

SEMINORMALITY OF OPERATORS FROM THEIR TENSOR PRODUCT

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(Communicated by Palle E. T. Jorgensen)

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ABSTRACT. The question of seminormality of tensor products of nonzero bounded linear operators on Hilbert spaces is investigated. It is shown that $A \otimes B$ is subnormal if and only if so are A and B .

INTRODUCTION

It is well known that some properties of Hilbert space operators are preserved by the operation of taking the two-fold tensor product. However, it may happen that $A \otimes B$ possesses a property P , though A and B do not (mainly because $A \otimes B = (cA) \otimes (c^{-1}B)$ for any nonzero $c \in \mathbb{C}$). The aim of this paper is to show that if P is related to seminormality, then the tensor product $A \otimes B$ of nonzero bounded linear operators possesses the property P if and only if so do cA and $c^{-1}B$ for some nonzero $c \in \mathbb{C}$. Since the same can be done for operators whose adjoints have the property P , we can concentrate on a more narrow class of properties. Let us recall some of them (cf. [2, 3, 7]).

We say that a bounded linear operator A on a complex Hilbert space is quasi-normal (resp. subnormal, hyponormal) if $AA^*A = A^*A^2$ (resp. A is a restriction of a bounded normal operator to its closed invariant subspace, $AA^* \leq A^*A$).

1. PRELIMINARIES

A sequence $\{\alpha_n\}_{n=0}^{\infty}$ of real numbers is said to be a *Stieltjes moment sequence* if there exists a positive Borel measure μ on the closed half-line $[0, +\infty)$ such that

$$(1.1) \quad \alpha_n = \int_0^{+\infty} t^n d\mu(t), \quad n \geq 0.$$

The measure μ is called a *representing measure* of $\{\alpha_n\}_{n=0}^{\infty}$.

Recall two basic properties of Stieltjes moment sequences (cf. [1] and [9]).

(S1) If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are Stieltjes moment sequences, then so is their product $\{\alpha_n \beta_n\}_{n=0}^{\infty}$.

Received by the editors July 6, 1994.

1991 *Mathematics Subject Classification*. Primary 47B20; Secondary 47A80.

Key words and phrases. Tensor product, seminormal operator, subnormal operator, Stieltjes moment sequence.

This work was supported by a grant of the Komitet Badań Naukowych, Warsaw.

(S2) The class of all Stieltjes moment sequences is closed with respect to the topology of pointwise convergence.

The following two lemmas play the crucial role in this paper.

Lemma 1.1. *If $\{\alpha_n\}_{n=0}^\infty$ is a Stieltjes moment sequence, then*

- (i) $\alpha_{n+1}^2 \leq \alpha_n \alpha_{n+2}$, $n \geq 0$,
- (ii) $\alpha_n = 0$ for every $n \geq 1$ if and only if there exists $k \geq 1$ such that $\alpha_k = 0$.

Proof. Applying the Cauchy–Schwarz inequality we get (i). (ii) follows easily from the integral representation (1.1). \square

Lemma 1.2. *Let $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be two sequences of positive real numbers. If $p = \limsup \alpha_{n+1} \alpha_n^{-1} < +\infty$ and $q = \limsup \beta_{n+1} \beta_n^{-1} < +\infty$, then the following conditions are equivalent:*

- (i) $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are Stieltjes moment sequences.
- (ii) For every integer $s \geq 0$, $\{\alpha_n \beta_{n+s}\}_{n=0}^\infty$ and $\{\alpha_{n+s} \beta_n\}_{n=0}^\infty$ are Stieltjes moment sequences.

Proof. The implication (i) \Rightarrow (ii) follows from (S1).

