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PROOF OF THE SIMON-ANDO THEOREM

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ABSTRACT. In 1961, Simon and Ando wrote a classical paper describing the convergence properties of nearly completely decomposable matrices. Basically, their work concerned a partitioned stochastic matrix e.g.

$$A = \begin{bmatrix} A_1 & E_1 \\ E_2 & A_2 \end{bmatrix}$$

where A_1 and A_2 are square blocks whose entries are all larger than those of E_1 and E_2 respectively.

Setting

$$A^k = \begin{bmatrix} A_1^{(k)} & E_1^{(k)} \\ E_2^{(k)} & A_2^{(k)} \end{bmatrix},$$

partitioned as in A, they observed that for some, rather short, initial sequence of iterates the main diagonal blocks tended to matrices all of whose rows are identical, e.g. $A_1^{(k)}$ to F_1 and $A_2^{(k)}$ to F_2. After this initial sequence, subsequent iterations showed that all blocks lying in the same column as those matrices tended to a scalar multiple of them, e.g.

$$\lim_{k\to\infty} A^k = \begin{bmatrix} \alpha F_1 & \beta F_2 \\ \alpha F_1 & \beta F_2 \end{bmatrix}$$

where $\alpha \geq 0$ and $\beta \geq 0$.

The purpose of this paper is to give a qualitative proof of the Simon-Ando theorem.

Let B be an $n \times n$ nonnegative matrix. Partition

$$B = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1r} \\ B_{21} & B_2 & \cdots & B_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ B_{s1} & B_{s2} & \cdots & B_s \end{bmatrix}.$$

Choose any column of blocks, say

$$\overline{B}_k = \begin{bmatrix} B_{1k} \\ \vdots \\ B_k \\ \vdots \\ B_{sk} \end{bmatrix}.$$

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If, for all j, any entry in the jth column of $\overline{B}_k$ which lies in B_k is at least as large as any entry in this column which lies in B_{ki}, for all i, then we say that B_k is block column dominant.

Throughout the paper we let

$$A = \begin{bmatrix} A_1 & \mathcal{E}_{12} & \cdots & \mathcal{E}_{1r} \\ \mathcal{E}_{21} & A_2 & \cdots & \mathcal{E}_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{r1} & \mathcal{E}_{r2} & \cdots & A_r \end{bmatrix}$$

be an $n \times n$ stochastic matrix with each A_i being $n_i \times n_i$ and column block dominant.

Let B be an $n \times m$ nonnegative matrix. Define

$$\alpha_j(B) = \max_k b_{kj} - \min_k b_{kj}.$$

Thus, α_j measures the spread of the entries in the jth column of B, and considering all j, the distance between the rows.

The next lemma measures the contraction of the spread under multiplication.

Lemma 1. *Let B be an $n \times n$ nonnegative matrix and c an $n \times 1$ nonnegative vector. Let $\overline{r}$ and $\underline{r}$ denote the largest and smallest row sums of B, respectively, and m the smallest entry in B. Then*

$$\alpha_1(Bc) \leq (\overline{r} - 2m)\alpha_1(c) + (\overline{r} - \underline{r}) \min_i c_i.$$

If $\overline{r} \leq 1$, then $\max_i (Bc)_i \leq \max_i c_i$.

Proof. First note that for any p,

$$\begin{aligned} &m \max_i c_i + \left(\sum_{k=1}^n b_{pk} - m \right) \min_i c_i \\ &\qquad \leq \sum_{k=1}^n b_{pk} c_k \leq \left(\sum_{k=1}^n b_{pk} - m \right) \max_i c_i + m \min_i c_i. \end{aligned}$$

Of course if $\overline{r} \leq 1$ the right side is bounded by $\max_i c_i$. For convenience, set $Bc = c'$. Then using the inequality above to measure spread,

$$\begin{aligned} \max_i c_i' - \min_i c_i' &\leq [(\overline{r} - m) \max_i c_i + m \min_i c_i] - [m \max_i c_i + (\underline{r} - m) \min_i c_i] \\ &\leq (\overline{r} - 2m) \max_i c_i - (\underline{r} - 2m) \min_i c_i \\ &\leq (\overline{r} - 2m)\alpha_1(c) + (\overline{r} - \underline{r}) \min_i c_i. \qquad \square \end{aligned}$$

