A BANACH SUBSPACE OF $L_{1/2}$ WHICH DOES NOT EMBED IN $L_1$ (ISOMETRIC VERSION)

ALEXANDER KOLDOBSKY

(Communicated by Dale Alspach)

Abstract. For every $n \geq 3$, we construct an $n$-dimensional Banach space which is isometric to a subspace of $L_{1/2}$ but is not isometric to a subspace of $L_1$. The isomorphic version of this problem (posed by S. Kwapien in 1969) is still open. Another example gives a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$. Note that, from the isomorphic point of view, all the spaces $L_q$ with $q < 1$ have the same Banach subspaces.

1. Introduction

A well-known fact is that the space $L_1$ is isometric to a subspace of $L_q$ for every $q < 1$. It is natural to ask whether the spaces $L_q$ with $q < 1$ contain any Banach space structure not generated by $L_1$. This question was first formulated in 1969 by Kwapien [6] in the following form: Need every Banach subspace of $L_0$ be also a subspace of $L_1$? Later the question was mentioned by Maurey [8, Question 124].

In 1970, Nikishin [9] proved that every Banach subspace of $L_0$ is isomorphic to a subspace of $L_q$ for every $q < 1$. Therefore, if we replace the space $L_0$ in Kwapien’s question by any of the spaces $L_q$ with $q < 1$ we get an equivalent question.

Since all the spaces $L_q$ with $q < 1$ embed in $L_0$, Nikishin’s result also shows that these spaces are all the same from the isomorphic Banach space point of view. Namely, every Banach space which is isomorphic to a subspace of $L_q$ with $q < 1$ is also isomorphic to a subspace of $L_p$ for every other $p < 1$.

In this paper we show that the answer to the isometric version of Kwapien’s question is negative. For every $n \in N$, $n \geq 3$, there exists an $n$-dimensional Banach space which is isometric to a subspace of $L_{1/2}$ but is not isometric to a subspace of $L_1$. Using this example it is easy to see that the spaces $L_q$ with $q < 1$ may be different from the isometric Banach space point of view. We give, however, a direct example illustrating the difference by constructing a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$.

The isomorphic version of Kwapien’s question is still open. The most recent related result seems to be a theorem of Kalton [2], who proved that a Banach space $X$ embeds in $L_1$ if and only if $\ell_1(X)$ embeds in $L_0$.

The isometric version of Kwapien’s question can be reformulated in the language of positive definite functions. In fact, a Banach space $(X, \| \cdot \|)$ is isometric to a subspace of $L_p$ with $0 < p \leq 2$ if and only if the function $\exp(-\|x\|p)$ is positive.

Received by the editors April 28, 1994 and, in revised form, July 13, 1994.

1991 Mathematics Subject Classification. Primary 46B04; Secondary 46E30, 60E10.
definite [1]. The main example of this paper gives a norm such that the function \( \exp(-\|x\|^{1/2}) \) is positive definite but the function \( \exp(-\|x\|) \) is not positive definite. This result is close to problems of Schoenberg's type (see [4]).

In this article, we consider real Banach spaces only.

## 2. The idea of the construction

Let \( f \) be an infinitely differentiable even function on the unit sphere \( S_n \) in \( \mathbb{R}^n \). We spoil the Euclidean norm \( \|x\|_2 \) in \( \mathbb{R}^n \) by means of the function \( f \). Namely, for \( \lambda > 0 \) consider the function

\[
N_\lambda(x) = \|x\|_2 (1 + \lambda f(\frac{x}{\|x\|_2})), \quad x \in \mathbb{R}^n.
\]

One can choose \( \lambda \) small enough so that \( N_\lambda \) is a norm in \( \mathbb{R}^n \). This follows from a simple one-dimensional consideration: if \( a, b \in \mathbb{R} \), \( g \) is a convex function on \([a, b]\) with \( g'' > \delta > 0 \) on \([a, b]\) and \( h \in C^2[a, b] \), then the functions \( g + \lambda h \) have positive second derivatives on \([a, b]\) for sufficiently small \( \lambda \)'s and, hence, are convex on \([a, b]\).