(ii) \Rightarrow (i). Put $p_n = \alpha_{n+1} \alpha_n^{-1}$ and $q_n = \beta_{n+1} \beta_n^{-1}$, $n \geq 0$. Applying Lemma 1.1(i) to the Stieltjes moment sequence $\{\alpha_n \beta_{n+s}\}_{n=0}^\infty$ ($s \geq 0$), we get

$$(1.2) \quad p_n q_{n+s} \leq p_{n+1} q_{n+s+1}, \quad n \geq 0, s \geq 0.$$

Thus the sequence $\{p_n q_n\}_{n=0}^\infty$ is weakly increasing, which in turn implies

$$0 < \lim_{n \rightarrow \infty} p_n q_n \leq \limsup_{n \rightarrow \infty} p_n \limsup_{n \rightarrow \infty} q_n = p q.$$

This means that $0 < p, q < +\infty$. It follows from (1.2) that

$$p_n = q^{-1} \limsup_{s \rightarrow \infty} p_n q_{n+s} \leq q^{-1} \limsup_{s \rightarrow \infty} p_{n+1} q_{n+s+1} = p_{n+1}, \quad n \geq 0.$$

Since the sequence $\{p_n\}_{n=0}^\infty$ is weakly increasing, there exists $\lim_{n \rightarrow \infty} p_n = p$. Thus

$$\lim_{s \rightarrow \infty} \frac{\alpha_{n+s}}{\alpha_s} = \lim_{s \rightarrow \infty} (p_s p_{s+1} \cdots p_{s+n-1}) = p^n, \quad n \geq 0,$$

which gives us

$$p^n \beta_n = \lim_{s \rightarrow \infty} \left(\frac{\alpha_{n+s}}{\alpha_s} \beta_n \right), \quad n \geq 0.$$

It follows from (S2) that $\{p^n \beta_n\}_{n=0}^\infty$ and consequently $\{\beta_n\}_{n=0}^\infty$ are Stieltjes moment sequences. Similar arguments apply to the sequence $\{\alpha_n\}_{n=0}^\infty$. This completes the proof. \square

2. MAIN RESULTS

From now on, H and K are fixed complex Hilbert spaces and $\mathcal{B}(H)$ stands for the C^* -algebra of all bounded linear operators on H .

Let us begin with proving two results that will be of use later.

Proposition 2.1. *If $A_1, A_2 \in \mathcal{B}(H)$ and $B_1, B_2 \in \mathcal{B}(K)$ are nonzero operators, then $A_1 \otimes B_1 = A_2 \otimes B_2$ if and only if there exists $c \in \mathbb{C} \setminus \{0\}$ such that $A_2 = c A_1$ and $B_2 = c^{-1} B_1$.*

Proof. First we show that if $f_1 \otimes g_1 = f_2 \otimes g_2 \neq 0$, then there exists $c \in \mathbb{C} \setminus \{0\}$ such that $f_2 = c f_1$ and $g_2 = c^{-1} g_1$. Indeed, due to our assumption $(f_1, \cdot) g_1 = (f_2, \cdot) g_2$ and $(g_1, \cdot) f_1 = (g_2, \cdot) f_2$ and consequently there are $c, d \neq 0$ such that $f_2 = c f_1$ and $g_2 = d g_1$. Substituting these vectors into the original equality we get $d = c^{-1}$.

Assume that $A_1 f \otimes B_1 g = A_2 f \otimes B_2 g$ for all $f \in H, g \in K$. Then one can show that $\ker A_1 = \ker A_2$ and $\ker B_1 = \ker B_2$. In virtue of the previous paragraph, we can attach to each pair of nonzero vectors $f \in H \ominus \ker A_1$ and $g \in K \ominus \ker B_1$ a unique nonzero complex scalar c such that $A_2 f = c A_1 f$ and $B_2 g = c^{-1} B_1 g$. It is easily seen that c does not depend on f 's and g 's, which completes the proof. \square

Proposition 2.2. *Let $A_1, A_2 \in \mathcal{B}(H)$ and $B_1, B_2 \in \mathcal{B}(K)$ be nonnegative operators. If $A_1 \neq 0$ and $B_1 \neq 0$, then the following conditions are equivalent:*

- (i) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.
- (ii) *There exists $c > 0$ such that $A_1 \leq c A_2$ and $B_1 \leq c^{-1} B_2$.*

Proof. Denote by C_j and D_j the square roots of A_j and B_j , respectively ($j = 1, 2$).

