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# ON COMB DOMAINS

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ABSTRACT. A result for comb domains is proved which is stronger than but in particular implies a conjecture of Rodin and Warschawski.

## 1

The term "comb domain" has been used in various contexts, but the most generally accepted definition is basically a domain $D$ whose boundary in the plane consists primarily of vertical slits

$$\begin{aligned} &a_n + iv, \qquad v \geq \frac{1}{2} - \theta_n, n = 1, 2, \ldots, a_n \uparrow +\infty \text{ as } n \to \infty, \theta_n < \frac{1}{2}, \\ &\tilde{a}_m + iv, \qquad v \leq -\frac{1}{2} + \tilde{\theta}_m, m = 1, 2, \ldots, \tilde{a}_m \uparrow +\infty \text{ as } m \to \infty, \tilde{\theta}_m < \frac{1}{2}. \end{aligned}$$

The principal question to be answered for such a domain is whether a function $f$ mapping the parallel strip $S$

$$-\infty < \mathcal{R}z < \infty, \qquad |\mathcal{I}z| < \frac{1}{2}$$

conformally onto $D$ (having suitable behavior at $+\infty$) has an angular derivative at $+\infty$. Rodin and Warschawski [4] made the following conjecture.

If $D$ is the domain in the plane whose boundary consists of the rays

$$a_n + iv, \qquad v \leq -\frac{1}{2} + \theta_n, v \geq \frac{1}{2} - \theta_n, n = 1, 2, \ldots, a_n \uparrow +\infty, 0 \leq \theta_n \leq \frac{1}{2},$$

and $f$ maps $S$ conformally onto $D$ so that the prime end of $S$ at $+\infty$ corresponds to the prime end of $D$ at $\infty$ determined by the real axis and $\sum_{n=1}^{\infty} \theta_n^2 < \infty$, then a necessary and sufficient condition for $f$ to have an angular derivative at the above prime end is that $\sum_{n=1}^{\infty}(a_{n+1} - a_n)^2 < \infty$. (We express this situation by saying that $D$ is conformal at infinity.) They proved that the condition is sufficient but were unable to prove that it is necessary.

Following concepts introduced by Burdzy [1], they studied the following situation. Let $D$ be a strip domain in the $w$-plane which contains the real axis and is such that $\partial D$ has nonempty intersection with the first and fourth quadrants.

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Lipschitz approximations for $D$ are defined as follows. Let $B^+$ be the family of Lipschitz-1 functions $g$ defined for $u > 0$ such that

$$\partial D \cap \{w | \mathcal{R}w > 0, \mathcal{I}w > 0\}$$

lies above the graph of $\frac{1}{2} + g(u)$, and let $B^-$ be the family of Lipschitz-1 functions $g$ defined for $u > 0$ such that

$$\partial D \cap \{w | \mathcal{R}w > 0, \mathcal{I}w < 0\}$$

lies below the graph of $-\frac{1}{2} + g(u)$. Let

$$h_+(u) = \underset{g \in B^+}{\mathrm{l.\,u.\,b.}}\, g(u), h_-(u) = \underset{g \in B^-}{\mathrm{g.\,l.\,b.}}\, g(u).$$

These are again Lipschitz-1 functions. Rodin and Warschawski [5] formulated the following statement.

RW. For a strip domain $D$ as above, under the assumption that

$$\int_0^\infty \min(h_+(u), 0)\, du > -\infty, \qquad \int_0^\infty \max(h_-(u), 0)\, du < +\infty,$$

a necessary and sufficient condition for $D$ to be conformal at $+\infty$ is that

$$\int_0^\infty \max(h_+(u), 0)\, du < +\infty, \qquad \int_0^\infty \min(h_-(u), 0)\, du > -\infty.$$

They were able to prove only the sufficiency part of statement RW.