Let \( \lambda_f = \sup\{ \lambda > 0 : N_\lambda \text{ is a norm in } \mathbb{R}^n \text{ for every } t \leq \lambda \} \). For each \( \lambda \leq \lambda_f \), we denote by \( X_\lambda \) the Banach space with the norm \( \|x\|_\lambda = N_\lambda(x) \).

Theorem 2 from the paper [5] shows that, for every \( q > 0 \) which is not an even integer, there exists a small enough number \( \lambda \) such that the space \( X_\lambda \) is isometric to a subspace of \( L_q \) for every \( t \leq \lambda \). This fact was used in [5] to prove that for every compact subset \( Q \) of \((0, \infty) \setminus \{2k, k \in \mathbb{N}\} \) there exists a Banach space different from Hilbert spaces which is isometric to a subspace of \( L_q \) for every \( q \in Q \).

For \( q \in (0, 1] \), let \( \lambda_q = \sup\{ \lambda > 0 : X_\lambda \text{ is isometric to a subspace of } L_q \text{ for every } t \leq \lambda \} \). If \( 0 < q < p \leq 2 \), then the space \( L_p \) is isometric to a subspace of \( L_q \); therefore, \( \lambda_p \leq \lambda_q \). In particular, \( \lambda_1 \leq \lambda_q \) for every \( q < 1 \). Clearly, \( \lambda_1 \leq \lambda_f \).

Now we can explain the idea of getting a Banach subspace of \( L_q \) with \( q < 1 \) which is not isometric to a subspace of \( L_1 \). Suppose we can find a function \( f \) so that \( \lambda_1 \) is strictly less than \( \lambda_f \), and also \( \lambda_1 \) is strictly less than \( \lambda_q \). Then, for every \( \lambda \in (\lambda_1, \min(\lambda_f, \lambda_q)) \), the space \( X_\lambda \) is a Banach space with the desired property.

Similarly, for \( q < p < 1 \), if we manage to find a function \( f \) so that \( \lambda_p < \lambda_q \) and \( \lambda_p < \lambda_f \) we get an example of a Banach space which embeds isometrically in \( L_q \) but does not embed in \( L_p \).

The construction in [5] is based on the use of spherical harmonics and, in general, does not give a chance to calculate the numbers \( \lambda_q \) exactly. We are, however, able to choose a function \( f \) for which it is possible to calculate the numbers \( \lambda_q \) for certain values of \( q \). Our calculations do not depend on the results from [5] mentioned above, so the paper [5] only shows a direction for constructing examples.

We shall use one simple characterization of finite-dimensional subspaces of \( L_q \).

**Proposition 1.** Let \( q \) be a positive number which is not an even integer, let \((X, \| \cdot \|)\) be an \( n \)-dimensional Banach space, and suppose there exists a continuous function \( b \) on the sphere \( S_n \) in \( \mathbb{R}^n \) such that, for every \( x \in \mathbb{R}^n \),

\[
\|x\|^q = \int_{S_n} |(x, \xi)|^q b(\xi) \, d\xi
\]

where \((x, \xi)\) stands for the scalar product in \( \mathbb{R}^n \).
Then $X$ is isometric to a subspace of $L_q$ if and only if $b$ is a non-negative (not identically zero) function.

Proof. If $b$ is a non-negative function we can assume without loss of generality that $\int_{S_n} b(\xi) \, d\xi = 1$. Choose any measurable (with respect to Lebesgue measure) functions $f_1, \ldots, f_n$ on $[0, 1]$ having the joint distribution $b(\xi)d\xi$. Then, by (2), the operator $x \mapsto \sum x_i f_i$, $x \in \mathbb{R}^n$, is an isometry from $X$ to $L_q([0,1])$.