As in [3], for any nonnegative matrix C, define

$$\mathcal{T}(C) = \frac{1}{2} \max_{i,j} \sum_k |c_{ik} - c_{jk}|$$

called a coefficient of ergodicity of C. If

$$\mathcal{E}_p = (\mathcal{E}_{p1}, \ldots, \mathcal{E}_{p\,p-1}, \mathcal{E}_{p\,p+1}, \ldots, \mathcal{E}_{pr}),$$

the $\mathcal{E}_{ij}$ blocks taken from A, then define

$$\varepsilon = \max_p \mathcal{T}(\mathcal{E}_p).$$

In addition, for the remainder of the paper, we let $\overline{r}$ and $\underline{r}$ denote the largest and smallest row sums of A_i and m the smallest entry in A_i, over all i. And also for the remainder of the paper we denote by B an $n \times n$ stochastic matrix

$$B = \begin{bmatrix} B_1 & \delta_{12} & \cdots & \delta_{1r} \\ \delta_{21} & B_2 & \cdots & \delta_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{r1} & \delta_{r2} & \cdots & B_r \end{bmatrix}$$

partitioned compatibly with A. The role of B in our iteration work is that if we write $A^{k+1} = AA^k$, then B will denote A^k.

The next two theorems are concerned with the contraction of the rows in all blocks of AB. Without loss of generality, we consider the $1,1$-block of this product. A preliminary lemma is needed.

Lemma 2. $\alpha_j(A_1B_1 + \sum_{k>1} \mathcal{E}_{1k}\delta_{k1}) \leq (\overline{r} - 2m)\alpha_j(B_1) + (\overline{r} - \underline{r}) + \varepsilon$.

Proof. Note that

$$\begin{aligned} &\alpha_j\left(A_1B_1 + \sum_{k>1} \mathcal{E}_{1k}\delta_{k1}\right) \\ &\qquad \leq \alpha_j(A_1B_1) + \alpha_j\left(\sum_{k>1} \mathcal{E}_{1k}\delta_{k1}\right) \\ &\qquad \leq \alpha_j(A_1B_1) + \max_p \sum_{r,k>1} \varepsilon_{pr}^{(1,k)}\delta_{rj}^{(k,1)} - \min_q \sum_{r,k>1} \varepsilon_{qr}^{(1,k)}\delta_{rj}^{(k,1)} \end{aligned}$$

where $\varepsilon_{pr}^{(1,k)}$ and $\delta_{rj}^{(k,1)}$ are entries in $\mathcal{E}_{1k}$ and δ_{k1}, respectively, with $\varepsilon_{pr}^{(1,k)}$ and $\delta_{rj}^{(k,1)}$ lying in rows p and r and columns r and j, respectively. Thus, using Lemma 1,

$$\begin{aligned} \alpha_j\left(A_1B_1 + \sum_{k>1} \mathcal{E}_{1k}\delta_{k1}\right) &\leq \alpha_j(A_1B_1) + \max_{p,q} \sum_{r,k>1} |\varepsilon_{pr}^{(1,k)} - \varepsilon_{qr}^{(1,k)}|\delta_{rj}^{(k,1)} \\ &\leq \alpha_j(A_1B_1) + \varepsilon \\ &\leq (\overline{r} - 2m)\alpha_j(B_1) + (\overline{r} - \underline{r}) + \varepsilon. \end{aligned}$$

The result follows. $\square$

Theorem 1. *Let γ be a nonnegative number with $\gamma < 1$. If $\frac{\overline{r}-\underline{r}+\varepsilon}{\gamma-\overline{r}+2m} \leq \alpha_j(B_1)$, then*

$$\alpha_j\left(A_1B_1 + \sum_k \mathcal{E}_{1k}\delta_{k1}\right) \leq \gamma\alpha_j(B_1).$$

Proof. By hypothesis,

$$\frac{\overline{r} - \underline{r} + \varepsilon}{\gamma - \overline{r} + 2m} \leq \alpha_j(B_1).$$

Thus,

$$\overline{r} - \underline{r} + \varepsilon \leq (\gamma - \overline{r} + 2m)\alpha_j(B_1).$$

Hence,

$$(\overline{r} - 2m)\alpha_j(B_1) + (\overline{r} - \underline{r}) + \varepsilon \leq \gamma\alpha_j(B_1).$$

And by Lemma 2,

$$\alpha_j\left(A_1B_1+\sum_k \mathcal{E}_{1k}\delta_{k1}\right)\le \gamma\alpha_j(B_1).$$

Note, if γ is chosen smallest in Theorem 1, then

$$\gamma=\frac{(\overline{r}-\underline{r}+\varepsilon)+(\overline{r}-2m)\alpha_j(B_1)}{\alpha_j(B_1)},$$

which is decreasing in $\alpha_j(B_1)$. So as $\alpha_j(B_1)$ gets smaller, γ gets bigger.

