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ANY BEHAVIOUR OF THE MITCHELL ORDERING OF NORMAL MEASURES IS POSSIBLE

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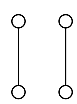
ABSTRACT. Let U_0, U_1 be two normal measures on κ. We say that U_0 is in the Mitchell ordering less than U_1, $U_0 \lhd U_1$, if $U_0 \in \mathrm{Ult}(V, U_1)$. The relation is well-known to be transitive and well-founded. It has been an open problem to find a model where $\lhd$ embeds the four-element poset . We find a generic extension where all well-founded posets are embeddable. Hence there is no structural restriction on the Mitchell ordering. Moreover we show that it is possible to have two $\lhd$-incomparable measures that extend in a generic extension into two $\lhd$-comparable measures.

We address the question of possible behaviours of the Mitchell ordering of normal measures.

In the well-known Mitchell's model $L[\overrightarrow{\mathcal{U}}]$ the ordering of measures on a cardinal κ is linear [Mi83]. S. Baldwin constructed a model where $\lhd$ is a pre-well-ordering [Ba85] (a well-founded poset is *pre-well-ordered* iff $\forall p, q \in P : p <_P q$ iff $o_P(p) < o_P(q)$ where $o_P(p)$ is the rank of p in P). Recently J. Cummings [Cu93] described the Mitchell ordering in a particular generic extension where it embeds any well-founded poset that does not embed the four-element poset:



We say that a well-founded poset P *embeds into the Mitchell ordering of normal measures on* κ if there are different measures $\{U_p; p \in P\}$ on κ so that $U_p \lhd U_q$ iff $p <_P q$. I show that there is a generic extension where any well-founded poset (with certain cardinality restrictions) embeds into $\lhd$. It proves that there is no structural restriction on the Mitchell ordering. However it still remains open to construct a model where all measures on κ ordered by the Mitchell ordering are isomorphic to a given poset, e.g. there are only four measures ordered according to the figure above. That would certainly need to go into inner models, possibly

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first starting with a generic extension. S. Baldwin in [Ba85] notes that his method hopelessly fails especially for posets that embed the four-element poset.

We say that a function $f : \kappa \to V_\kappa$ is *a Laver's function on* κ if

$$\forall x \in \mathcal{P}(\kappa^+) \exists U \text{ a measure on } \kappa : (j_U f)(\kappa) = x$$

where $j_U : V \to M_U = \mathrm{Ult}(V, U)$ is the canonical ultraproduct embedding. It is proved in [W94a] that there is a Laver's function on κ if $V = L[\overrightarrow{\mathcal{U}}]$ and $o^{\mathcal{U}}(\kappa) = \kappa^{++}$. It follows from the fact that the measures on κ cover $\mathcal{P}(\kappa^+)$ in the following sense

$$\forall x \in \mathcal{P}(\kappa^+) \exists \alpha < \kappa^{++} : x \in \mathrm{Ult}(V, U^\kappa_\alpha).$$

Theorem 1. *Assume that* $V = L[\overrightarrow{\mathcal{U}}]$, $o^{\mathcal{U}}(\kappa) = \kappa^{++}$. *Then there is a generic extension* $V[G]$ *preserving cardinalities, cofinalities, and GCH such that any well-founded poset in* $V[G]$ *of cardinality* $\leq \kappa^+$ *embeds into* $\lhd_\kappa^{V[G]}$. *Moreover there is still a Laver's function on* κ.

Let me first state a few well-known facts (see e.g. [Cu93] or [W94a]).

Fact 1. *Let* $U \lhd W$ *be two measures on* κ, $j_U : V \to M_U$, $j_W : V \to M_W$, *and* $j_U^{M_W} : M_W \to M'_U = \mathrm{Ult}(M_W, U)$. *Then* $j_U^{M_W} = j_U \restriction M_W$ *and* $V_{j_U(\kappa)+1} \cap M'_U = V_{j_U(\kappa)+1} \cap M_U$.

Fact 2. *Let* M, N *be two inner models of* ZFC, $M \subseteq N$, $N \cap {}^\kappa M \subseteq M$. *Let* $P \in M$ *be a forcing notion such that* $N \models$ *"P is κ^+-c.c. or P is κ-closed", and let* G *be* P*-generic over* N. *Then* $N[G] \cap {}^\kappa M[G] \subseteq M[G]$.