Conversely, if $X$ is a subspace of $L_q([0,1])$ choose any functions $f_1, \ldots, f_n \in L_q$ which form a basis in $X$, and let $\mu$ be the joint distribution of the functions $f_1, \ldots, f_n$ with respect to Lebesgue measure. Then, for every $x \in \mathbb{R}^n$,

$$\|x\|^q = \left\| \sum_{k=1}^n x_k f_k \right\|^q = \int_0^1 \left| \sum_{k=1}^n x_k f_k(t) \right|^q \, dt$$

$$= \int_{\mathbb{R}^n} |(x, \xi)|^q \, d\mu(\xi) = \int_{S_n} |(x, \xi)|^q \, d\nu(\xi)$$

where $\nu$ is the projection of $\mu$ to the sphere. (For every Borel subset $A$ of $S_n$, $\nu(A) = \int_A |x|^q \, d\mu(x)$.) It follows from (2) that

$$\int_{S_n} |(x, \xi)|^q \, b(\xi) \, d\xi = \int_{S_n} |(x, \xi)|^q \, d\nu(\xi)$$

for every $x \in \mathbb{R}^n$. Since $q$ is not an even integer, we can apply the uniqueness theorem for measures on the sphere from [3] to show that $d\nu(\xi) = b(\xi) \, d\xi$, which means that $b(\xi) \, d\xi$ is a measure and the function $b$ is non-negative.

The representation (2) exists for every smooth enough function on the sphere (see, for example, Theorem 1 from [5]). We are going to choose special smooth norms for which it is possible to calculate the function $b$ exactly and then check if $b$ is non-negative. In this way we calculate the numbers $\lambda_q$ for these norms.

We need the representation (2) for some simple functions on the sphere.

**Lemma 1.** For every $x = (x_1, \ldots, x_n)$ from the unit sphere $S_n$ in $\mathbb{R}^n$ and every $q > 0$ we have

$$x_n^2 = \frac{\Gamma((n+q)/2)}{2\pi^{(n-1)/2}\Gamma((q+1)/2)} \int_{S_n} |(x, \xi)|^q \left( \frac{n+q}{q} \xi_n^2 - 1 \right) \, d\xi.$$  

Therefore,

$$x_n^2 = \frac{\Gamma((2n+1)/4)}{2\pi^{(n-1)/2}\Gamma(3/4)} \int_{S_n} |(x, \xi)|^{1/2} \left( (2n+1)\xi_n^2 - 2 \right) \, d\xi$$

and

$$x_n^2 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)| \left( (n+1)\xi_n^2 - 1 \right) \, d\xi.$$  

Besides,

$$x_n^4 = \frac{\Gamma((n+1)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)| \left( -(n+3)(n+1)\xi_n^4 + 6(n+1)\xi_n^2 - 3 \right) \, d\xi.$$
Proof. It is a well-known simple fact (see, for example, [7]) that for every \( x \in \mathbb{R}^n \) and every \( k > 0 \),
\[
(x_1^2 + \cdots + x_n^2)^k = \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k+1)/2)} \int_{S_n} |(x, \xi)|^{2k} \, d\xi.
\]
Differentiate both sides of (7) by \( x \) twice, and then use the fact that \( x \in S_n \) to get
\[
1 + (2k - 2)x_n^2 = (2k - 1) \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k+1)/2)} \int_{S_n} |(x, \xi)|^{2k-2} \xi_n^2 \, d\xi.
\]
Use (7) with the exponent 2\( k - 2 \) instead of 2\( k \) to get
\[
(2k - 2)x_n^2 = \frac{1}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)|^{2k-2} \left( (2k - 1) \frac{\Gamma((n+2k)/2)}{\Gamma((2k+1)/2)} \xi_n^2 - \frac{\Gamma((n+2k-2)/2)}{\Gamma((2k-1)/2)} \right) \, d\xi.
\]
Now use the fact that \( \Gamma(x+1) = x\Gamma(x) \) and put 2\( k - 2 = q \) to get (3).