On the other hand we have the following. $\square$

Theorem 2. *Let γ be a nonnegative number with $\gamma<1$. If $\frac{\overline{r}-\underline{r}+\varepsilon}{\gamma-\overline{r}+2m}>\alpha_j(B_1)$, then*

$$\alpha_j\left(A_1B_1+\sum_k \mathcal{E}_{1k}\delta_{k1}\right)\le \gamma\left(\frac{\overline{r}-\underline{r}+\varepsilon}{\gamma-\overline{r}+2m}\right).$$

Proof. By hypothesis,

$$\frac{\overline{r}-\underline{r}+\varepsilon}{\gamma-\overline{r}+2m}>\alpha_j(B_1).$$

Then by Lemma 2,

$$\begin{aligned}\alpha_j\left(A_1B_1+\sum_k \mathcal{E}_{1k}\delta_{k1}\right)&\le(\overline{r}-2m)\alpha_j(B_1)+(\overline{r}-\underline{r})+\varepsilon\\ &\le(\overline{r}-2m)\left(\frac{\overline{r}-\underline{r}+\varepsilon}{\gamma-\overline{r}+2m}\right)+(\overline{r}-\underline{r})+\varepsilon\\ &\le(\overline{r}-\underline{r}+\varepsilon)\left[\frac{\overline{r}-2m}{\gamma-\overline{r}+2m}+1\right]\\ &\le(\overline{r}-\underline{r}+\varepsilon)\left(\frac{\overline{r}-2m+\gamma-\overline{r}+2m}{\gamma-\overline{r}+2m}\right)\\ &\le(\overline{r}-\underline{r}+\varepsilon)\left(\frac{\gamma}{\gamma-\overline{r}+2m}\right).\end{aligned}$$

Let $a=\frac{\overline{r}-\underline{r}+\varepsilon}{r-\overline{r}+2m}$. Putting Theorem 1 and Theorem 2 together we see that if $\alpha_j(B_1)\notin[0,a]$, then the next iterate AB, after B, shows contraction of the rows $A_1B_1+\sum_k \mathcal{E}_{1k}\delta_{k1}$. On the other hand if $\alpha_j(B_1)\in[0,a]$, then so is $\alpha_j(A_1B_1+\sum_k \mathcal{E}_{1k}\delta_{k1})$, the next iterate. Thus, under iteration, all blocks tend to flat matrices and when close remain so.

We now consider a fixed rate of convergence. Set

$$\gamma=\overline{r}-m.$$

Theorem 1 says that if

$$\frac{\overline{r}-\underline{r}+\varepsilon}{m}\le\alpha_j(B_1),$$

then

(i) $\alpha_j(A_1B_1+\sum_k \mathcal{E}_{1k}\delta_{k1})\le(\overline{r}-m)\alpha_j(B_1)$, so there is further contraction of rows in the 1,1-block of AB, and this block continues the trend toward a flat matrix.

By Theorem 2, if

$$\frac{\overline{r} - \underline{r} + \varepsilon}{m} > \alpha_j(B_1),$$

then

$$\text{(ii)} \qquad \alpha_j\left(A_1B_1 + \sum_k \mathcal{E}_{1k}\delta_{k1}\right) \le (\overline{r} - m)\left(\frac{\overline{r} - \underline{r} + \varepsilon}{m}\right) \le \frac{\overline{r} - \underline{r} + \varepsilon}{m},$$

so at this time, this block remains close to a flat matrix, other blocks are handled in the same way.

We have now shown that after some initial sequence of products, say A, $A^2, \dots, A^s$, the blocks of A^s are nearly flat, and in the remaining products A^{s+1}, $A^{s+2}, \dots$, these blocks stay nearly flat. We now show, in addition, that the blocks, which lie in the same block column, stay almost proportional to the corresponding main diagonal block in A^s.

