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THE L_p VERSION OF NEWMAN'S INEQUALITY FOR LACUNARY POLYNOMIALS

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ABSTRACT. The principal result of this paper is the establishment of the essentially sharp Markov-type inequality

$$\|xP'(x)\|_{L_p[0,1]} \le \left(1/p + 12\left(\sum_{j=0}^{n}(\lambda_j + 1/p)\right)\right)\|P\|_{L_p[0,1]}$$

for every $P \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ with distinct real exponents λ_j greater than $-1/p$ and for every $p \in [1, \infty]$.

A remarkable corollary of the above is the Nikolskii-type inequality

$$\|y^{1/p}P(y)\|_{L_\infty[0,1]} \le 13\left(\sum_{j=0}^{n}(\lambda_j + 1/p)\right)^{1/p}\|P\|_{L_p[0,1]}$$

for every $P \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ with distinct real exponents λ_j greater than $-1/p$ and for every $p \in [1, \infty]$.

Some related results are also discussed.

1. Introduction and notation

Let $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers. The span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

Elements of $M_n(\Lambda)$ are called Müntz (or lacunary) polynomials. We first present a simplified version of Newman's beautiful proof of a Markov-type inequality for Müntz polynomials. This modification gives a better constant, 9, than the constant 11 appearing in Newman's paper [5]. But more importantly, this modification allows us to prove the L_p analogues of Newman's Inequality. Some related results are stated in Section 4.

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2. New results

Theorem 2.1 (Newman's Inequality). *Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct nonnegative real numbers. Then*

$$\frac{2}{3}\sum_{j=0}^n \lambda_j \le \sup_{0\ne P\in M_n(\Lambda)} \frac{\|xP'(x)\|_{L_\infty[0,1]}}{\|P\|_{L_\infty[0,1]}} \le 9\sum_{j=0}^n \lambda_j .$$

Frappier [4] shows that the constant 11 in Newman's Inequality can be replaced by 8.29. We believe on the basis of considerable computation that the best possible constant in Newman's Inequality is 4. (We remark that an incorrect argument exists in the literature claiming that the best possible constant in Newman's Inequality is at least $4+\sqrt{15} = 7.87\dots$.)

Conjecture (Newman's Inequality with Best Constant). *Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct nonnegative real numbers. Then*

$$\|xP'(x)\|_{L_\infty[0,1]} \le 4\left(\sum_{j=0}^n \lambda_j\right)\|P\|_{L_\infty[0,1]}$$

for every $P \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

Theorem 2.2 (Newman's Inequality in L_p). *Let $p\in[1,\infty)$. Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct real numbers greater than $-1/p$. Then*

$$\|xP'(x)\|_{L_p[0,1]} \le \left(1/p + 12\left(\sum_{j=0}^n (\lambda_j + 1/p)\right)\right)\|P\|_{L_p[0,1]}$$

for every $P \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

The following Nikolskii-type inequality follows from Theorem 2.2 quite simply.

Theorem 2.3 (Nikolskii-Type Inequality). *Let $p\in[1,\infty)$. Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct real numbers greater than $-1/p$. Then*

$$\|y^{1/p}P(y)\|_{L_\infty[0,1]} \le 13\left(\sum_{j=0}^n (\lambda_j + 1/p)\right)^{1/p}\|P\|_{L_p[0,1]}$$

for every $P \in M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

Theorem 2.3 immediately implies the following result.

Theorem 2.4 (Müntz-Type Theorem in L_p). *Let $p\in[1,\infty)$. Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct real numbers greater than $-1/p$ satisfying*

$$\sum_{j=0}^\infty (\lambda_j + 1/p) < \infty .$$

Then

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is not dense in $L_p[0,1]$.

Much more about Müntz-type theorems in $L_p[0,1]$ may be found in [2] and the references therein.