(ii) \Rightarrow (i). Since $\|C_1 f\|^2 \leq c \|C_2 f\|^2$ for $f \in H$, there exists $S \in \mathcal{B}(H)$ such that $\|S\| \leq c^{1/2}$ and $C_1 = S C_2$ (cf. [2], Lemma 1.1). Likewise there exists $T \in \mathcal{B}(K)$ such that $\|T\| \leq c^{-1/2}$ and $D_1 = T D_2$. Consequently

$$\begin{aligned} (A_1 \otimes B_1 h, h) &= \|C_1 \otimes D_1 h\|^2 = \|(S \otimes T)(C_2 \otimes D_2) h\|^2 \\ &\leq \|S\|^2 \|T\|^2 \|(C_2 \otimes D_2) h\|^2 \leq (A_2 \otimes B_2 h, h), \quad h \in H \otimes K. \end{aligned}$$

(i) \Rightarrow (ii). It follows from (i) that

$$(2.1) \quad \|C_1 f\| \|D_1 g\| \leq \|C_2 f\| \|D_2 g\|, \quad f \in H, g \in K.$$

First notice that $\ker C_2 \subseteq \ker C_1$ and $\ker D_2 \subseteq \ker D_1$. Indeed, if $C_2 f = 0$, then by (2.1) we have $\|C_1 f\| \|D_1 g\| = 0$ for every $g \in K$. However $D_1 \neq 0$, so $C_1 f = 0$ (similarly we show the other inclusion). The above remark and (2.1) imply that there exists $0 < c < +\infty$ such that

$$\sup_{C_1 f \neq 0} \|C_1 f\| \|C_2 f\|^{-1} \leq c^{1/2} \leq \inf_{D_1 g \neq 0} \|D_2 g\| \|D_1 g\|^{-1}.$$

Hence

$$(A_1 f, f) = \|C_1 f\|^2 \leq c \|C_2 f\|^2 = c (A_2 f, f), \quad f \in H.$$

Similarly we show that $(B_1 g, g) \leq c^{-1} (B_2 g, g)$ for $g \in K$, which completes the proof. \square

In the proof of the main result of this paper we will make use of the following version of Lambert's criterion for subnormality of bounded operators.

Proposition 2.3. *If $A \in \mathcal{B}(H)$, then the following conditions are equivalent:*

- (i) *A is subnormal.*
- (ii) *For every $f \in H$, $\{\|A^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence.*
- (iii) *For every $f \in H \ominus \ker A$, $\{\|A^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence.*

Proof. The equivalence (i) \Leftrightarrow (ii) is essentially due to Lambert (cf. [6, 10]).

(iii) \Rightarrow (ii). If $f \in H$, then $f = g \oplus h$, where $g \in \ker A$, $h \in H \ominus \ker A$. Therefore $\|A^n f\|^2 = \int_0^\infty t^n d(\mu + \|g\|^2 \delta_0)(t)$ for $n \geq 0$, where μ is a representing measure of $\{\|A^n h\|^2\}_{n=0}^\infty$ and δ_0 is the probability measure concentrated on $\{0\}$. This completes the proof. \square

We are now in a position to prove the main result of the paper.

Theorem 2.4. *If $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ are nonzero operators, then*

- (i) *$A \otimes B$ is isometric (resp. unitary, selfadjoint, nonnegative) if and only if so are cA and $c^{-1}B$ for some $c \in \mathbb{C} \setminus \{0\}$,*
- (ii) *$A \otimes B$ is normal (resp. quasinormal, subnormal, hyponormal) if and only if so are A and B .*

Proof. (i) Among the properties mentioned in (i), only the positivity needs some explanation, as the other ones can be easily deduced from Proposition 2.1 (see also *Normality* below).

Positivity. If $A \geq 0$ and $B \geq 0$, then $A \otimes B = (A^{1/2} \otimes B^{1/2})^2 \geq 0$. Conversely, if $A \otimes B \geq 0$, then by (i) there exists $c \neq 0$ such that cA and $c^{-1}B$ are selfadjoint. Therefore we can assume without loss of generality that A and B are selfadjoint and

$$(2.2) \quad (Af, f)(Bg, g) \geq 0, \quad f \in H, g \in K.$$

Since $A \neq 0$, there exists f_0 such that $(Af_0, f_0) \in \mathbb{R} \setminus \{0\}$. If $(Af_0, f_0) > 0$, then by (2.2) we have $B \geq 0$. However, $B \neq 0$, so there exists g_0 such that $(Bg_0, g_0) > 0$. Applying once more (2.2) we obtain $A \geq 0$. The other case $(Af_0, f_0) < 0$ leads to $-A \geq 0$ and $-B \geq 0$.