For this work we need a preliminary lemma. We use that e is a column vector all of whose entries are 1. □

Lemma 3. *Suppose, for all i and j, the largest entry in $\mathcal{E}_{ij}e$ is less than α. Set $B = A^k$. Then the entries in $\delta_{ij}e$ are less than $k\alpha$.*

Proof. The proof is by induction on k. For $k = 1$, the result is obvious.

Suppose the result holds for $k = \overline{k}$. Set $B = A^{\overline{k}}$. Then $A^{\overline{k}+1} = AB$. A typical off-diagonal block is

$$A_i\delta_{ij} + \sum_{r\ne i,j} \mathcal{E}_{ir}\delta_{rj} + \mathcal{E}_{ij}B_j.$$

Note that

$$\begin{aligned}
&\left(A_i\delta_{ij} + \sum_{r\ne i,j} \mathcal{E}_{ir}\delta_{rj} + \mathcal{E}_{ij}B_j\right) e \\
&\quad = A_i\delta_{ij}e + \sum_{r\ne i,j} \mathcal{E}_{ir}\delta_{rj}e + \mathcal{E}_{ij}B_je \\
&\quad \le \overline{k}\alpha A_ie + \overline{k}\alpha \sum_{r\ne i,j} \mathcal{E}_{ir}e + \mathcal{E}_{ij}e \\
&\quad \le \overline{k}\alpha\left(A_i + \sum_{r\ne i,j} \mathcal{E}_{ir}\right) e + \mathcal{E}_{ij}e;
\end{aligned}$$

since A is stochastic, this is no greater than

$$(\overline{k}\alpha)e + \alpha e \le (\overline{k} + 1)\alpha e. \quad \square$$

This lemma gives a bound on the size of the off-diagonal blocks of A^s. When these blocks are sufficiently small, we can prove the theorem about the blocks of future iterates being nearly proportional to the main diagonal block in A^s, lying in its column.

Theorem 3. *Let $\overline{\varepsilon} > 0$ be given. Suppose that after s iterates the rows in the main diagonal blocks of A^s are within $\overline{\varepsilon}$ and the entries in the off-diagonal blocks are no larger than $\overline{\varepsilon}$. Let f_i be the average of the rows of the ith main diagonal block of A^s and F_{ij} the $n_i \times n_j$ matrix whose rows are f_i. Then, for all $k \geq s$, there are diagonal matrices $D_{ij}^{(k)}$ and a matrix E_k such that*

$$A^k = \begin{bmatrix} D_{11}^{(k)} F_{11} & D_{12}^{(k)} F_{12} & \cdots & D_{1r}^{(k)} F_{1r} \\ D_{21}^{(k)} F_{21} & D_{22}^{(k)} F_{22} & \cdots & D_{2r}^{(k)} F_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r1}^{(k)} F_{r1} & D_{r2}^{(k)} F_{r2} & \cdots & D_{rr}^{(k)} F_{rr} \end{bmatrix} + E_k$$

where the entries in E_k are no larger than $\overline{\varepsilon}$.

Proof. By the hypothesis,

$$A^s = \begin{bmatrix} F_{11} & \overline{\varepsilon} F_{12} & \cdots & \overline{\varepsilon} F_{1r} \\ \overline{\varepsilon} F_{21} & F_{22} & \cdots & \overline{\varepsilon} F_{2r} \\ \vdots & \vdots & & \vdots \\ \overline{\varepsilon} F_{r1} & \overline{\varepsilon} F_{r2} & \cdots & \overline{\varepsilon} F_{rr} \end{bmatrix} + E_s$$

where E_s has entries no larger than $\overline{\varepsilon}$. Now,

$$A^{s+1} = \left[(\mathcal{E}_{i1} \ldots \mathcal{E}_{ii-1} A_i \mathcal{E}_{ii+1} \cdots \mathcal{E}_{ir}) \begin{pmatrix} \overline{\varepsilon} F_{1j} \\ \vdots \\ \overline{\varepsilon} F_{ij} \\ F_{jj} \\ \overline{\varepsilon} F_{i+1j} \\ \vdots \\ \overline{\varepsilon} F_{rj} \end{pmatrix} \right] + AE_s$$

where the ijth block is displayed in the first matrix of the right side. This block can be written as

$$(\overline{\varepsilon}\mathcal{E}_{i1} \cdots \overline{\varepsilon}\mathcal{E}_{ii-1} A_i \overline{\varepsilon}\mathcal{E}_{ii+1} \cdots \overline{\varepsilon}\mathcal{E}_{ir}) \begin{pmatrix} F_{1j} \\ \vdots \\ F_{ij} \\ F_{jj} \\ F_{i+1j} \\ \vdots \\ F_{rj} \end{pmatrix}.$$

