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ON FACTOR STATES OF C^*-ALGEBRAS AND THEIR EXTENSIONS

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ABSTRACT. We obtain some results on the unique extension of (factor) states of C^*-algebras which complement various existing results. Our results also lead to a class of C^*-algebras whose states are σ-convex sums of factor states.

1. INTRODUCTION

Recently Pfitzner [19] proved that every von Neumann algebra is a Grothendieck space, thus solving a long-standing open problem (cf. [9; p.104] and see also [1]). Using this interesting result, we obtain some new information concerning the unique extension of (factor) states of C^*-algebras which complements the existing results in [2, 4, 7, 14, 15]. Extensions of factor states have been investigated by many authors (e.g., [3, 6, 17, 20, 21, 23, 25]). Our approach to unique extension, however, is based on a simple observation (Lemma 1) that if states φ of certain type on a subalgebra extend uniquely to states $\overline{\varphi}$ of certain type on the containing algebra, then the natural map $\varphi \to \overline{\varphi}$ is weak*-continuous. This observation enables us to apply Pfitzner's result to study unique extension of states if the containing algebra is a von Neumann algebra. We show, for instance, that given a separable C^*-subalgebra B of a von Neumann algebra M, then B must be scattered if every factor state of B extends uniquely to a weak*-limit of factor states on M. Further, if B is abelian, then every pure state of B extends uniquely to a pure state of M if and only if $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n \oplus \cdots$, where each p_n is a minimal projection in M and the sum is a c_0-sum and may be finite. This latter result complements the results of Kadison and Singer [14], Anderson [2], and Archbold, Bunce and Gregson [7]. Other extension result also leads us to study a class of C^*-algebras whose states are σ-convex sums of factor states.

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2. EXTENSION OF FACTOR STATES

Let A be a C^*-algebra and let $S(A)$ be its state space. Given a subset $K \subset A^*$, we will denote by $\overline{K}$ the weak* closure of K in A^*, and by $\overline{\overline{K}}$ the norm-closure of K in A^*. Let $P(A)$ be the set of pure states of A and let $F(A)$ be the set of factor states of A.

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The works of Anderson and Bunce [3], Longo [17] and Popa [20] show that if B is a *separable* C^*-subalgebra of A, then every $\varphi \in F(B)$ extends to a $\overline{\varphi} \in F(A)$. Also Archbold [5] has shown that for any C^*-subalgebra $B \subset A$, every $\varphi \in \overline{F(B)}$ extends to a $\overline{\varphi} \in \overline{F(A)}$. Naturally, one is interested in the question of when these extensions are unique. Of course, if B separates points of $F(A)$ (respectively $\overline{F(A)}$), then every $\varphi \in F(B)$ extends uniquely to a $\overline{\varphi} \in F(A)$ (respectively $\overline{F(A)}$); but in this case, the Stone-Weierstrass theorem of Longo [17] and Popa [20] (respectively Glimm [11]) implies $B = A$. In the following, we will show various necessary conditions for the unique extension of factor states. If B is abelian, the question of unique factor state extension is equivalent to that of unique pure state extension which has been considered in [2, 4, 7, 14]; nevertheless, we are able to describe B with such unique extension explicitly if it is separable and if A is a von Neumann algebra. We begin with a simple but useful lemma.

Lemma 1. *Let A be a C^*-algebra and B be a C^*-subalgebra of A. Let K_B be a subset of B^* and K_A be a weak*-compact set of A^*. Suppose that every $\varphi \in K_B$ extends uniquely to $\overline{\varphi} \in K_A$. Then the natural map $\varphi \in K_B \to \overline{\varphi} \in K_A$ is weak*-continuous.*

Proof. Let $\Phi : K_B \to K_A$ be the natural injection defined by $\Phi(\varphi) = \overline{\varphi}$. Let F be a weak*-closed subset of K_A. We show that $\Phi^{-1}(F)$ is weak*-closed in K_B. Let $\varphi = w^* - \lim_\alpha \varphi_\alpha$ with $\varphi_\alpha \in \Phi^{-1}(F)$. By weak* compactness of K_A, there exists a subnet $\{\varphi_\beta\}$ such that $\Phi(\varphi_\beta)$ converges to $\psi \in F$, say, in the weak* topology. We have

$$\psi|_B = w^* - \lim_\beta \Phi(\varphi_\beta)|_B = \lim_\beta \varphi_\beta = \varphi.$$

By unique extension, we have $\Phi(\varphi) = \overline{\varphi} = \psi \in F$. □

Let $K_B \subset B^*$ have the weak* topology as above, and let $C(K_B)$ denote the C^*-algebra of complex bounded continuous functions on K_B. Identifying K_B as a subset of $C(K_B)^*$, we have the following general criterion for unique extension.