To prove (6), differentiate both sides of (7) four times by \( x \) (remember that \( x_1^2 + \cdots + x_n^2 = 1 \); do not factor the second and the third derivatives!), and then put \( k = 5/2 \):
\[
-x_n^4 + 6x_n^2 + 3 = \frac{4\Gamma((n+5)/2)}{2\pi^{(n-1)/2}} \int_{S_n} |(x, \xi)|^{4} \xi_n^4 \, d\xi.
\]
Now use (5), (7) with \( k = 1/2 \), and the fact that \( \Gamma(x+1) = x\Gamma(x) \) to get (6). \( \square \)

3. Examples

For every \( \lambda > 0 \) define a function \( N_\lambda \) on \( \mathbb{R}^n \) by
\[
N_\lambda(x) = (x_1^2 + \cdots + x_n^2)^{1/2} \left( 1 + \lambda \frac{x_1^2 + \cdots + x_{n-1}^2 - 2x_n^2}{x_1^2 + \cdots + x_n^2} \right)^2, \quad x \in \mathbb{R}^n.
\]

Lemma 2. \( N_\lambda \) is a convex function if and only if \( \lambda \leq \frac{1}{11} \).

Proof. The function \( N_\lambda \) is convex if and only if the following function of two variables is convex:
\[
g(x, y) = (x^2 + y^2)^{1/2} \left( 1 + \lambda \frac{x^2 - 2y^2}{x^2 + y^2} \right)^2.
\]
Calculating the second derivatives of the function \( g \) we get
\[
a^2 \frac{\partial^2 g}{\partial x^2} + 2ab \frac{\partial^2 g}{\partial x \partial y} + b^2 \frac{\partial^2 g}{\partial y^2}
= (x^2 + y^2)^{-7/2} (ay - bx)^2 
\cdot \left( x^4(1 - 10\lambda - 11\lambda^2) + x^2y^2(2 - 2\lambda + 104\lambda^2) + y^4(1 + 8\lambda - 20\lambda^2) \right).
\]
The function \( g \) is convex if and only if the latter expression is non-negative for every choice of \( a, b, x, y \). Clearly, it happens if and only if \( \frac{\lambda}{10} \leq \lambda \leq \frac{1}{11} \). \( \square \)

For \( \lambda \leq \frac{1}{11} \) denote by \( X_\lambda \) the Banach space with the norm \( N_\lambda \).
Theorem 1. Let \( n \geq 3 \). If
\[
\alpha_n = \frac{(18n^2 - 18n)^{1/2} - 3n + 1}{9n^2 - 12n - 1} < \lambda \leq \frac{1}{6n - 4},
\]
then the Banach space \( X_\lambda \) is isometric to a subspace of \( L_{1/2} \) and, at the same time, \( X_\lambda \) is not isometric to a subspace of \( L_1 \).

Proof. Let us first prove that the space \( X_\lambda \) is isometric to a subspace of \( L_{1/2} \) if and only if \( \lambda \leq \frac{1}{6n-4} \). For every \( x \in S_n \), use (4) and (7) with \( k = 1/4 \) to get the representation (2) (with \( q = 1/2 \)) for the norm \( N_x \):
\[
N_{1/2}(x) = 1 + \lambda (1 - 3x_n^2)
\]
\[
= \frac{\Gamma((2n + 1)/4)}{2\pi^{(n-1)/2}\Gamma(3/4)} \int_{S_n} |(x, \xi)|^{1/2} (1 + 7\lambda - (6n + 3)\lambda \xi_n^2) \, d\xi.
\]
Clearly, the function \( b(\xi) = 1 + 7\lambda - (6n + 3)\lambda \xi_n^2 \) is non-negative on \( S_n \) if and only if \( \lambda \leq \frac{1}{6n-4} \), and the fact we need follows from Proposition 1.