3. Proofs

Proof of Theorem 2.1 (*Newman's proof modified*). It is equivalent to prove that

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \le \sup_{0\ne P\in E_n(\Lambda)} \frac{\|P'\|_{[0,\infty)}}{\|P\|_{[0,\infty)}} \le 9\sum_{j=0}^{n}\lambda_j, \tag{3.1}$$

where $E_n(\Lambda)$ is the linear span of $\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots, e^{-\lambda_n t}\}$ over $\mathbb{R}$. Without loss of generality we may assume that $\lambda_0 = 0$. By a change of scale we may also assume that $\sum_{j=0}^{n}\lambda_j = 1$. We may also assume that $n \ge 2$, otherwise the theorem is trivial. We begin with the first inequality. We define the Blaschke product

$$B(z) := \prod_{j=1}^{n}\frac{z-\lambda_j}{z+\lambda_j}$$

and the function

$$T(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{-zt}}{B(z)}\,dz, \quad \text{where} \quad \Gamma := \{z\in\mathbb{C} : |z-1| = 1\}. \tag{3.2}$$

By the Residue Theorem

$$T(t) := \sum_{j=1}^{n}(B'(\lambda_j))^{-1}e^{-\lambda_j t},$$

so $T \in E_n(\Lambda)$. We claim that

$$|B(z)| \ge \frac{1}{3}, \quad z\in\Gamma. \tag{3.3}$$

Indeed, it is easy to see that $0 < \lambda_j \le 1$ implies

$$\left|\frac{z-\lambda_j}{z+\lambda_j}\right| \ge \frac{2-\lambda_j}{2+\lambda_j} = \frac{1-\lambda_j/2}{1+\lambda_j/2}, \qquad z\in\Gamma,$$

so, for $z\in\Gamma$,

$$|B(z)| \ge \prod_{j=1}^{n}\frac{1-\lambda_j/2}{1+\lambda_j/2} \ge \frac{1-\frac{1}{2}\sum\limits_{j=1}^{n}\lambda_j}{1+\frac{1}{2}\sum\limits_{j=1}^{n}\lambda_j} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}.$$

Here the inequality

$$\begin{aligned}\frac{1-x}{1+x}\,\frac{1-y}{1+y} &= \frac{1-(x+y)}{1+(x+y)} + \frac{2xy(x+y)}{(1+x)(1+y)(1+(x+y))}\\ &\ge \frac{1-(x+y)}{1+(x+y}, \quad x,y\ge 0,\end{aligned}$$

is used. From (3.2) and (3.3) we deduce that

$$|T(t)| \le \frac{1}{2\pi}\int_\Gamma \left|\frac{e^{-zt}}{B(z)}\right| |dz| \le \frac{1}{2\pi} 3(2\pi) = 3, \quad t \ge 0, \tag{3.4}$$

$$T'(t) = \frac{1}{2\pi i}\int_\Gamma \frac{-ze^{-zt}}{B(z)}\, dz$$

and

$$T'(0) = -\frac{1}{2\pi i}\int_\Gamma \frac{z}{B(z)}\, dz = -\frac{1}{2\pi i}\int_{|z|=1} \frac{z}{B(z)}\, dz. \tag{3.5}$$

Also, for $|z| > \max_{1\le j\le n} \lambda_j$ we have the Laurent series expansion

$$\begin{aligned} \frac{z}{B(z)} &= z\prod_{j=1}^n \frac{1+\lambda_j/z}{1-\lambda_j/z} = z\prod_{j=1}^n \left(1 + 2\sum_{k=1}^n (\lambda_j/z)^k\right) \\ &= z\left(1 + 2\left(\sum_{j=1}^n \lambda_j\right) z^{-1} + 2\left(\sum_{j=1}^n \lambda_j\right)^2 z^{-2} + \cdots\right) \\ &= z + 2 + 2z^{-1} + \cdots , \end{aligned} \tag{3.6}$$

which, together with (3.5), yields that $T'(0) = -2$. Hence, by (3.4)

$$\frac{|T'(0)|}{\|T\|_{[0,\infty)}} \ge \frac{2}{3} = \frac{2}{3}\sum_{j=1}^n \lambda_j,$$

so the lower bound of the theorem is proved.