Proposition 2. *Let A be a C^*-algebra and B a C^*-subalgebra of A. Let K_B be a subset of B^* and let K_A be a weak*-compact set in A^*. The following conditions are equivalent:*

(i) *Every $\varphi \in K_B$ extends uniquely to a $\overline{\varphi} \in K_A$;*

(ii) *There is a linear map $Q : A \to C(K_B)$ such that for every $\varphi \in K_B$ with extension $\overline{\varphi} \in K_A$, we have $\overline{\varphi} = \varphi \circ Q$.*

Proof. It is evident that (ii) $\Longrightarrow$ (i). Assume condition (i). By Lemma 1, the map $Q : A \to C(K_B)$ given by

$$Q(a)(\varphi) = \overline{\varphi}(a) \qquad (a \in A,\ \varphi \in K_B)$$

is well defined and satisfies condition (ii). □

Remark. In the above result, Q is continuous if K_B is a norm-bounded subset of B^*.

We have the following immediate corollary which has been obtained by Anderson [2] under the assumption that A is unital and with a different method.

Corollary 3. *Let B be an abelian C^*-subalgebra of a C^*-algebra A. The following conditions are equivalent:*

(i) *Every $\varphi \in P(B) \cup \{0\}$ extends uniquely to a $\overline{\varphi} \in P(A) \cup \{0\}$;*

(ii) *There is a contractive projection $Q : A \to B$ such that $\overline{\varphi} = \varphi \circ Q$ for every $\varphi \in P(B) \cup \{0\}$ with extension $\overline{\varphi}$ on A.*

Proof. We note that if a pure state has a unique pure state extension, then it has unique state extension. We identify B with $C_0(P(B)) = \{b \in C(P(B) \cup \{0\}) : b(0) = 0\}$. Letting $K_B = P(B) \cup \{0\}$ and $K_A = \overline{P(A) \cup \{0\}}$ in Proposition 2, we see that the map $Q : A \to C(K_B)$ has range $Q(A) \subset C_0(P(B)) = B \subset Q(A)$ and Q is a desired contractive projection. □

Now we are going to show how Lemma 1 together with the aforementioned result of Pfitzner can be used to obtain new information on the unique extension of states. Recall that a Banach space A is called a *Grothendieck space* [9] if every weak*-convergent sequence in A^* is $\sigma(A^*, A^{**})$-convergent. It is evident that a quotient space of a Grothendieck space is also a Grothendieck space and that a separable Grothendieck space must be reflexive by the weak compactness of the dual ball.

Theorem 4 (Pfitzner [19]). *Every von Neumann algebra is a Grothendieck space.*

It follows from Pfitzner's theorem that there is no surjective continuous linear map from a von Neumann algebra onto an infinite-dimensional *separable* C^*-algebra. This interesting fact will be used to deduce Theorem 6 below. We note that if M is an infinite-dimensional von Neumann algebra, then its predual M_* (and hence M^*) is not a Grothendieck space. Indeed, M contains a copy of c_0 and so M_* contains a complemented copy of l_1 [16; Proposition 2.e.8] which is not a Grothendieck space. Also $M_* \supset l_1$ implies that M^* contains a bounded sequence which has *no* weak*-convergent subsequence [16, Theorem 2.e.7]. Therefore, in order to apply Pfitzner's result, we need to assume the separability condition in some of the following arguments.

We note that there are C^*-algebras, other than von Neumann algebras, which are Grothendieck spaces. For instance, the Calkin algebra $B(H)/K(H)$ is such a space. If K is a compact Hausdorff space in which any two disjoint open F_σ sets have disjoint closures, then $C(K)$ is a Grothendieck space [22]. Also, given a C^*-algebra A which is a Grothendieck space, it is easy to verify that the matrix algebra $M_n(A)$ over A is also a Grothendieck space. In fact, one can deduce from [4] that a C^*-algebra A is a Grothendieck space if and only if there is no surjective continuous linear map from A onto c_0. In Theorem 6, Theorem 7 and Corollary 8 below, one can actually replace the von Neumann algebra there by a C^*-algebra which is a Grothendieck space.

