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ON A QUESTION OF MAKAR-LIMANOV

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ABSTRACT. Let K be an uncountable field, let $K \subset F$ be a field extension, and let A be an associative K-algebra. We show that if $F \otimes_K A$ contains a non-commutative free algebra, then so does A.

Throughout this note K will be a field and K_0 will be the prime subfield of K. Let A be an associative K-algebra. By this we mean, in particular, that K is contained in the center of A and $1_K = 1_A$. We would like to know whether or not A contains (a) a free semi-group and (b) a free K_0-algebra. Both of these free objects are presumed to be non-commutative on two generators. For a more detailed discussion of free subobjects of associative algebras we refer the reader to [L1], [L2], [LM] and [K].

The following result was conjectured by Makar-Limanov.

Conjecture 1. *Let D be a skew field, let K be a subfield of its center, and let F be a field extension of K. If $F \otimes_K D$ contains a free K_0-algebra, then so does D.*

In this note we prove this conjecture under the additional assumption that K is an uncountable field. Our main result is the following theorem.

Theorem 1. *Let K be an uncountable field, A an associative K-algebra, and F a field extension of K. Denote the common prime field of K and F by K_0.*

(a) *If $F \otimes_K A$ contains a copy of the free semi-group, then so does A.*

(b) *If $F \otimes_K A$ contains a copy of the free K_0-algebra, then so does A.*

We remark that by [LM, Lemma 1] elements $x, y \in A$ generate a free subalgebra over K_0 if and only if they generate a free subalgebra over K. Note that since we are assuming $1_A = 1_K$, the argument of [LM, Lemma 1] goes through even if A is not a domain.

Our proof of Theorem 1 is based on a general position argument. The condition that a pair of elements generates a free object in A is given by a countable number of polynomial inequalities; see Lemmas 2 and 3. Thus over an uncountable field the set of all such pairs behaves very much like an open set in the Zariski topology. In particular, we can prove the existence of a K-point by exhibiting an F-point.

We now make these ideas precise.

Lemma 1. *Let K be an uncountable field and let $X_1, X_2, \dots$ be a countable number of Zariski closed subsets of K^n. If $\bigcup_{i=1}^{\infty} X_i = K^n$, then $X_i = K^n$ for some $i \geq 1$.*

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Proof. We proceed by induction on n. If $n = 0$, there is nothing to prove. Let $n \geq 1$ be the smallest dimension where the lemma fails. Choose Zariski closed subsets $X_i \neq K^n$ such that $\bigcup_{i=1}^{\infty} X_i = K^n$. Note that each X_i contains at most a finite number of hyperplanes. Since K is uncountable, we can choose a hyperplane H which is not contained in X_i for any $i \geq 1$. Let $Y_i = X_i \cap H$. Then $H = \bigcup_{i=1}^{\infty} Y_i$. By our choice of H, $Y_i \neq H$ for any $i \geq 1$. Since each Y_i is closed in $H \simeq K^{n-1}$, this contradicts the minimality of n. $\square$

Let A be an associative K-algebra and let F be a field extension of K. If $z \in F$ and $a \in A$ we shall write $za \in F \otimes_K A$ instead of $z \otimes a$. Let $a_1, \dots, a_r, b_1, \dots, b_s \in A$. For $x = (x_1, \dots, x_r) \in F^r$ and $y = (y_1, \dots, y_s) \in F^s$ let $a_x = \sum_{i=1}^{r} x_i a_i$ and $b_y = \sum_{j=1}^{s} y_j b_j$ be elements of $F \otimes_K A$. In the sequel we shall denote the free associative K-algebra in two variables by $K\{z, w\}$.

Lemma 2. *Let $f \in K\{z, w\}$ be a polynomial in two variables. Then the pairs (x, y), such that $f(a_x, b_y) = 0$, form a Zariski closed subset of F^{r+s} defined over K.*

Proof. Denote the degree of f by d. Let V be the K-vector subspace of A spanned by all monomials of degree $\leq d$ in $a_1, \dots, a_r, b_1, \dots, b_s$. Let $e_1, \dots, e_n$ be a K-basis for V. Then we can write

$$f(a_x, b_y) = \sum_{i=1}^{n} p_i(x, y) e_i \ ,$$

where each p_i is a (commutative) polynomial in $r + s$ variables with coefficients in K. Thus $f(a_x, b_y) = 0$ if and only if $p_i(x, y) = 0$ for every $i = 1, \dots, n$. $\square$

Lemma 3. *Let $f_1, \dots, f_m \in K\{z, w\}$ be polynomials in two variables. Then the pairs (x, y), such that $f_1(a_x, b_y), \dots, f_m(a_x, b_y)$ are F-linearly dependent, form a Zariski closed subset of F^{r+s} defined over K.*