To prove the upper bound in (2.1), first we show that if

$$U(t) := \frac{1}{2\pi i}\int_\Gamma \frac{e^{-zt}}{(1-z)B(z)}\, dz, \qquad \Gamma := \{z \in \mathbb{C} : |z-1| = 1\},$$

then

$$\int_0^\infty |U''(t)|\, dt \le 6. \tag{3.7}$$

Indeed, observe that if $z = 1 + e^{-\theta}$, then $|z|^2 = 2 + 2\cos\theta$, so (3.3) and Fubini's Theorem yield that

$$\begin{aligned} \int_0^\infty |U''(t)|\, dt &= \int_0^\infty \frac{1}{2\pi}\left|\int_\Gamma \frac{z^2 e^{-zt}}{(1-z)B(z)}\, dz\right| dt \\ &\le \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \frac{|z|^2|e^{-zt}|}{|B(z)|}\, d\theta\, dt \\ &\le \frac{3}{2\pi}\int_0^\infty \int_0^{2\pi} (2+2\cos\theta)e^{-(1+\cos\theta)t}\, d\theta\, dt \\ &= \frac{3}{2\pi}\int_0^{2\pi} (2+2\cos\theta)\frac{1}{1+\cos\theta}\, d\theta = 6. \end{aligned}$$

Now we show that

$$\int_0^\infty e^{-\lambda_j t} U''(t)\, dt = 3 - \lambda_j. \tag{3.8}$$

To see this we write the left-hand side as

$$\begin{aligned}\int_0^\infty e^{-\lambda_j t} U''(t)\, dt &= \int_0^\infty e^{-\lambda_j t} \frac{1}{2\pi i} \int_\Gamma \frac{z^2 e^{-zt}}{(1-z)B(z)}\, dz\, dt \\ &= \frac{1}{2\pi i} \int_0^\infty \int_\Gamma \frac{z^2 e^{-(z+\lambda_j)t}}{(1-z)B(z)}\, dz\, dt = \frac{1}{2\pi i} \int_\Gamma \frac{z^2}{(z+\lambda_j)(1-z)B(z)}\, dz \\ &= \frac{1}{2\pi i} \int_{|z|=2} \frac{z}{z+\lambda_j}\, \frac{z}{1-z}\, \frac{1}{B(z)}\, dz,\end{aligned}$$

where in the third equality Fubini's Theorem is used. Here, for $|z| > 1$, we have the Laurent series expansions

$$\begin{aligned}\frac{z}{z+\lambda_j} &= 1 - \lambda_j z^{-1} + \lambda_j^2 z^{-2} + \cdots, \\ \frac{z}{1-z} &= 1 + z^{-1} + z^{-2} + \cdots\end{aligned}$$

and, as in (3.6),

$$\frac{1}{B(z)} = 1 + 2z^{-1} + 2z^{-2} + \cdots .$$

Now (3.8) follows from the Residue Theorem. Let $P \in E_n(\Lambda)$ be of the form

$$P(t) = \sum_{j=0}^n c_j e^{-\lambda_j t}, \quad c_j \in \mathbb{R}.$$

Then

$$\begin{aligned}\int_0^\infty P(t+a) U''(t)\, dt &= \int_0^\infty \sum_{j=0}^n c_j e^{-\lambda_j a} e^{-\lambda_j t} U''(t)\, dt \\ &= \sum_{j=0}^n c_j e^{-\lambda_j a} \int_0^\infty e^{-\lambda_j t} U''(t)\, dt = \sum_{j=0}^n c_j (3-\lambda_j) e^{-\lambda_j a} \\ &= -3P(a) - P'(a),\end{aligned}$$

therefore

$$|P'(a)| \le 3|P(a)| + \int_0^\infty |P(t+a)\, U''(t)|\, dt. \tag{3.9}$$

Combining this with (3.7), we obtain

$$\|P'\|_{[0,\infty)} \le 3\, \|P\|_{[0,\infty)} + 6 \|P\|_{[0,\infty)} = 9\, \|P\|_{[0,\infty)}$$