Lemma 5. *Let φ and ψ be two states of a C^*-algebra A. Suppose there is a minimal projection p in A such that $\varphi(p) = \psi(p) = 1$. Then $\varphi = \psi$.*

Proof. Note that φ is a normal state of A^{**} which has identity 1. For $x \in A$, we have $|\varphi((1-p)x)|^2 \leq \varphi(1-p)\varphi(x^*x) = 0$, which gives $\varphi(px) = \varphi(x)$. Likewise $\varphi(xp) = \varphi(x)$. Therefore $\varphi(pxp) = \varphi(xp) = \varphi(x)$. Similarly we obtain that $\psi(pxp) = \psi(x)$. Hence, for $x \in A$ with $pxp = \lambda p$ where $\lambda \in \mathbb{C}$, we have $\varphi(x) = \varphi(pxp) = \lambda = \psi(pxp) = \psi(x)$. □

Given a sequence $\{A_n\}_{n=1}^{\infty}$ of C^*-algebras, their c_0-sum is denoted by $A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus \cdots$ which is the C^*-algebra consisting of all sequences $\{a_n\}$ such that $a_n \in A_n$ and for any $\varepsilon > 0$ the set $\{n : \|a_n\| \geq \varepsilon\}$ is finite, with the coordinatewise algebraic operations and the supremum norm. For a finite sequence, it is the usual C^*-direct sum.

Theorem 6. *Let B be a separable abelian C^*-subalgebra of a von Neumann algebra M. The following conditions are equivalent:*

(i) *Every pure state of B extends uniquely to a pure state of M.*

(ii) $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n \oplus \cdots$, *where each p_n is a minimal projection in M and the sum may be finite.*

Proof. (i) $\Longrightarrow$ (ii). If B is unital, then $B = C(P(B))$ and the identity of B is a projection $p \in M$. Every pure state of B extends to a unique pure state of pMp. By Corollary 3 and the remarks after Theorem 4, we infer that B is finite-dimensional and hence $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n$, where $p_1, p_2, ..., p_n$ are mutually orthogonal projections in M.

Suppose that B is nonunital. Then $B = C_0(P(B)) = \{b \in C(P(B) \cup \{0\}) : b(0) = 0\}$, where $X = P(B) \cup \{0\}$ is a compact metric space. We have only to show that $P(B)$ is countable and discrete. Let $\{V_n\}$ be a decreasing sequence of open neighbourhoods of 0 such that $\overline{V_{n+1}} \subset V_n$ and $\bigcap_{n=1}^{\infty} V_n = \{0\}$. Then $K_n = X \backslash V_n$ is a compact subset of $P(B)$. Define a linear map $Q : M \to C(K_n)$ by

$$Q(a)(\varphi) = \overline{\varphi}(a) \quad (a \in M,\ \varphi \in K_n),$$

where $\overline{\varphi} \in P(M)$ is the unique extension of φ. By Lemma 1, Q is well defined. Also Q is continuous since $\|Q(a)\| \leqq \|a\|$. Moreover Q is surjective. Indeed, given $h \in C(K_n)$, there exists $b \in C_0(P(B)) \subset M$ such that $b|_{K_n} = h$ which gives $Q(b)(\varphi) = \overline{\varphi}(b) = \varphi(b) = h(\varphi)$ for $\varphi \in K_n$, that is, $Q(b) = h$. Since $C(K_n)$ is separable, K_n must be finite by the remark after Theorem 4. Hence $X = \{0\} \cup \bigcup_{n=1}^{\infty} K_n$ is countable. For each n, $X \backslash \overline{V_n}$ is a finite open subset of $P(B)$ since it is contained in K_n. It follows that $P(B) = \bigcup_{n=1}^{\infty} (X \backslash \overline{V_n})$ is discrete. Thus we obtain that $B = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \cdots \oplus \mathbb{C}p_n \oplus \cdots$, where $p_1, p_2, ..., p_n, ...$ are mutually orthogonal projections in M.