Fact 3 (Easton's lemma). *Let a notion of forcing* P *be* κ*-c.c., and let* Q *be* $< \kappa$*-closed. Then*
(i) $Q \Vdash$ *"$\check{P}$ is $\check{\kappa}$-c.c.",*
(ii) $P \Vdash$ *"$\check{Q}$ is $< \check{\kappa}$-distributive",*
(iii) *Generics over* V *for* P *and* Q *are mutually generic.*

Let $\mathrm{Add}(\alpha, \kappa)$ denote the Cohen forcing adding α Cohen subsets of κ : a condition $p \in \mathrm{Add}(\alpha, \kappa)$ is a function $p : \mathrm{dom}\,(p) \to \{0, 1\}$ such that $\mathrm{dom}\,(p) \subseteq \alpha \times \kappa$, $|\mathrm{dom}\,(p)| < \kappa$, $p < q$ iff $p \supseteq q$. If κ is a regular cardinal, then $\mathrm{Add}(\alpha, \kappa)$ is κ^+-c.c. and $< \kappa$-closed. An $\mathrm{Add}(\alpha, \kappa)$-generic filter G is best represented by a subset of $\alpha \times \kappa$. If $A \subseteq \alpha$, then $G \cap (A \times \kappa)$ naturally gives an $\mathrm{Add}(\mathrm{o.t.}(A), \kappa)$-generic filter. Any bijection $f : \alpha \times \kappa \to \beta \times \kappa$ gives an isomorphism of $\mathrm{Add}(\alpha, \kappa)$ and $\mathrm{Add}(\beta, \kappa)$; if $G \subseteq \alpha \times \kappa$ is $\mathrm{Add}(\alpha, \kappa)$-generic, then $f[G]$ is $\mathrm{Add}(\beta, \kappa)$-generic.

Proof of Theorem 1. Let P_κ be an Easton product of $\langle \mathrm{Add}(1, \lambda^+); \lambda \in \mathrm{Reg}(\kappa) \rangle$; we use the trivial forcing if $\lambda < \kappa$ is not a regular cardinal. Since P_κ is a direct limit, it can be considered to be a subset of V_κ. For regular λ factor P_κ as $P_\lambda \times P_{\lambda,\kappa}$; then P_λ is λ^+-c.c. and $P_{\lambda,\kappa}$ is λ-closed. Consequently by Easton's Lemma

$$P_\lambda \Vdash \text{“}\check{P}_{\lambda,\kappa} \text{ is } \check{\lambda}\text{-distributive”}.$$

It follows that P_κ preserves cardinalities, cofinalities, and GCH. As a matter of fact P_κ is the well-known Kunen-Paris forcing [KuP71]. Let G be P_κ-generic over V.

Assume $P \in V[G]$ is a given well-founded poset of cardinality $\leq \kappa^+$. I claim that P can be embedded into $\lhd_\kappa^{V[G]}$.

Claim 1. *Let $U_0 \in V$ be a measure on κ, $j_0 : V \to \mathrm{Ult}(V, U_0) = M_0$. Then j_0 can be lifted to an elementary embedding $j_0^* : V[G] \to M_0[G \times \tilde{H}_0]$, $j_0^*(G) = G \times \tilde{H}_0$, uniquely determined by $\tilde{H}_0 \in V[G]$.*

Proof. Factor $j_0(P_\kappa)$ as $P_\kappa \times R$ where $R = (j_0 P_\kappa)_{\kappa, j_0\kappa}$. Since P_κ is a direct limit, it is enough to find an R-generic filter $\tilde{H}_0 \in V$ over M_0. By Easton's Lemma $\tilde{H}_0$ is then R-generic over $M_0[G]$. The number of R-antichains in M_0 is $j_0(\kappa^+)$, hence the number of these antichains computed in V is only κ^+. Moreover R is κ-closed in M_0, thus in V, enabling us to build up an $\tilde{H}_0 \in V$. □

The embedding j_0^* defines a measure $U_0^* \in V[G]$ extending U_0; it is easy to see that j_0^* is the ultraproduct embedding given by $U_0^* : M_0 = \{(j_0 g)(\kappa); g \in {}^\kappa V \cap V\}$, and it is enough to prove that $M_0[G \times \tilde{H}_0] = \{(j_0^* g)(\kappa); g \in {}^\kappa V[G] \cap V[G]\}$. Let $x = i_{G \times \tilde{H}_0}(\dot{x}) \in M_0[G \times \tilde{H}_0]$, find $g \in {}^\kappa V \cap V$ such that $(j_0 g)(\kappa) = \dot{x}$, and define $g^*(\alpha) = i_G(g(\alpha))$ if $g(\alpha)$ as a P_κ-name, then $g^* \in {}^\kappa V[G] \cap V[G]$ and $(j_0^* g^*)(\kappa) = x$.