and the theorem is proved. $\square$

Proof of Theorem 2.2. First we show that it is sufficient to prove that if $\Gamma := \{\gamma_i\}_{i=0}^\infty$ is a sequence of distinct positive real numbers, then

$$\|P'\|_{L_p[0,\infty)} \leq 12 \left(\sum_{j=0}^n \gamma_j\right) \|P\|_{L_p[0,\infty)} \tag{3.10}$$

for every $P \in E_n(\Gamma)$ and $p \in [1,\infty)$, where $E_n(\Gamma)$ is, as before, the linear span of $\{e^{-\gamma_0 t}, e^{-\gamma_1 t}, \dots, e^{-\gamma_n t}\}$ over $\mathbb{R}$.

Indeed, if $\{\lambda_i\}_{i=0}^\infty$ is a sequence of distinct real numbers greater than $-1/p$ and $\gamma_i := \lambda_i + 1/p$ for each i, then $\{\gamma_i\}_{i=1}^\infty$ is a sequence of distinct positive real numbers. Let $Q \in M_n(\Lambda)$. Applying (3.10) with $P(t) := Q(e^{-t})e^{-t/p} \in E_n(\Gamma)$ and using the substitution $x = e^{-t}$, we obtain that

$$\left(\int_0^1 \left| x\left(x^{1/p}Q(x)\right)'\right|^p x^{-1}\,dx\right)^{1/p} \leq 12\left(\sum_{j=0}^n (\lambda_j + 1/p)\right)\|Q\|_{L_p[0,1]}.$$

Now the product rule of differentiation and Minkowski's Inequality yield

$$\|xQ'(x)\|_{L_p[0,1]} \leq \left(1/p + 12\left(\sum_{j=0}^n (\lambda_j + 1/p)\right)\right)\|Q\|_{L_p[0,1]},$$

which is the inequality of the theorem.

Let $P \in E_n(\Gamma)$ and $p \in [1,\infty)$ be fixed. As in the proof of Theorem 3.1, by a change of scale, without loss of generality we may assume that $\sum_{j=0}^n \gamma_j = 1$. It follows from (3.9) and Hölder's Inequality that

$$\begin{aligned} |P'(a)|^p &\leq 2^{p-1}\left(3^p|P(a)|^p + \left(\int_0^\infty |P(t+a)U''(t)|\,dt\right)^p\right) \\ &\leq 6^p|P(a)|^p + 2^{p-1}\left(\left(\int_0^\infty |P(t+a)|^p|U''(t)|\,dt\right)^{1/p}\left(\int_0^\infty |U''(t)|\,dt\right)^{1/q}\right)^p \end{aligned}$$

for every $a \in [0,\infty)$, where $q \in (1,\infty]$ is the conjugate exponent to p defined by $1/p + 1/q = 1$. Combining the above inequality with (3.7), we obtain that

$$|P'(a)|^p \leq 6^p|P(a)|^p + 2^{p-1}6^{p/q}\int_0^\infty |P(t+a)|^p|U''(t)|\,dt$$

for every $a \in [0,\infty)$. Integrating with respect to a, then using Fubini's Theorem and (3.7), we conclude

$$\begin{aligned} \|P'\|^p_{L_p[0,\infty)} \leq& 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q}\int_0^\infty\int_0^\infty |P(t+a)|^p|U''(t)|\,dt\,da \\ \leq& 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q}\int_0^\infty\int_0^\infty |P(t+a)|^p|U''(t)|\,da\,dt \\ \leq& 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q}\|P\|^p_{L_p[0,\infty)}\int_0^\infty |U''(t)|\,dt \\ \leq& 6^p\|P\|^p_{L_p[0,\infty)} + 2^{p-1}6^{p/q+1}\|P\|^p_{L_p[0,\infty)} \\ =& (6^p + 2^{p-1}6^p)\|P\|^p_{L_p[0,\infty)} \leq 12^p\|P\|^p_{L_p[0,\infty)}, \end{aligned}$$