Assume, for contradiction, that dim $p_1 M p_1 \geq 2$. Then there are distinct pure states φ and ψ on $p_1 M p_1$. As $\varphi(p_1) = \psi(p_1) = 1$, it follows that the pure state ω on B with $\omega(p_1) = 1$ has distinct pure state extensions on M, which contradicts (i). So p_1, and likewise, $p_2, ..., p_n, ...$ are all minimal projections in M.

(ii) $\Longrightarrow$ (i). Let $\omega \in P(B)$. Then there exists i such that $\omega(p_i) = 1$ and $\omega(p_j) = 0$ for $j \neq i$. Let $\varphi, \psi \in P(M)$ be extensions of ω. Then $\varphi(p_i) = \omega(p_i) = \psi(p_i) = 1$. So $\varphi = \psi$ by Lemma 5. □

Remarks. 1. As already mentioned, Theorem 6 holds if M is replaced by, for instance, the Calkin algebra $B(H)/K(H)$. The referee has kindly pointed out an alternative approach to Theorem 6 which does not depend on M being a Grothendieck space but makes use of its ample supply of projections.

2. Theorem 6 is false for nonabelian subalgebras. In fact, it is even false for type I_0 subalgebras. Let M be the full operator algebra on an infinite-dimensional separable Hilbert space H and let B be the C^*-algebra of compact operators on H. Then B satisfies condition (i) in Theorem 6.

3. Kadison and Singer [14] have shown that if B is a maximal abelian subalgebra of $B(H)$ isomorphic to $L^\infty(0,1)$, then there exists a pure state on B with distinct extensions to $B(H)$.

4. Although every state of a C^*-algebra B extends uniquely to a normal state on B^{**}, Theorem 6 shows that a pure state on B need not extend uniquely to a pure state on B^{**}.

Let K be a subset of the state space of a C^*-algebra. We will denote by $\mathrm{co}(K)$ the convex hull of K throughout this paper.

Let A be a C^*-algebra, and let φ be a state of A. We denote by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of A associated with φ, that is, π_φ is a represenatation of A on the Hilbert space H_φ with the canonical cyclic vector ξ_φ and the inner product defined by $\langle \pi_\varphi(x)\xi_\varphi,\ \pi_\varphi(y)\xi_\varphi\rangle = \varphi(y^*x)$ for $x, y \in A$ ([10, 18]). A state φ is called a *type* I *state* if the GNS representation associated with φ is type I.

We recall that a C^*-algebra A is *dual* if and only if it is isomorphic to a C^*-subalgebra of the C^*-algebra $K(H)$ of compact operators, or equivalently, every maximal abelian subalgebra of A is generated by minimal projections [10, 4.7.20]. Note that dual C^*-algebras are scattered (cf. [12]).

Theorem 7. *Let B be a separable C^*-subalgebra of a von Neumann algebra M. If every $\varphi \in F(B)$ extends uniquely to a $\overline{\varphi} \in \overline{F(M)}$, then B is a scattered C^*-algebra.*

Proof. Let $\varphi \in F(B)$. We show that φ is type I. Let $F_I(B)$ be the set of all type I factor states of B. By [5, Corollary 3.4] and by separability of B, there is a sequence $\{\varphi_n\}$ in $F_I(B)$ such that $\varphi = w^* - \lim_{n\to\infty} \varphi_n$. By Lemma 1, we have $\overline{\varphi} = w^* - \lim_{n\to\infty} \overline{\varphi_n}$ in M^*. By Theorem 4, $\overline{\varphi} = \lim_{n\to\infty} \overline{\varphi_n}$ in the $\sigma(M^*, M^{**})$-topology and hence $\overline{\varphi} \in \overline{\overline{\mathrm{co}}}(\{\overline{\varphi_n} : n = 1, 2, ...\})$. It follows that $\varphi = \overline{\varphi}|_B \in \overline{\overline{\mathrm{co}}}(\{\varphi_n : n = 1, 2, ...\}) \subset \overline{\overline{\mathrm{co}}}(F_I(B))$. Therefore φ is type I since type I states form a norm-closed convex set by [13, Theorem 4.1].