Proof. Let d be the maximum of the degrees of $f_1, \dots, f_m$ and let $e_1, \dots, e_n$ be a basis of the K-vector subspace of A spanned by all monomials of degree $\leq d$ in $a_1, \dots, a_r, b_1, \dots, b_s$. Then for $i = 1, \dots, m$ we can write

$$f_i(a_x, b_y) = \sum_{j=1}^{n} p_{ij}(x, y) e_j \ ,$$

where each p_{ij} is a (commutative) polynomial in $r+s$ variables defined over K. Thus $f_1(a_x, b_y), \dots, f_m(a_x, b_y)$ are F-linearly dependent if and only if the $m \times n$ matrix $(p_{ij}(x, y))$ has rank $\leq m-1$. This is equivalent to the vanishing of all $m \times m$ minors of this matrix. Each minor is a (commutative) polynomial in $x_1, \dots, x_r, y_1, \dots, y_s$ defined over K. $\square$

We are now ready to finish the proof of Theorem 1.

Proof of Theorem 1. (a) Suppose the elements

$$a_u = \sum_{i=1}^{r} u_i a_i \quad \text{and} \quad b_v = \sum_{j=1}^{s} v_j b_j$$

generate a free semi-group in $F \otimes_K A$ for some $u = (u_1, \dots, u_r) \in F^r$, $v = (v_1, \dots, v_s) \in F^s$ and $a_i, b_j \in A$. We shall write $a_x = \sum_{i=1}^{r} x_i a_i \in F \otimes_K A$ and

$b_y = \sum_{j=1}^{s} y_j b_j \in F \otimes_K A$ where $x = (x_1, \dots, x_r) \in F^r$ and $y = (y_1, \dots, y_s) \in F^s$ as before. Note that if $x \in K^r$, then $a_x \in A$. Similarly if $y \in K^s$, then $b_y \in A$.

Let M, N be distinct monomials in two variables. Denote by $X_{M,N}$ the set of all pairs $(x, y) \in F^{r+s}$ such that $M(a_x, b_y) = N(a_x, b_y)$. As usual, the set of K-points of $X_{M,N}$ will be denoted by $X_{M,N}(K)$. That is, $X_{M,N}(K) = X_{M,N} \cap K^{r+s}$.

By Lemma 2, $X_{M,N}$ is a closed subset of F^{r+s} defined over K. We want to show that a_x and b_y generate a free semi-group for some $(x, y) \in K^{r+s}$. Assume the contrary: for every $(x, y) \in K^{r+s}$ there exists a pair of distinct monomials M, N so that $M(a_x, b_y) = N(a_x, b_y)$. In other words, K^{r+s} is the union of $X_{M,N}(K)$ as M, N range over the (countable) set of pairs of distinct monomials in two variables. By Lemma 1, $X_{M_0,N_0}(K) = K^{r+s}$ for some distinct monomials M_0 and N_0. Since K^{r+s} is Zariski dense in F^{r+s}, this implies $X_{M_0,N_0} = F^{r+s}$. In other words, (a_x, b_y) satisfy $M_0(a_x, b_y) = N_0(a_x, b_y)$ for every $(x, y) \in F^{r+s}$. This contradicts our assumption that a_u and b_v generate a free semi-group in $F \otimes_K A$.

(b) We use a similar argument, with Lemma 2 replaced by Lemma 3. Suppose the elements

$$a_u = \sum_{i=1}^{r} u_i a_i \quad \text{and} \quad b_v = \sum_{j=1}^{s} v_j b_j$$

generate a free K_0-subalgebra of $F \otimes_K A$ for some $u = (u_1, \dots, u_r) \in F^r$, $v = (v_1, \dots, v_s) \in F^s$ and $a_i, b_j \in A$. As we remarked after the statement of Theorem 1, these elements also generate a free F-subalgebra.

We shall use the symbols a_x and b_y as above. For $d \geq 1$ let $X_d \in F^{r+s}$ be the set of all pairs (x, y) such that the monomials in a_x, b_y of degree $\leq d$ are F-linearly dependent. By Lemma 3, X_d is a closed subset of F^{r+s} defined over K. We want to show that a_x and b_y generate a free K-algebra for some $(x, y) \in K^{r+s}$. Assume the contrary: for every $(x, y) \in K^{r+s}$ the elements a_x and b_y satisfy some non-zero polynomial in two variables. If the degree of this polynomial is d, then $(a_x, b_y) \in X_d$. In other words, K^{r+s} is the union of $X_d(K)$ as d ranges over the positive integers. By Lemma 1, $X_n(K) = K^{r+s}$ for some integer n. Since K^{r+s} is Zariski dense in F^{r+s}, this implies $X_n = F^{r+s}$. That is, for every $(x, y) \in F^{r+s}$ the elements a_x and b_y satisfy some polynomial of degree n over F. This contradicts our assumption that a_u and b_v generate a free subalgebra of $F \otimes_K A$. $\square$

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