and (7) with $k = 1/2$ to get the representation (2) for $\mathcal{N}_\lambda(x)$ with $q = 1$:

$$\mathcal{N}_\lambda(x) = \Gamma((n+1)/2) \int_{S_n} |(x,\xi)|\ b(\xi_n^2)\ d\xi$$

where

$$b(\xi_n^2) = (1 + 8\lambda - 20\lambda^2) + \xi_n^2(n+1)(48\lambda^2 - 6\lambda) - 9\xi_n^4(n+3)(n+1)\lambda^2.$$

The function b is a quadratic function of ξ_n^2 with negative first coefficient. By Proposition 1, the space X_λ embeds in L_1 if and only if b is non-negative for every $\xi_n^2 \in [0,1]$. Clearly, it happens if and only if both numbers $b(0) = 1 + 8\lambda - 20\lambda^2$ and $b(1) = 1 + \lambda(2 - 6n) + \lambda^2(1 + 12n - 9n^2)$ are non-negative. Since $1 + 12n - 9n^2$ is a negative number for every $n \geq 2$ and we consider only positive numbers λ, it is clear that the condition is that $\lambda \leq \alpha_n$. To prove the theorem, it suffices to note that, for every $n \geq 3$, $\frac{1}{6n-4}$ is less than $\frac{1}{11}$ and $\alpha_n < \frac{1}{6n-2} < \frac{1}{6n-4}$. □

Remarks. 1. Since every two-dimensional Banach space is isometric to a subspace of L_1, it is impossible to construct a two-dimensional space with the property of Theorem 1. In our example, $\alpha_2 = \frac{1}{11}$, which coincides with the bound from Lemma 2.

2. The author is unable to apply the scheme from Section 2 to every $q < 1$, although it is very likely the same idea works. For instance, getting an example for $q = 3/4$ is a matter of calculating the eighth derivative of the function $(x_1^2 + \cdots + x_n^2)^k$.

Theorem 1 shows also that the spaces L_q with $q < 1$ may have different Banach subspaces. The Banach subspaces of $L_{1/2}$ constructed in Theorem 1 cannot be isometric to subspaces of L_q for all $q < 1$. In fact, if a Banach space $(X, \|\cdot\|)$ is isometric to a subspace of L_q for every $q < 1$, then, by a theorem from [1], the function $\exp(-\|x\|^q)$ is positive definite for every $q < 1$. The function $\exp(-\|x\|)$

is then positive definite as a pointwise limit of positive definite functions, and the space X (by the same result from [1]) is isometric to a subspace of L_1.

On the other hand, we can show the difference directly. For every $\lambda > 0$ define a function $\mathcal{N}_\lambda$ on $\mathbb{R}^3$ by

$$\mathcal{N}_\lambda(x) = (x_1^2 + x_2^2 + x_3^2)^{1/2}\Big(1 + \lambda\frac{x_1^2 + x_2^2 - 2x_3^2}{x_1^2 + x_2^2 + x_3^2}\Big)^4, \ x \in \mathbb{R}^3.$$

The proof of the following theorem is similar to the proofs of Lemma 2 and Theorem 1.

Theorem 2. *The function $\mathcal{N}_\lambda$ is a norm if and only if $\lambda \le \frac{1}{23}$. The corresponding Banach space X_λ is isometric to a subspace of $L_{1/4}$ if and only if $\lambda \le \frac{1}{26}$, and X_λ is not isometric to a subspace of $L_{1/2}$ if $\lambda > \frac{1}{28}$. Thus, for $\frac{1}{28} < \lambda \le \frac{1}{26}$, the space X_λ is a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$.*

References

1. J. Bretagnolle, D. Dacunha-Castelle, and J. L. Krivine, *Lois stables et espaces L_p*, Ann. Inst. H. Poincaré Probab. Statist. **2** (1966), 231–259. MR **34:**3605
2. N. Kalton, *Banach spaces embedding into L_0*, Israel J. Math. **52** (1985), 305–319. MR **87k:**46045
3. M. Kanter, *The L_p-norms of sums of translates of a function*, Trans. Amer. Math. Soc. **179** (1973), 35–47. MR **50:**14062
4. A. Koldobsky, *Schoenberg's problem on positive definite functions*, Algebra and Analysis **3** (1991), 78–85; English transl., St. Petersburg Math. J. **3** (1992), 563–570. MR **93c:**42014
5. A. Koldobsky, *Common subspaces of L_p-spaces*, Proc. Amer. Math. Soc. **122** (1994), 207–212. MR **94k:**46060
6. S. Kwapien, *Problem* 3, Studia Math. **38** (1970), 469.
7. P. Levy, *Théory de l'addition de variable aléatoires*, Gauthier-Villars, Paris, 1937.
8. B. Maurey, *Theoremes de factorisation pour les operateurs lineaires a valeurs dans les espaces L_p*, Asterisque **11** (1974). MR **49:**9670
9. E. M. Nikishin, *Resonance theorems and superlinear operators*, Russian Math. Surveys **25** (1970), 124–187.

Division of Mathematics and Statistics, University of Texas at San Antonio, San Antonio, Texas 78249

E-mail address: koldobsk@ringer.cs.utsa.edu