Since B is type I, it contains a nonzero hereditary abelian C^*-subalgebra C (cf. [18, 6.1]) and by [18, 3.1.6], every pure state of C extends uniquely to one of M. By Theorem 6, C is generated by minimal projections of M and, in particular, the set P of all those minimal projections of M contained in B is nonempty. Let I be the norm-closed ideal in B generated by P and let J be the norm-closed ideal in M generated by P. Then I and J are dual C^*-algebras and $I = B \cap J$. Now pass to the inclusion $B/I \subset M/J$ and we can repeat the above argument since Theorem 6 is valid for the Grothendieck space M/J as remarked before. By transfinite induction, B has a composition series in which successive quotients are dual C^*-algebras. Hence B is scattered. □

Remark. The arguments in the last paragraph are due to the referee.

Corollary 8. *Let B be a separable hereditary C^*-subalgebra of a von Neumann algebra M. Then B is a dual C^*-algebra generated by minimal projections of M.*

Proof. By Theorem 7, B is scattered. Let C be any maximal abelian subalgebra of B. Then C is generated by projections. But if $p \in B$ is a projection, then pMp is contained in B and is therefore finite-dimensional by separability. So C is generated by minimal projections and it follows that B is a dual C^*-algebra generated by minimal projections of M. □

We recall that given a C^*-subalgebra B of a C^*-algebra A, then B is hereditary in A if and only if every $\varphi \in S(B)$ extends uniquely to a $\overline{\varphi} \in S(A)$ (cf. [15], [18]). Let B act on a Hilbert space H and let $V(B) = \{\omega_\xi : \xi \in H, \|\xi\| = 1\}$ be the set of vector states of B where $\omega_\xi(\cdot) = \langle \cdot \xi, \xi \rangle$ is defined on B. Then $\mathrm{co}(V(B)) \subset S(B) \subset \overline{\mathrm{co}}(V(B))$ [10, 3.4.1]. In the following, we will consider unique extension of states in $\mathrm{co}(V(B))$. Given $K \subset S(B)$, we denote by

$$\sigma(K) = \{\sum_{n=1}^{\infty} \lambda_n \varphi_n : \varphi_n \in K, \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1\}$$

the σ-convex hull of K, where the infinite sum is norm-convergent.

We note that $\sigma(K) \subset \overline{\overline{\mathrm{co}}}(K)$. Indeed, let $\varphi = \sum_{n=1}^{\infty} \lambda_n \varphi_n \in \sigma(K)$ and let $\varepsilon > 0$. Then there exists j such that $\| \sum_{n=1}^{j} \lambda_n (\varphi - \varphi_n)\| < \varepsilon$ and $\lambda = \sum_{n=1}^{j} \lambda_n > \frac{1}{2}$. So $\|\varphi - \sum_{n=1}^{j} \frac{\lambda_n}{\lambda} \varphi_n\| < \frac{\varepsilon}{\lambda} < 2\varepsilon$ and hence $\varphi \in \overline{\overline{\mathrm{co}}}(K)$.

We also note that $\sigma(\sigma(K)) = \sigma(K)$. To see this, let $\mu = \sum_{n=1}^{\infty} \alpha_n \mu_n$ with $\alpha_n \geq 0$, $\sum_{n=1}^{\infty} \alpha_n = 1$ and $\mu_n \in \sigma(K)$. We show $\mu \in \sigma(K)$. Let $\mu_n = \sum_{k=1}^{\infty} \lambda_k^n \nu_k^n$ with $\lambda_k^n \geq 0$, $\sum_{k=1}^{\infty} \lambda_k^n = 1$ and $\nu_k^n \in K$. Since the norm is additive on $S(B)$, we have $\sum_{k=1}^{\infty} \lambda_k^n \|\nu_k^n\| = \|\mu_n\| = 1 = \sum_{n=1}^{\infty} \alpha_n \|\mu_n\|$. Let $\varphi_m = \sum_{n+k \leq m} \alpha_n \lambda_k^n \nu_k^n$. Then $\sum_{m=2}^{\infty} \|\varphi_m\| < \infty$ since it has bounded partial sums. By absolute convergence, we have, for $b \in B$,

$$\sum_{m=2}^{\infty} \varphi_m(b) = \sum_{m=2}^{\infty} \sum_{n+k \leq m} \alpha_n \lambda_k^n \nu_k^n(b) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_n \lambda_k^n \nu_k^n(b) = \sum_{n=1}^{\infty} \alpha_n \mu_n(b) = \mu(b).$$

So $\mu = \sum_{m=2}^{\infty} \varphi_m \in \sigma(K)$.