Since the columns of the F_{ij}'s are identical, we can write this matrix as $D_{ij}^{(s+1)} F_j$ where $D_{ij}^{(s+1)}$ is a diagonal matrix whose tth main diagonal entry is the tth row sum of $(\overline{\varepsilon}\mathcal{E}_{i1} \ldots \overline{\varepsilon} A_i \ldots \overline{\varepsilon}\mathcal{E}_{ir})$.

Since A is stochastic, $E_{s+1} = AE_s$ has entries no larger than $\overline{\varepsilon}$. Thus, A^{s+1} has the form of the theorem. Now, duplicating the argument above shows that A^k has the form described in the theorem for all $k \geq s$. □

Putting the results together we have the following. Theorem 1 and Theorem 2 assure that during some initial sequence $A^1, A^2, \ldots, A^s$ all blocks tend toward being nearly flat and then stay nearly flat. For a convergence rate of $\overline{r} - m$, $\frac{\overline{r} + \underline{r} + \varepsilon}{m}$

determines an upper bound on this nearness to being flat. Thus, if $(\overline{r}-m)^s \leq \frac{\overline{r}-\underline{r}+\varepsilon}{m}$, then for $k \geq s$ the blocks of A^k remain within $\frac{\overline{r}+\underline{r}+\varepsilon}{m}$. And if $(\overline{r}-m)^s > \frac{\overline{r}-\underline{r}+\varepsilon}{m}$, then for all $k \geq s$ the blocks of A^k remain within $(\overline{r}-m)^s$.

Now Lemma 3 assures that in A^s the off-diagonal blocks have entries no larger than $s\alpha$. Thus, if we choose the largest s such that $(\overline{r}-m)^s > s\alpha$, then

(i) A^s has all blocks within the larger of $(\overline{r}-m)^s$ and $\frac{\overline{r}-\underline{r}+\varepsilon}{m}$ of a flat matrix while

(ii) the off-diagonal blocks of A^s are smaller than $(\overline{r}-m)^s$.

Thus, by Theorem 3, for this s the blocks of A^k, for $k \geq s$, are within the larger of $(\overline{r}-m)^s$ and $\frac{\overline{r}-\underline{r}+\varepsilon}{m}$ of

$$\begin{bmatrix} D_{11}^{(k)}F_{11} & \cdots & D_{1r}^{(k)}F_{1r} \\ \cdots & \cdots & \cdots \\ D_{r1}^{(k)}F_{r1} & \cdots & D_{rr}^{(k)}F_{rr} \end{bmatrix}$$

for some diagonal matrices $D_{ij}^{(k)}$. Thus, the row in the jth column blocks are near scalar multiples of f_j. This is the nature of the Simon-Ando theorem.

An example may be helpful.

Example. Let

$$P = \left[\begin{array}{cc:cc} .6999 & .3000 & .0001 & 0 \\ .2000 & .7996 & 0 & .0004 \\ \hdashline .0003 & 0 & .7997 & .2000 \\ 0 & .0007 & .1000 & .8993 \end{array}\right]$$

as given in [2]. Here $m = .1$, $\overline{r} = .9999$, $\underline{r} = .9993$ and $\varepsilon = .0004$. For the contraction rate of the blocks we take $\overline{r} - m = .8999$. And,

$$\frac{\overline{r}-\underline{r}+\varepsilon}{m} = .011.$$

Now solving

$$(.8999)^s \geq (.0007)s$$

for the largest such s yields that $s = 35$ and

$$(.8999)^{35} = .0249344.$$

Hence in the sequence $A, A^2, \ldots, A^{35}$ the blocks tend to flat matrices, their rows ending within .0249344. Then for the remaining iterations $A^{36}, A^{37}, \ldots$ the blocks remain at least this close to flat matrices. And the rows in the ith block column remain nearly proportional f_i.

Of course, as is well known, in the long run the iterates also tend to the flat matrix whose rows are the stochastic eigenvector for A. □

In conclusion, adjustments can be made on the bounds used in this paper to obtain a corresponding result for nonhomogeneous stochastic products.

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