(ii) *Normality.* If $A \otimes B$ is normal, then by Proposition 2.1 there exists $c \neq 0$ such that $A^*A = cAA^*$ and $B^*B = c^{-1}BB^*$. Consequently $c > 0$, $\|A\|^2 = c\|A\|^2$ and $\|B\|^2 = c^{-1}\|B\|^2$, which is possible only if $c = 1$. Thus A and B are normal. The converse implication is obvious.

Quasinormality. If $A \otimes B$ is quasinormal, then by Proposition 2.1 there exists $c \neq 0$ such that $AA^*A = cA^*A^2$ and $BB^*B = c^{-1}B^*B^2$. This in turn implies that $(A^*A)^2 = c(A^2)^*A^2$ and $(B^*B)^2 = c^{-1}(B^2)^*B^2$. Consequently $c > 0$, $\|A\|^4 = c\|A^2\|^2 \leq c\|A\|^4$ and $\|B\|^4 = c^{-1}\|B^2\|^2 \leq c^{-1}\|B\|^4$, which yields $c = 1$. Therefore A and B are quasinormal. The converse implication is obvious.

Subnormality. Assume that $A \otimes B$ is subnormal. Put $\alpha_n(f) := \|A^n f\|^2$ and $\beta_n(g) := \|B^n g\|^2$ for $f \in H$, $g \in K$ and $n \geq 0$. Then $\alpha_{n+i}(f)\beta_{n+j}(g) = \|(A \otimes B)^n (A^i f \otimes B^j g)\|^2$ for $i, j \geq 0$, so by Proposition 2.3 we have

$$(2.3) \quad \{\alpha_{n+i}(f)\beta_{n+j}(g)\}_{n=0}^\infty \text{ is a Stieltjes moment sequence} \\ \text{for all } f \in H, g \in K \text{ and } i, j \geq 0.$$

Note that if $f \notin \ker A$ and $g \notin \ker B$, then $\alpha_n(f) > 0$ and $\beta_n(g) > 0$ for every $n \geq 0$. Indeed, if $\alpha_k(f) = 0$ for some $k \geq 1$ ($\alpha_0(f) \neq 0$!), then by $B \neq 0$ there exists $h \notin \ker B$. It follows from (2.3) and Lemma 1.1(ii) that $\|Af\|^2\|Bh\|^2 = \alpha_1(f)\beta_1(h) = 0$, which contradicts $Af \neq 0$ and $Bh \neq 0$. Similar arguments apply to $\{\beta_n(g)\}_{n=0}^\infty$.

Take arbitrary nonzero vectors $f \in H \ominus \ker A$ and $g \in K \ominus \ker B$. Then $\limsup_{n \rightarrow \infty} \alpha_{n+1}(f) \alpha_n^{-1}(f) \leq \|A\|^2 < +\infty$ and $\limsup_{n \rightarrow \infty} \beta_{n+1}(g) \beta_n^{-1}(g) \leq \|B\|^2 < +\infty$. This and condition (2.3) imply—via Lemma 1.2—that $\{\alpha_n(f)\}_{n=0}^\infty$ and $\{\beta_n(g)\}_{n=0}^\infty$ are Stieltjes moment sequences. Hence, by Proposition 2.3, A and B are subnormal. The converse implication is obvious.

Hyponormality. The sufficiency can be directly derived from Proposition 2.2. If $A \otimes B$ is hyponormal, then $(A A^* \otimes B B^*) \leq A^* A \otimes B^* B$ and consequently—by Proposition 2.2—there exists $c > 0$ such that $A A^* \leq c A^* A$ and $B B^* \leq c^{-1} B^* B$. Hence $\|A\|^2 \leq c \|A\|^2$ and $\|B\|^2 \leq c^{-1} \|B\|^2$. Since both A and B are nonzero, we come to the conclusion that $c = 1$. This means that A and B are hyponormal. \square

3. COMMENTS AND CONCLUDING REMARKS

The following example shows that the condition (ii) of Lemma 1.2 cannot be weakened.