Let us show that \( X_\lambda \) is isometric to a subspace of \( L_1 \) if and only if \( \lambda \leq \alpha_n \). If \( x \in S_n \), then \( N_x(x) = 1 + 2\lambda(1 - 3x_n^2) + \lambda^2(1 - 6x_n^2 + 9x_n^4) \). We use (5), (6) and (7) with \( k = 1/2 \) to get the representation (2) for \( N_x(x) \) with \( q = 1 \):
\[
N_x(x) = \Gamma((n+1)/2) \int_{S_n} |(x, \xi)| b(\xi_n^2) \, d\xi
\]
where
\[
b(\xi_n^2) = (1 + 8\lambda - 20\lambda^2) + \xi_n^2(n + 1)(48\lambda^2 - 6\lambda) - 9\xi_n^4(3n)(n + 1)\lambda^2.
\]
The function \( b \) is a quadratic function of \( \xi_n^2 \) with negative first coefficient. By Proposition 1, the space \( X_\lambda \) embeds in \( L_1 \) if and only if \( b \) is non-negative for every \( \xi_n^2 \in [0, 1] \). Clearly, it happens if and only if both numbers \( b(0) = 1 + 8\lambda - 20\lambda^2 \) and \( b(1) = 1 + \lambda(2 - 6n) + \lambda^2(1 + 12n - 9n^2) \) are non-negative. Since \( 1 + 12n - 9n^2 \) is a negative number for every \( n \geq 2 \) and we consider only positive numbers \( \lambda \), it is clear that the condition is that \( \lambda \leq \alpha_n \). To prove the theorem, it suffices to note that, for every \( n \geq 3 \), \( \frac{1}{6n-4} < \frac{1}{11} \) and \( \alpha_n < \frac{1}{6n-2} < \frac{1}{6n-4} \). \( \square \)

Remarks. 1. Since every two-dimensional Banach space is isometric to a subspace of \( L_1 \), it is impossible to construct a two-dimensional space with the property of Theorem 1. In our example, \( \alpha_2 = \frac{1}{11} \), which coincides with the bound from Lemma 2.

2. The author is unable to apply the scheme from Section 2 to every \( q < 1 \), although it is very likely the same idea works. For instance, getting an example for \( q = 3/4 \) is a matter of calculating the eighth derivative of the function \( (x_1^2 + \cdots + x_n^2)^k \).

Theorem 1 shows also that the spaces \( L_q \) with \( q < 1 \) may have different Banach subspaces. The Banach subspaces of \( L_{1/2} \) constructed in Theorem 1 cannot be isometric to subspaces of \( L_q \) for all \( q < 1 \). In fact, if a Banach space \( (X, \| \cdot \|) \) is isometric to a subspace of \( L_q \) for every \( q < 1 \), then, by a theorem from [1], the function \( \exp(-\|x\|^q) \) is positive definite for every \( q < 1 \). The function \( \exp(-\|x\|) \)
is then positive definite as a pointwise limit of positive definite functions, and the space $X$ (by the same result from [1]) is isometric to a subspace of $L_1$.

On the other hand, we can show the difference directly. For every $\lambda > 0$ define a function $N_\lambda$ on $\mathbb{R}^3$ by

$$N_\lambda(x) = \left( x_1^2 + x_2^2 + x_3^2 \right)^{1/2} \left( 1 + \lambda \frac{x_1^2 + x_2^2 - 2x_3^2}{x_1^2 + x_2^2 + x_3^2} \right)^4, \quad x \in \mathbb{R}^3.$$

The proof of the following theorem is similar to the proofs of Lemma 2 and Theorem 1.

**Theorem 2.** The function $N_\lambda$ is a norm if and only if $\lambda \leq \frac{1}{23}$. The corresponding Banach space $X_\lambda$ is isometric to a subspace of $L_{1/4}$ if and only if $\lambda \leq \frac{1}{20}$, and $X_\lambda$ is not isometric to a subspace of $L_{1/2}$ if $\lambda > \frac{1}{28}$. Thus, for $\frac{1}{28} < \lambda \leq \frac{1}{26}$, the space $X_\lambda$ is a Banach subspace of $L_{1/4}$ which does not embed isometrically in $L_{1/2}$.

**References**