Proposition 9. *For any C^*-algebra B, we have $\overline{\overline{\mathrm{co}}}(F(B)) = \sigma(F(B))$.*

Proof. By the above remarks, we need only show that $\sigma(F(B))$ is norm-closed. Let $\varphi = \lim_{n \to \infty} \varphi_n$ in the norm topology where $\varphi_n \in \sigma(F(B))$. Let $\psi = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n$. Then $\psi \in \sigma(\sigma(F(B))) = \sigma(F(B))$ and hence its GNS representation π_ψ is a (countable) direct sum of factor representations (cf. Theorem 11). Let $p \in B^{**}$ be the support of ψ and let $q \in B^{**}$ be the support of φ. Then $\varphi_n(1-p) = 0$ for all n implies $\varphi(1-p) = 0$. Hence $q \leq p$ and π_φ is equivalent to a subrepresentation of π_ψ. It follows that π_φ is a (countable) direct sum of factor representations and so $\varphi \in \sigma(F(B))$. □

Remark. One can show analogously that $\overline{\overline{\mathrm{co}}}(P(B)) = \sigma(P(B))$.

Theorem 10. *Let B be a separable C^*-algebra acting on a Hilbert space H such that its weak closure M is a direct sum of factors. Suppose that every $\varphi \in \text{co}(V(B))$ extends uniquely to a $\overline{\varphi} \in S(M)$. Then $S(B) = \sigma(F(B))$.*

Proof. Let $K(H)$ be the C^*-algebra of compact operators on H. Let $\varphi \in S(B)$. Then $\varphi = \lambda\phi + (1-\lambda)\psi$ where $0 \leq \lambda \leq 1, \|\phi|_{B\cap K(H)}\| = 1$ and $\psi|_{B\cap K(H)} = 0$. Since $B \cap K(H)$ is a scattered C^*-algebra [12], $\phi|_{B\cap K(H)}$ is a σ-convex sum of pure states of $B \cap K(H)$. As $B \cap K(H)$ is an ideal in B, by unique extension, we conclude that $\phi \in \sigma(P(B))$. By [11, Theorem 2] and by separability of B, we have $\psi = w^* - \lim_{n\to\infty} \omega_n$ where $\omega_n \in V(B)$. By Lemma 1, the simultaneous extension map $\omega \in \text{co}(V(B)) \to \overline{\omega} \in S(M)$ is weak*-continuous and unique extension also implies that the map $\omega - \omega' \in V(B) - V(B) \to \overline{\omega} - \overline{\omega'} \in S(M) - S(M)$ is well defined and weak*-continuous. It follows that $\overline{\omega_m} - \overline{\omega_n} \to 0$ in the weak* topology as $m, n, \to \infty$ and therefore $\omega = w^* - \lim_{n\to\infty} \overline{\omega_n}$ exists. By Theorem 4, we have $\omega = \lim_{n\to\infty} \overline{\omega_n}$ in the $\sigma(M^*, M^{**})$-topology which gives $\omega \in \overline{\overline{\text{co}}}(\{\overline{\omega_n} : n = 1, 2, ...\})$. So

$$\psi = \omega|_B \in \overline{\overline{\text{co}}}(\{\omega_n : n = 1, 2, ...\}).$$

By the hypothesis, the identity representation of B is a direct sum of factor representations. Since the GNS representation induced by ω_n is equivalent to a subrepresentation of the identity representation, it follows that $\omega_n \in \sigma(F(B))$. By Proposition 9, we have

$$\varphi = \lambda\phi + (1-\lambda)\psi \in \sigma(F(B)). \qquad \square$$

A special example of Theorem 10 is the C^*-algebra $B = K(H)$ in which case we even have $S(B) = \sigma(P(B))$ [10, 4.1.3]. We study below those algebras satisfying the conclusion of Theorem 10.

3. A class of C^*-algebras whose states are σ-convex sums of factor states

In Theorem 10, a class of C^*-algebras A occur for which $S(A) = \sigma(F(A))$. We derive some properties of these algebras in this section.