The concept of angular derivative can be defined also for mappings of a half-plane onto appropriate domains. For a detailed discussion of this and its relationship to the concept for strip domains, we refer to the paper [2]. Burdzy [1] introduced the concept of Lipschitz majorant in this context and gave a necessary and sufficient condition for the existence of an angular derivative in this case. Rodin and Warschawski considered that they had given an alternative proof for the sufficiency part of this result. This was based on the belief that "the differences are not essential" [5, p. 252]. They did not justify this remark and it is almost certainly unjustifiable. Thus they had actually proved a new result.

Burdzy [1, p. 106] asserted that he gave an affirmative answer to the necessity part of the Rodin-Warschawski conjecture on comb domains, but this is not the case. Here the assertion is clearly fallacious since the domains considered by Burdzy when transferred to the strip formulation are evidently not comb domains. He made the same questionable remark as Rodin and Warschawski [1, p. 106].

## 2

Recently Swati Sastry [6] has proved the necessity part of statement RW using the method of the extremal metric. It is now possible to settle the conjecture for comb domains. In fact the conditions imposed by Rodin and Warschawski are unnecessarily restrictive. First of all the existence of an angular derivative depends only on the behavior of $D$ in a right-hand half-plane which may be taken as $\mathcal{R}w > 0$. We assume that the boundary of $D$ therein consists of the vertical slits

$$a_n + iv, \qquad v \geq \frac{1}{2} - \theta_n, n = 1, 2, \ldots, a_n \uparrow +\infty \text{ as } n \to \infty, \theta_n < \frac{1}{2},$$

$$\tilde{a}_m + iv, \qquad v \leq -\frac{1}{2} + \tilde{\theta}_m, m = 1, 2, \ldots, \tilde{a}_m \uparrow +\infty \text{ as } m \to \infty, \tilde{\theta}_m < \frac{1}{2}.$$

The result to be proved is the following.

**Theorem.** *Let* $\sum_{n=1}^{\infty} \theta_n^2 < \infty, \sum_{m=1}^{\infty} \tilde{\theta}_m^2 < \infty$. *Then for* $D$ *to be conformal at* $+\infty$ *it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n)^2 < \infty, \qquad \sum_{m=1}^{\infty} (\tilde{a}_{m+1} - \tilde{a}_m)^2 < \infty.$$

The proof is carried out by utilizing the geometric necessary and sufficient condition for the existence of an angular derivative and the necessity part of statement RW. As regards the first let $\sigma_u$, $u > 0$, denote the intersection of $D$ with the line $\mathcal{R}w = u$. Let $D^*(a, b)$, $0 < a < b$, be the quadrangle whose domain is $D \cap \{a < \mathcal{R}w < b\}$ with $\sigma_a, \sigma_b$ as a pair of opposite sides. Let $m(a, b)$ be the module of $D^*(a, b)$ for the family of curves joining the other pair of opposite sides. Then [2, 3] for $D$ to be conformal at $+\infty$ it is necessary and sufficient that

$$m(a, b) = (b - a) + o(1)$$

as $a, b, a < b$, tend to $+\infty$.

## 3

The sufficiency proof of the theorem is carried out on somewhat the lines of [4]. However, in this special case the proofs can be done much more simply. Let $a_n < a \le a_{n+1}, a_N \le b < a_{N+1}, \tilde{a}_m < a \le \tilde{a}_{m+1}, \tilde{a}_M \le b < \tilde{a}_{M+1}$. Clearly, we can assume $n + 1 < N, m + 1 < M$ since otherwise both $m(a, b)$ and $(b - a)$ are $o(1)$. Also we can assume $\theta_j < \frac{1}{4}$, $j = n + 1, \ldots, N$, $\tilde{\theta}_k < \frac{1}{4}$, $k = m + 1, \ldots, M$. About every point $a_j + (\frac{1}{2} - \theta_j)i$, $j = n + 1, \ldots, N$, with $\theta_j > 0$, we draw circumferences of radius $\theta_j, 2\theta_j$ and about every point $\tilde{a}_k - (\frac{1}{2} - \tilde{\theta}_k)i$, $k = m + 1, \ldots, M$, with $\tilde{\theta}_k > 0$, we draw circumferences of radius $\tilde{\theta}_k, 2\tilde{\theta}_k$. Let $R$ denote the rectangle $a < \mathcal{R}w < b, -\frac{1}{2} < \mathcal{I}w < \frac{1}{2}$. In