and the proof is finished. $\square$

Proof of Theorem 2.3. After the scaling $x \to yx$ and the substitution $x = e^{-t}$, it is sufficient to prove that if $\Gamma := \{\gamma_i\}_{i=0}^{\infty}$ is a sequence of distinct positive real numbers, then

$$|P(0)| \leq 13 \left(\sum_{j=0}^{n} \gamma_j\right)^{1/p} \|P\|_{L_p[0,\infty)} \tag{3.11}$$

for every $P \in E_n(\Gamma)$ and $p \in [1,\infty)$, where $E_n(\Gamma)$ is, as before, the linear span of $\{e^{-\gamma_0 t}, e^{-\gamma_1 t}, \dots, e^{-\gamma_n t}\}$ over $\mathbb{R}$.

Indeed, if $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$ and $\gamma_i := \lambda_i + 1/p$ for each i, then $\{\gamma_i\}_{i=1}^{\infty}$ is a sequence of distinct positive real numbers. Let $Q \in M_n(\Lambda)$ and $y \in [0,1]$. Applying (3.11) with $P(t) := Q(ye^{-t})e^{-t/p} \in E_n(\Gamma)$ and using the substitution $x = e^{-t}$, we obtain that

$$\begin{aligned} |y^{1/p}Q(y)| \leq & 13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p)\right)^{1/p} \|Q\|_{L_p[0,y]} \\ \leq & 13 \left(\sum_{j=0}^{n} (\lambda_j + 1/p)\right)^{1/p} \|Q\|_{L_p[0,1]}, \end{aligned}$$

which is the inequality of the theorem.

Now let $P \in E_n(\Gamma)$. As in the proof of Theorem 3.1, by a change of scale, without loss of generality we may assume that $\sum_{j=0}^{n} \gamma_j = 1$. Using Hölder's Inequality, we obtain that

$$\begin{aligned} |P(0)| \leq & \|P(t)e^{-t}\|_{L_\infty[0,\infty)} \leq \|(P(t)e^{-t})'\|_{L_1[0,\infty)} \\ \leq & \|P'(t)e^{-t}\|_{L_1[0,\infty)} + \|P(t)e^{-t}\|_{L_1[0,\infty)} \\ \leq & \|P'\|_{L_p[0,\infty)} + \|P\|_{L_p[0,\infty)}. \end{aligned}$$

Combining this with (3.10) and $\sum_{j=0}^{n} \gamma_j = 1$, we conclude (3.11). □

Remarks. Theorem 2.1 is due to Newman [5] with 11 instead of 9. We presented a modified version of Newman's original proof of Theorem 2.1. He worked with T instead of U, and instead of (3.9) he established a more complicated identity involving the second derivative of P. Therefore, he needed an application of Kolmogorov's Inequality to finish his proof.

4. Related results

In this section we state some related results without proof. Proofs will be presented in [1].

Theorem 4.1 (sharpness of Theorem 2.2). *Let $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers with $\lambda_0 = 0$ and $\lambda_k \geq k$ for each k. Then there exists an absolute constant $c > 0$ (independent of Λ or p) so that*

$$\sup_{P \in M_n(\Lambda)} \frac{\|xP'(x)\|_{L_p[0,1]}}{\|P\|_{L_p[0,1]}} \geq c \sum_{j=0}^{n} \lambda_j$$

for every $p \in [2,\infty)$.

It can be proved that under a growth condition $\|xp'(x)\|_{L_\infty[0,1]}$ in Newman's Inequality can be replaced by $\|p'\|_{L_\infty[0,1]}$. More precisely, the following result holds.

Theorem 4.2 (Newman's Inequality without the factor x). *Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct real numbers with $\lambda_0 = 0$ and $\lambda_k \geq k$ for each k. Then*

$$\|P'\|_{[0,1]} \leq 18 \left(\sum_{j=1}^n \lambda_j \right) \|P\|_{[0,1]}$$

for every $P \in M_n(\Lambda)$.