Let A be a C^*-algebra and let I be a closed two-sided ideal in A. We denote as before by $(\pi_\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of A associated with a state φ. Given a state φ on I with unique extension $\overline{\varphi}$ to A, we have $\varphi \in F(I)$ if and only if $\overline{\varphi} \in F(A)$ since $\pi_\varphi(I)'' = \pi_{\overline{\varphi}}(A)''$ [8, Lemma 4.1.33]. Further, if $q : A \to A/I$ is the quotient map and if $\psi \in S(A/I)$, then by [24, Theorem 1.4] $\psi \in F(A/I)$ if and only if $\psi \circ q \in F(A)$.

Theorem 11. *Let A be a C^*-algebra and let I be a closed two-sided ideal of A. The following conditions are equivalent:*

(i) $S(A) = \sigma(F(A))$.

(ii) *Every nondegenerate representation of A is equivalent to a subrepresentation of a direct sum of factor representations.*

(iii) A^{**} *is a direct sum of factors.*

(iv) $S(I) = \sigma(F(A))$ *and* $S(A/I) = \sigma(F(A/I))$.

Proof. (i) $\Longrightarrow$ (ii). Given a cyclic representation $\pi_\varphi : A \to B(H_\varphi)$ with $\varphi \in S(A)$ and $\varphi = \sum_{n=1}^{\infty} \lambda_n \varphi_n \in \sigma(F(A))$ where $\varphi_n \in F(A)$ and $0 < \lambda_n < 1$, we have

$$\begin{aligned}\langle \pi_\varphi(\cdot)\xi_\varphi,\ \xi_\varphi\rangle = \varphi(\cdot) = \sum_{n=1}^{\infty} \lambda_n \varphi_n &= \sum_{n=1}^{\infty} \langle \pi_{\varphi_n}(\cdot)\sqrt{\lambda_n}\xi_{\varphi_n},\ \sqrt{\lambda_n}\xi_{\varphi_n}\rangle \\ &= \langle \bigoplus_n \pi_{\varphi_n}(\cdot)\xi,\ \xi\rangle\end{aligned}$$

where $\xi = \bigoplus_n \sqrt{\lambda_n}\xi_{\varphi_n} \in \bigoplus_n H_{\varphi_n}$ and π_{φ_n} is a factor representation.

(ii) $\Longrightarrow$ (iii). The universal representation is a direct sum of factor representations.

(iii) $\Longrightarrow$ (i). Let $\varphi \in S(A)$ and let $e \in A^{**}$ be its central support. Then $e = \sum_\alpha e_\alpha$ where e_α is a minimal central projection in A^{**}. Since $\sum_\alpha \varphi(e_\alpha) = \varphi(e) = 1$, $\sum_\alpha \varphi(e_\alpha)$ is a countable sum. Relabel $\{e_\alpha\}$ as $\{e_n\}$ so that $\lambda_n = \varphi(e_n) > 0$ with $\sum_n \lambda_n = 1$ and $\varphi(a) = \varphi(ae) = \sum_n \varphi(ae_n)$ for $a \in A$. Let $\varphi_n(\cdot) = \frac{1}{\lambda_n}\varphi(\cdot\, e_n)$. Then $\varphi_n \in S(A)$ and e_n is its central support. Indeed, let $p \in A^{**}$ be a central projection such that $\varphi_n(a) = \varphi_n(ap)$ for $a \in A$. Then $\varphi(ape_n) = \lambda_n \varphi_n(ap) = \varphi(ae_n)$. Let $e' = \sum_m e'_m$ where

$$e'_m = \begin{cases} e_m & \text{if} \quad m \neq n, \\ pe_n & \text{if} \quad m = n. \end{cases}$$

Then $\varphi(ae') = \sum_m \varphi(ae'_m) = \sum_m \varphi(ae_m) = \varphi(ae) = \varphi(a)$, which gives $e' \geq e$ and so $pe_n \geq e_n$. As e_n is a minimal central projection, we have $\varphi_n \in F(A)$. So $\varphi = \sum_n \lambda_n \varphi_n \in \sigma(F(A))$.

(i) $\Longleftrightarrow$ (iv). This follows from the remarks before Theorem 11 and the fact that given $\varphi \in S(A)$ and given a closed two-sided ideal $I \subset A$, we have $\varphi = \varphi_1 + \varphi_2$ with $\|\varphi_1\| = \|\varphi|_I\|$ and $\varphi_2(I) = \{0\}$. $\square$

Remark. A C^*-algebra B is scattered if and only if it is type I and $S(B) = \sigma(F(B))$.

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