$$R \cap \left( {\bigcap_{j=n+1}^{N}}' \left( \left| w - \left( a_j + \left( \frac{1}{2} - \theta_j \right) i \right) \right| > 2\theta_j \right) \right)$$
$$\cap \left( {\bigcap_{k=m+1}^{M}}' \left( \left| w - \left( \tilde{a}_k - \left( \frac{1}{2} - \tilde{\theta}_k \right) i \right) \right| > 2\tilde{\theta}_k \right) \right)$$

we set $\rho(w) = 1$. In

$$R \cap \left[ \left( {\bigcup_{j=n+1}^{N}}' \left( \theta_j < \left| w - \left( a_j + \left( \frac{1}{2} - \theta_j \right) i \right) \right| < 2\theta_j \right) \right) \right.$$
$$\left. \cup \left( {\bigcup_{k=m+1}^{M}}' \left( \tilde{\theta}_k < \left| w - \left( \tilde{a}_k - \left( \frac{1}{2} - \tilde{\theta}_k \right) i \right) \right| < 2\tilde{\theta}_k \right) \right) \right]$$

we set $\rho(w) = 2$. (The primes denote that the sums are taken over those $\theta_j > 0, \tilde{\theta}_k > 0$.) Elsewhere we set $\rho(w) = 0$.

In $D^*(a, b)$ this is an admissible metric for the module problem for $m(a, b)$ and we get

$$m(a, b) \le (b - a) + \frac{3}{2}\pi {\sum_{j=n+1}^{N}}' \theta_j^2 + \frac{3}{2}\pi {\sum_{k=m+1}^{M}}' \tilde{\theta}_k^2.$$

Finally

$$m(a,b) \le (b-a) + \frac{3}{2}\pi \left( \sum_{j=n+1}^{N} \theta_j^2 + \sum_{k=m+1}^{M} \tilde{\theta}_k^2 \right). \tag{1}$$

To get a lower bound for $m(a,b)$ we get an upper bound for $m(a,b)^{-1}$ the module of $D^*(a,b)$ for the family $\Gamma$ of curves joining the sides $\sigma_a, \sigma_b$. We set $\delta_n = a_{n+1} - a_n, \tilde{\delta}_m = \tilde{a}_{m+1} - \tilde{a}_m$. We set $\theta_j^* = |\theta_j|$ for $\theta_j < 0, \theta_j^* = 0$ for $\theta_j \ge 0$ and $\tilde{\theta}_k^* = |\tilde{\theta}_k|$ for $\tilde{\theta}_k < 0, \tilde{\theta}_k^* = 0$ for $\tilde{\theta}_k \ge 0$. In the intersection $\widehat{D}(a,b)$ of $D^*(a,b)$ with the set whose upper boundary is given for $a_j \le \mathcal{R}w < a_{j+1}$, $j = n, \dots, N+1$, by

$$\mathcal{I}w = \frac{1}{2} + \delta_j + \theta_j^* + \theta_{j+1}^*$$

and whose lower boundary is given for $\tilde{a}_k \le \mathcal{R}w < \tilde{a}_{k+1}$, $k = m, \dots, M+1$, by

$$\mathcal{I}w = -\frac{1}{2} - \tilde{\delta}_k - \tilde{\theta}_k^* - \tilde{\theta}_{k+1}^*$$

we set $\hat{\rho}(w) = 1$. Elsewhere we set $\hat{\rho}(w) = 0$.