The next result shows that the growth condition in Theorem 4.2 cannot be dropped in general.

Theorem 4.3. *For every $\delta \in (0,1)$ there exists a sequence $\Lambda := \{\lambda_i\}_{i=0}^\infty$ with $\lambda_0 = 0$, $\lambda_1 \geq 1$, and $\lambda_{i+1} - \lambda_i \geq \delta$, $i = 0, 1, 2, \ldots$, so that*

$$\lim_{n \to \infty} \sup_{P \in M_n(\Lambda)} \frac{|P'(0)|}{\left(\sum_{j=0}^n \lambda_j \right) \|P\|_{[0,1]}} = \infty.$$

The following L_2 version of Newman's Inequality is proved by Borwein, Erdélyi, and J. Zhang [3]. This theorem offers an L_2 analogue of Theorem 3.1 even for complex exponents. It also improves the multiplicative constant 12 in the L_2 inequality of Theorem 3.2.

Theorem 4.4 (Newman's Inequality in $L_2[0,1]$). *If $\Lambda := \{\lambda_i\}_{i=0}^\infty$ is a set of distinct complex numbers with $\mathrm{Re}(\lambda_i) > -1/2$ and $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ over $\mathbb{C}$, then*

$$\sup_{0 \neq P \in M_n(\Lambda)} \frac{\|xP'(x)\|_{L_2[0,1]}}{\|P\|_{L_2[0,1]}} \leq \left(\sum_{j=0}^n |\lambda_j|^2 + \sum_{j=0}^n (1 + 2\mathrm{Re}(\lambda_j)) \sum_{k=j+1}^n (1 + 2\mathrm{Re}(\lambda_k)) \right)^{1/2}$$

for every $n \in \mathbb{N}$.

If $0 \leq \lambda_0 < \lambda_1 < \cdots$ are real, and $M_n(\Lambda)$ denotes the linear span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

over $\mathbb{R}$, then

$$\frac{1}{2\sqrt{30}} \sum_{j=0}^n \lambda_j \leq \sup_{0 \neq P \in M_n(\Lambda)} \frac{\|xP'(x)\|_{L_2[0,1]}}{\|P\|_{L_2[0,1]}} \leq \frac{1}{\sqrt{2}} \sum_{j=0}^n (1 + 2\lambda_j)$$

for every $n \in \mathbb{N}$.

Note that the interval $[0,1]$ plays a special role in the study of Müntz polynomials. Analogues of the results on $[a,b]$, $a > 0$, cannot be obtained by a linear transformation. We can however prove the following result.

Theorem 4.5 (Newman's Inequality on $[a,b] \subset [0,\infty)$). *Let $\Lambda := \{\lambda_i\}_{i=1}^{\infty}$ be a sequence of nonnegative real numbers. Assume that there exists an $\alpha > 0$ so that $\lambda_i - \lambda_{i-1} \geq \alpha$ for each i. Suppose that $[a,b] \subset [0,\infty)$. Then there exists a constant $c(a,b,\alpha)$ depending only on a, b, and α so that*

$$\|P'\|_{[a,b]} \leq c(a,b,\alpha) \left(\sum_{j=0}^{n} \lambda_j \right) \|P\|_{[a,b]}$$

for every $p \in M_n(\Lambda)$, where $M_n(\Lambda)$ denotes the linear span of $\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$ over $\mathbb{R}$.

It is also shown that the above result does not necessarily hold without the gap condition $\lambda_i - \lambda_{i-1} \geq \alpha > 0$.

When $p = 2$ the best possible constant in the Nikolskii-type inequality of Corollary 2.3 is found in [3]. We have the following result.

Theorem 4.6. *Let $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers greater than $-1/2$. Then*

$$\max_{P \in M_n(\Lambda)} \frac{|P(1)|}{\|P\|_{L_2[0,1]}} = \left(\sum_{j=0}^{n} (1 + 2\lambda_j) \right)^{1/2}.$$

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