We consider only one-step extensions of this type. Notice that the forcing R defined above is a product of forcings that always starts with $\mathrm{Add}(1, \kappa^+)^V$. Thus we can always factor $\tilde{H}_0$ as $H_0^\kappa \times H_0$ where H_0^κ is $\mathrm{Add}(1, \kappa^+)$-generic over $M_0[G]$. Now we need to find some sufficient and necessary conditions for $\lhd$ on those extensions.

Claim 2. *Let U_0^*, U_1^* be extensions of U_0, U_1 given by $H_0^\kappa \times H_0$ and $H_1^\kappa \times H_1$. If $U_0^* \lhd U_1^*$, then $U_1 \neq U_0, U_1 \not\lhd U_0$, and $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. On the other hand if $U_0 \lhd U_1$ and $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$, then $U_0^* \lhd U_1^*$.*

Proof. Assume $U_0 \lhd U_1$ and $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. Extend $j_0 \restriction M_1 = j_0' : M_1 \to M_0^1 = \mathrm{Ult}(M_1, U_0)$ to $\tilde{\jmath}_0 : M_1[G] \to M_0^1[G \times H_0^\kappa \times H_0]$ in $M_1[G \times H_1^\kappa]$. Then $\tilde{\jmath}_0 = j_0^* \restriction M_1[G]$ defines the measure U_0^* in $M_1[G \times H_1^\kappa]$ because subsets of κ are the same in $V[G]$ and $M_1[G]$.

Assume that $U_0^* \lhd U_1^*$; then $j_0(\kappa) < j_1(\kappa)$, and so U_1 cannot be less than U_0 in the Mitchell ordering. $U_0^* \in M_1[G \times H_1^\kappa]$ since the rest of the forcing $(j_0 P_\kappa)_{\kappa+1, j_0\kappa}$ is sufficiently closed. Consequently $j_0^*(G) = G \times H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. □

Let $P = (\Theta, <_P)$, $\Theta < \kappa^{++}$, so that the ordering of ordinals extends $<_P$. Using the κ-c.c. of P_κ find a P_κ-name $\dot{P} \subseteq \Theta \times \Theta \times P_\kappa$ for P of cardinality $\leq \kappa^+$. Since the measures on κ cover $\mathcal{P}(\kappa^+)$, there is a $\beta < \kappa^{++}$ so that $\dot{P} \in \mathrm{Ult}(V, U_\beta^\kappa)$ and $\Theta < \kappa^{++\mathrm{Ult}(V, U_\beta^\kappa)}$. Fix a sequence of measures $U_0 \lhd U_1 \lhd \cdots \lhd U_\alpha \lhd \cdots$ $(\alpha < o(P))$ starting with $U_0 = U_\beta^\kappa$. Denote $j_\alpha : V \to \mathrm{Ult}(V, U_\alpha) = M_\alpha$; then $P \in M_\alpha[G]$ and $\Theta < \kappa^{++M_\alpha}$ for all $\alpha < o(P)$ by the choice of U_0 and Fact 1. I am going to find $U_p^* \in V[G]$ (for $p \in P$) extending $U_{o_P(p)}$ so that $U_p^* \lhd U_q^*$ iff $p <_P q$.