Using this as a comparison metric we have

$$\int_\gamma \hat{\rho}(w)|dw| \ge (b-a)$$

for $\gamma \in \Gamma$ while

$$\iint_{\widehat{D}(a,b)} \hat{\rho}^2(w)\, dA \le (b-a) + \sum_{j=n}^{N+1} \delta_j^2 + \sum_{k=m}^{M+1} \tilde{\delta}_k^2 + 2\sum_{j=n}^{N+1} \delta_j \theta_j^* + 2\sum_{k=m}^{M+1} \tilde{\delta}_k \tilde{\theta}_k^*$$

which we denote by $(b-a) + \xi$. Thus

$$m(a,b)^{-1} \le (b-a)^{-2}[(b-a) + \xi]$$

and

$$m(a,b) \ge (b-a)^2[(b-a)+\xi]^{-1} = (b-a)\left[1 + \frac{\xi}{(b-a)}\right]^{-1} \ge (b-a) - \xi. \tag{2}$$

Combining (1) and (2) we have

$$m(a,b) = (b-a) + o(1)$$

as $a, b, a < b$, tend to $+\infty$.

## 4

The necessity proof of the theorem is carried out by using the necessity part of statement RW. The graph of $\frac{1}{2} + h_+(u)$ is the lower envelope of the rays

$$a_j + \left(\frac{1}{2} - \theta_j\right) i + (\pm 1 + i)t, \qquad t \ge 0, j = 1, 2, \dots .$$

It is made up of segments of alternate slopes $+1$ and $-1$. Some of the points $a_j + (\frac{1}{2} - \theta_j)i$ will occur as vertices of this graph. Let us denote them as $a'_l + (\frac{1}{2} - \theta'_l)i$, $l = 1, 2, \dots$. Only those with $\theta'_l > 0$ contribute to $\int_0^\infty \min(h_+(u), 0)\, du$ and this satisfies

$$\int_0^\infty \min(h_+(u), 0)\, du \ge -\sum^{+} \theta_l'^2$$

where the sum extends over these $\theta'_l$. Thus the first subsidiary condition in statement RW is satisfied. Likewise the second subsidiary condition is satisfied.

We denote

$$\delta'_l = a'_{l+1} - a'_l.$$

We have to determine the contribution to $\int_0^\infty \max(h_+(u,0)\,du$ (which is known to be finite) from the various intervals $[a'_l, a'_{l+1}]$. If both $\theta'_l, \theta'_{l+1}$ are positive and $\delta'_l \le \theta'_l + \theta'_{l+1}$, there is no contribution but clearly the corresponding sum $\sum \delta'^2_l$ converges. If both $\theta'_l, \theta'_{l+1}$ are negative, the contribution is more than $\frac{1}{2}\delta'^2_l$. Thus the corresponding sum $\sum \delta'^2_l$ converges. If both $\theta'_l, \theta'_{l+1}$ are positive but $\delta'_l > \theta'_l + \theta'_{l+1}$, the contribution is $\frac{1}{4}(\delta'_l - \theta'_l - \theta'_{l+1})^2$; this is at least $\frac{1}{8}\delta'^2_l - \frac{1}{4}(\theta'_l + \theta'_{l+1})^2$, thus the corresponding sum $\sum \delta'^2_l$ converges. In case one of $\theta'_l, \theta'_{l+1}$ is positive, one negative the contribution is at least as much as if the negative term were zero. The previous argument then shows that the corresponding sum $\sum \delta'^2_l$ converges. Thus $\sum_{l=1}^\infty \delta'^2_l < \infty$. However, each $\delta'_l$ is a sum of consecutive terms $\delta_t + \delta_{t+1} + \cdots + \delta_T$, therefore $\sum_{j=1}^\infty \delta_j^2 < 0$.

The proof that $\sum_{k=1}^\infty \tilde{\delta}_k^2 < \infty$ is just the same.

## 5

For a general strip domain one may also investigate unrestricted conformality at $+\infty$. From the necessary and sufficient conditions given in [2] it is clear that this never occurs for comb domains.

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