Example 3.1. For every integer $s \geq 1$, there exist two sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ of positive real numbers such that

- (i) $\limsup_{n \rightarrow \infty} \alpha_{n+1} \alpha_n^{-1} < +\infty$ and $\limsup_{n \rightarrow \infty} \beta_{n+1} \beta_n^{-1} < +\infty$,
- (ii) $\{\alpha_n\}_{n=0}^\infty$ is not a Stieltjes moment sequence,
- (iii) for every $j = 0, 1, \dots, s$, $\{\alpha_n \beta_{n+j}\}_{n=0}^\infty$ and $\{\alpha_{n+j} \beta_n\}_{n=0}^\infty$ are Stieltjes moment sequences.

Take an arbitrary Stieltjes moment sequence $\{p_n\}_{n=0}^\infty$ of positive real numbers. Choose α_0 and β_0 so that $\alpha_0 \beta_0 = p_0 \cdots p_{s-1}$. Put $\alpha_n = \beta_0^{-1} p_0 \cdots p_{n+s-1}$ and $\beta_n = \beta_0 (p_0 \cdots p_{n-1})^{-1}$ for $n \geq 1$. Then $\{\alpha_n \beta_{n+j}\}_{n=0}^\infty$ is the constant sequence for $j = s$ and the finite product $\prod_{k=j}^{s-1} p_{n+k}$ of Stieltjes moment sequences for $j < s$. It follows from (S1) that $\{\alpha_n \beta_{n+j}\}_{n=0}^\infty$, $j = 0, 1, \dots, s$, are Stieltjes moment sequences. The same reasoning applies to the sequences $\{\alpha_{n+j} \beta_n\}_{n=0}^\infty$, $j = 0, 1, \dots, s$. If additionally there exist $\delta > 0$ and $M > 0$ such that $\delta \leq p_n \leq M$ for all $n \geq 0$ (or equivalently $\{p_n\}_{n=0}^\infty$ is a Hausdorff moment sequence whose representing measure has 1 as an atom), then (i) holds. To get (ii) it is sufficient to assume that $\{p_n\}_{n=0}^\infty$ is strictly decreasing. Indeed, if $\{\alpha_n\}_{n=0}^\infty$ were a Stieltjes moment sequence, the sequence $\{\alpha_{n+1} \alpha_n^{-1}\}_{n=0}^\infty$ would be weakly increasing (apply Lemma 1.1(i)), which would contradict the equality $\alpha_{n+1} \alpha_n^{-1} = p_{n+s}$, $n \geq 0$. The sequence $p_n := 1 + (n+1)^{-1}$ meets all the requirements.

A part of Theorem 2.4(ii) concerning subnormality of $A \otimes B$ can be directly inferred from the definition of subnormality, provided A and B have nonempty point spectrum. Among operators with empty point spectrum there are injective weighted shifts. This particular case is precisely covered by Lemma 1.2. Indeed, since $A \otimes B$ is unitarily equivalent to $(\sum_{k=0}^\infty \oplus A_k) \oplus (\sum_{k=1}^\infty \oplus B_k)$, where A , B , A_k and B_k are weighted shifts with weights $\{\lambda_n\}_{n=0}^\infty$, $\{\mu_n\}_{n=0}^\infty$, $\{\lambda_{n+k} \mu_n\}_{n=0}^\infty$ and $\{\lambda_n \mu_{n+k}\}_{n=0}^\infty$, respectively, we can apply the Berger-Gellar-Wallen criterion for subnormality of weighted shifts (cf. [4, 5, 8]).

We conclude the paper with the following

Open Question. Is it true that if the two-fold symmetric tensor product of a nonzero bounded linear operator A by itself is subnormal, then so is A ?

We know the affirmative answer for injective weighted shifts. Indeed, in this particular case the symmetric tensor product of A by itself is unitarily equivalent

to the orthogonal sum $\sum_{k=0}^{\infty} \bigoplus S_k$ of weighted shifts S_k with weights $\{\lambda_n \lambda_{n+k}\}_{n=0}^{\infty}$, $k \geq 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ are weights of A . Applying the Berger-Gellar-Wallen theorem and Lemma 1.2 we get the conclusion.

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