Claim 3. *There is $\tilde{H} \in V[G]$ simultaneously $\mathrm{Add}(1, \kappa^+)$-generic over all $M_\alpha[G]$ for $\alpha < o(P)$.*

Proof. The number of $\mathrm{Add}(1, \kappa^+)$-antichains in $M_\alpha[G]$ computed in $V[G]$ is κ^+ for a fixed α, hence still κ^+ for all $\alpha < o(P) < \kappa^{++}$ together. $\mathrm{Add}(1, \kappa^+)$ is κ-closed; thus the generic $\tilde{H}$ can be constructed in $V[G]$. □

Let $\pi : 1 \times \kappa^+ \to \Theta \times \kappa^+$ be a bijection in M_0. Then $\pi[\tilde{H}]$ is $\mathrm{Add}(\Theta, \kappa^+)$-generic over all $M_\alpha[G]$. For $q \in P$ let

$$H_q^\kappa = \pi_q^{-1}[\pi[\tilde{H}] \cap (\{p; p \leq_P q\} \times \kappa^+)]$$

where $\pi_q : 1 \times \kappa^+ \to \{p; p \le_P q\} \times \kappa^+$ is a bijection in $M_0[G]$ (and so in all $M_\alpha[G]$). H_q^κ is obviously $\mathrm{Add}(1,\kappa^+)$-generic over all $M_\alpha[G]$. Moreover for $p \in P$ let g_p denote the p-th $\mathrm{Add}(1,\kappa^+)$-generic of $\pi[\tilde{H}]$,

$$g_p = \{\alpha < \kappa; (p,\alpha) \in \pi[\tilde{H}]\}.$$

If $p \le_P q$, then obviously $g_p \in M_\alpha[G \times H_q^\kappa]$ and, on the other hand, if $p \not\le_P q$, then g_p is $\mathrm{Add}(1,\kappa^+)$-generic over $M_\alpha[G \times H_q^\kappa]$, and so $g_p \notin M_\alpha[G \times H_q^\kappa]$. To complete the definition of U_q^* we need to find an appropriate $(j_\alpha P_\kappa)_{\kappa+1, j_\alpha\kappa}$-generic filter over M_α, where $\alpha = o_P(q)$. Consider $j'_\alpha : M_{\alpha+1} \to \mathrm{Ult}(M_{\alpha+1}, U_\alpha) = M'_\alpha$, and find an $H_q \in M_{\alpha+1}$ which is $(j_\alpha P_\kappa)_{\kappa+1, j_\alpha\kappa}$-generic over M'_α as in Claim 1. By Fact 1 $V^{M_\alpha}_{j_\alpha(\kappa)+1} = V^{M'_\alpha}_{j_\alpha(\kappa)+1}$, thus H_q is generic over M_α as well. $G \times H_q^\kappa \times H_q$ is $j_\alpha P_\kappa$-generic over M_α, defining an extension of j_α

$$j_\alpha^* : V[G] \to M_\alpha[G \times H_q^\kappa \times H_q].$$

The embedding j_α^* defines a measure $U_q^* \in V[G]$ extending U_α; U_q^* is actually an element of $M_{\alpha+1}[G \times H_q^\kappa]$.

Claim 4. *$U_p^* \lhd U_q^*$ iff $p <_P q$.*

Proof. Let $U_p^* \lhd U_q^*$, then by Claim 2 $\alpha = o_P(p)$ must be strictly less than $\beta = o_P(q)$, and $H_p^\kappa \times H_p \in M_\beta[G \times H_q^\kappa]$. Since g_p can be decoded from H_p^κ using $\pi_p \in M_\beta[G]$, it is in $M_\beta[G \times H_q^\kappa]$, which is possible only if $p \le_P q$, i.e. $p <_P q$.

Let $p <_P q$; then $\alpha = o_P(p) < o_P(q) = \beta$. According to Claim 2 all we need to prove is that $H_p^\kappa \times H_p \in M_\beta[G \times H_q^\kappa]$. The mappings π_p, π_q are both in $M_\beta[G]$. Consequently we can compute

$$H_p^\kappa = \pi_p^{-1}[\pi_q[H_q^\kappa] \cap (\{p'; p' \le_P p\} \times \kappa^+)]$$

in $M_\beta[G \times H_q^\kappa]$. By the construction H_p is in $M_{\alpha+1}$, and so in M_β by Fact 1. □

Let us prove that there is a Laver's function on κ in $V[G]$.

First find a sequence of bijections $\pi_\lambda : P_\lambda \to \lambda$ ($\lambda \le \kappa$ inaccessible) coherent in the following sense:

$$\forall \lambda \le \kappa \forall \alpha < o^{\mathcal{U}}(\lambda) : j_{U_\alpha^\lambda}(\langle \pi_{\lambda'}; \lambda' < \lambda\rangle)(\lambda) = \pi_\lambda. \tag{1}$$

Assume $\pi_{\lambda'}$ has been defined for $\lambda' < \lambda$. If $o^{\mathcal{U}}(\lambda) = 0$ pick any bijection $\pi_\lambda : P_\lambda \to \lambda$. If $o^{\mathcal{U}}(\lambda) > 0$, then π_λ can be defined as $j_{U_0^\lambda}(\langle \pi_{\lambda'}; \lambda' < \lambda\rangle)(\lambda)$ since $j_{U_0^\lambda}(\langle P_{\lambda'}; \lambda' < \lambda\rangle)(\lambda) = P_\lambda$. Condition (1) is then satisfied because $\overrightarrow{\mathcal{U}}$ forms a coherent sequence of measures. Notice that if σ_λ is the bijection $\lambda^+ \times \lambda \to \lambda^+$ given by the maximolexicographical ordering of $\lambda^+ \times \lambda$, then also

$$j_{U_\alpha^\lambda}(\langle \sigma_{\lambda'}; \lambda' < \lambda\rangle)(\lambda) = \sigma_\lambda \quad \text{for any } \alpha < o^{\mathcal{U}}(\lambda).$$

A P_λ-name $\dot{x} \subseteq \lambda^+ \times P_\lambda$ for a subset of λ^+ can be uniquely coded using π_λ and σ_λ as a subset of λ^+. If $f(\lambda)$ codes in this way a P_λ-name $\dot{x}$ for a subset of λ^+ define $f^*(\lambda) = i_{G\restriction P_\lambda}(\dot{x})$. It is easy to verify that this defines a Laver's function on κ in $V[G]$. Theorem 1 is proved. □

We can still ask what well-founded κ^{++}-like posets can be embedded into $\lhd$. For example, can we embed the poset consisting of a chain of length κ^{++} and one incomparable element? The answer is positive; however, in this case we have to destroy the covering of $\mathcal{P}(\kappa^+)$.

Theorem 2. *Assume that* $V = L[\overrightarrow{\mathcal{U}}]$, $o^{\mathcal{U}}(\kappa) = \kappa^{++}$. *Then there is a generic extension* $V[G]$ *preserving cardinalities, cofinalities, and GCH such that all well-founded* κ^{++}*-like posets in* V *and all well-founded posets of cardinality* $\leq \kappa^+$ *in* $V[G]$ *are embeddable into* $\lhd_\kappa^{V[G]}$.

Proof. Let $G = H \times \tilde{G}$ be $\mathrm{Add}(\kappa^{++}, \kappa^+) \times P_\kappa$-generic over V where P_κ is the Kunen-Paris forcing. Cardinalities, cofinalities, and GCH are obviously preserved in $V[G]$.

If $U_0 \in V$ is a measure on κ, $j_0 : V \to \mathrm{Ult}(V, U_0) = M_0$, and $j_0^* : V[\tilde{G}] \to M_0[\tilde{G} \times H_0^\kappa \times H_0]$ an extension of j_0 defined in $V[\tilde{G} \times H]$, then $U_0^* = \{X \in V[\tilde{G}]; X \subseteq \kappa \text{ and } \kappa \in j_0^*(X)\}$ is a measure in $V[\tilde{G} \times H]$ because H does not add any new subsets of κ. Similarly as in the proof of Theorem 1 if U_0^*, U_1^* are two such extensions of U_0, U_1 given by $H_0^\kappa \times H_0$, $H_1^\kappa \times H_1$, then $U_0^* \lhd U_1^*$ implies $U_1 \neq U_0$, $U_1 \not\lhd U_0$, $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. On the other hand $U_0 \lhd U_1$, $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$ implies $U_1^* \lhd U_1^*$.

Let $P \in V$ be a κ^{++}-like poset.

Claim. *P can be enumerated as $\langle p_\alpha; \alpha < \kappa^{++} \rangle$ so that $p_\alpha <_P p_\beta$ implies $\alpha < \beta$. Consequently there is a sequence of measures $U_0 \lhd \cdots \lhd U_\alpha \lhd \cdots$ $(\alpha < \kappa^{++})$ on κ so that*

$$\forall \alpha < \kappa^{++} : M_\alpha \models \text{“}|\alpha| \leq \kappa^+\text{” and } \{\gamma; p_\gamma \leq_P p_\alpha\} \in M_\alpha,$$

where $j_\alpha : V \to M_\alpha = \mathrm{Ult}(V, U_\alpha)$.

Proof. Define the enumeration by induction: Start with any well-ordering $\prec$ of P of order type κ^{++}. Assume that $\langle p_\alpha; \alpha < \mu \rangle$ has been defined so that

$$\forall \alpha < \mu : \{q \in P; q <_P p_\alpha\} \subseteq \{p_\gamma; \gamma < \alpha\}.$$

Let p_μ be the $\prec$-first element of $P \setminus \{p_\alpha; \alpha < \mu\}$ that is minimal with respect to $<_P$. Then the sequence $\langle p_\alpha; \alpha < \kappa^{++} \rangle$ clearly exhausts all elements of P : Assume that $\{p_\alpha; \alpha < \kappa^{++}\} \subsetneq P$; let $p \in P \setminus \{p_\alpha; \alpha < \kappa^{++}\}$ be a $<_P$-minimal element. Find $\mu < \kappa^{++}$ such that

$$\{q \in P; q <_P p\} \cup \{p_\alpha; \alpha < \kappa^{++} \& p_\alpha \prec p\} \subseteq \{p_\alpha; \alpha < \mu\}.$$

Then p is still $<_P$-minimal in $P \setminus \{p_\alpha; \alpha < \mu\}$ and so $p_\mu \preceq p$—a contradiction.

The second part of the claim follows easily from the fact that the measures on κ cover $\mathcal{P}(\kappa^+)$. □

Define $U_{p_\alpha}^* \in V[G]$ extending U_α as follows: Put

$$H_\alpha^\kappa = \pi_\alpha[H \cap (\{\gamma; p_\gamma \leq_P p_\alpha\} \times \kappa^+)]$$

where $\pi_\alpha : \{\gamma; p_\gamma \leq_P p_\alpha\} \times \kappa^+ \to 1 \times \kappa^+$ is a bijection in M_α. H_α^κ is $\mathrm{Add}(1, \kappa^+)$-generic over $M_\alpha[\tilde{G}]$—the point is that H is now generic over all $M_\alpha[\tilde{G}]$ as $\alpha < \kappa^{++}$. Then find an $H_\alpha \in M_{\alpha+1}$ that is $(j_\alpha P_\kappa)_{\kappa+1, j_\alpha \kappa}$-generic over $\mathrm{Ult}(M_{\alpha+1}, U_\alpha)$. That defines a measure $U_{p_\alpha}^*$ extending U_α. Similarly as in the proof of Theorem 1 $U_{p_\alpha}^* \lhd U_{p_\beta}^*$ iff $p_\alpha <_P p_\beta$.

Finally let $P \in V[G]$ be a well-founded poset of cardinality κ^+. Then by the κ^{++}-c.c. of $\mathrm{Add}(\kappa^{++}, \kappa^+)$ there is $\vartheta < \kappa^{++}$ so that $P \in V[\tilde{G} \times (H \restriction \vartheta)]$. Represent

P as $([\vartheta,\Theta),<_P)$, $\vartheta<\Theta<\kappa^{++}$, and find $U_0\lhd\cdots\lhd U_\alpha\lhd\cdots$ $(\alpha<o(P))$ so that $P\in M_0[\tilde{G}\times(H\restriction\vartheta)]$ and $\vartheta<\kappa^{++M_0}$. Then define U^*_q extending the $U_{o_P(q)}$ setting

$$H^\kappa_q=\pi_q[H\cap((\vartheta\cup\{p;p<_P q\})\times\kappa^+)]$$

where $\pi_q:(\vartheta\cup\{p;p<_P q\})\times\kappa^+\to 1\times\kappa^+$ is a bijection in $M_0[\tilde{G}\times(H\restriction\vartheta)]$ such that $\pi_q\restriction(\vartheta\times\kappa^+)$ is in M_0. Find H_q as in the proof of Theorem 1. Then similarly as in the proof $H^\kappa_p\in M_\beta[\tilde{G}\times H^\kappa_q]$ iff $p<_P q$ iff $U^*_p\lhd U^*_q$. □

Remark 1. J. Cummings in [Cu93] starts with $V=K[\overrightarrow{\mathcal{U}}_{\max}]$ $(o^{\mathcal{U}}(\kappa)<\kappa^{++})$, then applies a forcing similar to the Kunen-Paris forcing, and classifies all measures in $V[G]$ using special properties of $K[\overrightarrow{\mathcal{U}}_{\max}]$. More specifically the measures are divided into blocks of incomparable measures $M(\alpha,\beta)$ $(\alpha<\beta\le o^{\mathcal{U}}(\kappa))$ of cardinality κ^+ or κ^{++} so that for $U\in M(\alpha,\beta)$, $W\in M(\gamma,\delta)$: $U\lhd W$ iff $\beta\le\gamma$.

Starting with $V=K[\overrightarrow{\mathcal{U}}_{\max}]$ we can use the same method to classify measures in $V[G]$ where G is P_κ-generic over V.

A finite normal iteration $j:V\to N$ of length $n+1$ is an iteration of ultraproducts by measures on $\kappa=\kappa_0<\kappa_1<\cdots<\kappa_n$. Any finite normal iteration $j:V\to N$ that starts with a measure U gives κ^{++} extensions U^* in $V[G]$ of U such that $j_{U^*}\restriction V=j$, $j_{U^*}(G)=G*H^\kappa*H$. All measures in $V[G]$ are produced in this way. We can give sufficient and necessary conditions for $\lhd$ in $V[G]$: Let U^*_0,U^*_1 extending U_0,U_1 be given by finite normal iterations $j_0:V\to N_0$, $j_1:V\to N_1$ and $H^\kappa_0*H_0,H^\kappa_1*H_1$. Then $U^*_0\lhd U^*_1$ iff $j_0\restriction\mathrm{Ult}(V,U_1)$ is an internal iteration in $\mathrm{Ult}(V,U_1)$ and

$$H^\kappa_0*H_0\in\mathrm{Ult}(V,U_1)[G*H^\kappa_1].$$

However, we can hardly describe the Mitchell ordering $\lhd$ in $V[G]$ in a simpler manner.

That is illustrated by the following: Let U_0 be the minimal measure on κ in V, and let U^*_0 be its one-step extension using $H^\kappa_0*H_0\in V[G]$. We have seen that there may be measures above U^*_0 even if $H^\kappa_0*H_0\notin M_\alpha[G]$ for all $\alpha<o(\kappa)$ $(M_\alpha=\mathrm{Ult}(V,U^\kappa_\alpha))$. However it is also possible that there are no measures above U^*_0. It follows from the following joint lemma with J. Zapletal.

Lemma. *There is $H^\kappa_0\in V[G]$ $\mathrm{Add}(1,\kappa^+)$-generic over $M_0[G]$ such that for any $\alpha<o^{\mathcal{U}}(\kappa)$, $\alpha>0$, there is no $H^\kappa_1\in V[G]$ $\mathrm{Add}(1,\kappa^+)$-generic over $M_\alpha[G]$, $\alpha<o(\kappa)$, satisfying $H^\kappa_0\in M_\alpha[G][H^\kappa_1]$.*

Proof. Let $R\subseteq\kappa^+\times\kappa^+$ be a well-ordering of order type γ where $\gamma>\kappa^{++M_\alpha}$ for all $\alpha<o(\kappa)<\kappa^{++}$. Notice that if H^κ_1 is any $\mathrm{Add}(1,\kappa^+)$-generic over $M_\alpha[G]$, then $R\notin M_\alpha[G][H^\kappa_1]$; otherwise γ would be less than κ^{++M_α}. So it would be enough to code R into H^κ_0. Let $\langle a_\alpha;\alpha<\kappa^+\rangle\in M_1[G]$ canonically enumerate $\kappa^+\times\kappa^+$ and $\langle D_\alpha;\alpha<\kappa^+\rangle\in M_1[G]$ enumerate all $M_0[G]$-dense subsets of $\mathrm{Add}(1,\kappa^+)$, each set D_α enumerated by ordinals $<\kappa^+$. Construct a descending sequence of conditions $\langle p_\alpha;\alpha<\kappa^+\rangle\in V[G]$ as follows: Assume $\langle p_\delta;\delta<\alpha\rangle$ has been constructed, then find the first $q\in D_\alpha$ extending $\bigcup\{p_\delta;\delta<\alpha\}$, let $\eta=\sup\{\xi+1;\xi\in\mathrm{dom}\,(q)\}$, and put $p\restriction\eta=q$ and

$$p(\eta)=\begin{cases}1 & \text{iff } a_\alpha\in R,\\ 0 & \text{otherwise.}\end{cases}$$

That gives an $\mathrm{Add}(1,\kappa^+)$-generic filter H_0^κ over $M_0[G]$ such that for $\alpha > 0$

$$H_0^\kappa \in M_\alpha[G][H_1^\kappa]$$

implies $R \in M_\alpha[G][H_1^\kappa]$. □

Remark 2. It is not true in general that $U_0^* \lhd U_1^*$ implies $U_0 \lhd U_1$ as Claim 2 in the proof of Theorem 1 might suggest: There is a model M, two measures $U_0, U_1 \in M$ and their extensions U_0^*, U_1^* in $M[\tilde{G}]$, where $\tilde{G}$ is P_κ-generic over M, so that

$$U_0 \not\lhd U_1 \text{ but } U_0^* \lhd U_1^*.$$

Proof. Start with two measures $U_0 \lhd U_1$ in V, and with the corresponding canonical embeddings $j_0 : V \to M_0$, $j_1 : V \to M_1$. Let $G \times \tilde{G}$ be $P_\kappa \times P_\kappa$-generic over V. Then find $H_0^\kappa \times H_1^\kappa \times H_2^\kappa \in V$ a filter $\mathrm{Add}(3,\kappa^+)$-generic over M_1, $H_1 \times \tilde{H}_1 \in V$ a filter $(j_1 P_\kappa)_{\kappa+1, j_1\kappa} \times (j_1 P_\kappa)_{\kappa+1, j_1\kappa}$-generic over M_1, and $H_0 \times \tilde{H}_0 \in M_1$ a filter $(j_0 P_\kappa)_{\kappa+1, j_0\kappa} \times (j_0 P_\kappa)_{\kappa+1, j_0\kappa}$-generic over M_0. Using Easton's Lemma $G, \tilde{G}, H_0^\kappa, H_1^\kappa, H_2^\kappa, H_1, \tilde{H}_1$ are mutually generic over M_1 and $G, \tilde{G}, H_0^\kappa, H_1^\kappa, H_2^\kappa, H_0, \tilde{H}_0$ are mutually generic over M_0. First extend U_0, U_1 to U_0^*, U_1^* in $M = V[G]$ so that $j_1^*(G) = G \times H_1^\kappa \times H_1$ and $j_0^*(G) = G \times (H_0^\kappa \otimes H_1^\kappa) \times H_0$ where $H_0^\kappa \otimes H_1^\kappa$ denotes a coding of H_0^κ, H_1^κ into an $\mathrm{Add}(1,\kappa^+)$-generic. Obviously $U_0^* \not\lhd U_1^*$ since $H_0^\kappa \notin M_1[j_1^*(G)]$. Then extend U_0^*, U_1^* to U_0^{**}, U_1^{**} in $M[\tilde{G}] = V[G \times \tilde{G}]$ so that $j_1^{**}(\tilde{G}) = \tilde{G} \times (H_0^\kappa \otimes H_2^\kappa) \times \tilde{H}_1$ and $j_0^{**}(\tilde{G}) = \tilde{G} \times H_2^\kappa \times \tilde{H}_0$. Then $U_0^{**} \lhd U_1^{**}$ since $U_0^*, j_0^{**}(\tilde{G}) \in M_1[j_1^{**}(G \times \tilde{G})]$. □

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References

[Ba85] S. Baldwin, *The $\lhd$-ordering on normal ultrafilters*, J. Symbolic Logic **51** (1985), 936–952. MR **87d:**03124

[Cu93] J. Cummings, *Possible behaviors for the Mitchell ordering*, Ann. Pure Appl. Logic (to appear).

[KuP71] K. Kunen and J. B. Paris, *Boolean extensions and measurable cardinals*, Ann. of Math. Logic **2** (1971), 359–377. MR **43:**3114

[Mi83] W.J. Mitchell, *Sets constructible from sequences of measures: revisited*, J. Symbolic Logic **48** (1983), 600–609. MR **85j:**03052

[W94a] J. Witzany, *Possible behaviours of the reflection ordering of stationary sets*, J. Symbolic Logic (to appear).

[W94b] J. Witzany, *Reflection of stationary sets and the Mitchell ordering of normal measures*, Ph.D. thesis, Pennsylvania State University, 1994.

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