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RADICALS OF ALGEBRAS GRADED BY CANCELLATIVE LINEAR SEMIGROUPS

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ABSTRACT. We consider algebras over a field of characteristic zero, and prove that the Jacobson radical is homogeneous in every algebra graded by a linear cancellative semigroup. It follows that the semigroup algebra of every linear cancellative semigroup is semisimple.

Recently it has been shown that several theorems concerning various ring constructions can be obtained in the more general situation of graded rings. For example, Saorín [20] carried certain results due to Passman [15] and Reid [19] to strongly group graded rings. More general graded arguments may also clarify and unify the proofs of various facts (see [17, p. 708], [7, p. 159], [5], [8]). It would be interesting to determine which of the known results on group algebras extend to algebras of cancellative semigroups and are of graded nature.

One of the long-standing problems on group algebras is that of whether every group algebra over any field of characteristic zero is semiprimitive. The answer is known to be positive for linear groups. We obtain a graded analog of this result.

Group algebras of linear groups have been explored by many authors. Passman and Zalesskii investigated the Jacobson radical and semiprimitivity of these algebras, see [16]. A systematic study of the semigroup algebras of linear semigroups has been started by Okniński and Putcha ([12], [13], [14]). In particular, in [14] the radicals of algebras of connected algebraic monoids were described. For algebras over a field of characteristic zero, it is shown in [13, §3] that the radical of every algebra of a linear semigroup is nilpotent, and the algebra of the full matrix semigroup is semiprimitive. For the full matrix semigroup M over a finite field F and for a field K of characteristic different from that of F, it follows from Fadeev's Theorem (see Kovács [10]) that the semigroup algebra KM is semiprimitive.

Let S be a semigroup. An algebra $R = \bigoplus_{s\in S} R_s$ is said to be S-*graded* if $R_sR_t \subseteq R_{st}$ for all $s, t \in S$. An ideal I of R is said to be *homogeneous* if $I = \bigoplus_{s\in S} I_s$, where $I_s = I \cap R_s$.

Theorem 1. *Let S be a cancellative linear semigroup. Then the Jacobson radical of every S-graded algebra over a field of characteristic zero is homogeneous.*

The radical of a (not necessarily cancellative) semigroup algebra over a field cannot contain nonzero homogeneous elements. Indeed, the factor of the semigroup

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algebra FS by the augmentation ideal

$$I = \left\{ \sum_{s\in S} f_s s | \sum_{s\in S} f_s = 0 \right\}$$

is isomorphic to F. Therefore the radical is contained in I. However, I does not contain nonzero homogeneous elements. Hence we get

Corollary 2. *If F is a field of characteristic zero and S a linear cancellative semigroup, then FS is semiprimitive.*

For algebras over a field F with $\operatorname{char} F = p > 0$, the assertion analogous to Corollary 2 is not valid, and so the restriction on characteristic cannot be removed from Theorem 1. In [3] it is proved that if S is not cancellative, then there exists an S-graded algebra whose radical is not homogeneous. Therefore cancellativity cannot be dropped either. Note that if one extends Theorem 1 to arbitrary groups, then the positive solution to the semiprimivity problem for group algebras mentioned above will follow.

We shall use the previous results on radicals of graded rings ([7]), on linear groups ([7, Lemma 49.8]), and on linear semigroups ([11, Chapter 1]). Let R be an S-graded algebra. The Jacobson radical of R is denoted by $\mathcal{J}(R)$. If $r = \sum_{s\in S} r_s$ where $r_s \in R_s$, then we put $\operatorname{supp}(r) = \{s | r_s \neq 0\}$, and $H(R) = \{r_s | r_s \neq 0\}$. By the *length of* r we mean $|\operatorname{supp}(r)|$. For $T \subseteq S$, put $R_T = \bigoplus_{s\in T} R_s$, and $r_T = \sum_{s\in T} r_s$. For $P \subseteq R$, let $H(P) = \{H(r) | r \in P\}$. Then $H(R) = \bigcup_{s\in S} R_s$ is the set of all *homogeneous* elements. If I is a homogeneous ideal of R and $I \subseteq \mathcal{J}(R)$, then $R/I = \bigoplus_{s\in S} R_s/I_s$ is S-graded and $\mathcal{J}(R/I) = \mathcal{J}(R)/I$. Using this during the proof we shall be able to factor out homogeneous ideals contained in $\mathcal{J}(R)$. In what follows, "algebra" will always mean "algebra over a field of characteristic zero".

Let us begin with a few known lemmas.

Lemma 3 ([7, Corollary 22.8])**.** *Let G be a group, N a normal semigroup of G, and R a G-graded algebra. Then $R = \bigoplus_{gN\in G/N} R_{gN}$ is G/N-graded.*

Lemma 4 ([7, Corollary 30.11])**.** *Let G be a finite group with identity e, and let R be a G-graded algebra. Then $\mathcal{J}(R)$ is the largest homogeneous ideal of R with the property that $\mathcal{J}(R) \cap R_e = \mathcal{J}(R_e)$.*

Next we collect all the information on the structure of the full matrix semigroup needed for our proof. We present only the final conclusions.

Lemma 5. *Let F be a field and $M_n(F)$ the set of all $n \times n$ matrices over F. For $k = 0, 1, \ldots, n$ denote by I_k the set of all matrices of rank $\leq k$. Then*

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n = M_n(F)$$

are the only ideals of the multiplicative semigroup $M_n(F)$. For every $k = 1, \ldots, n$, the set $I_k \backslash I_{k-1}$ is a disjoint union of subsets $G_{\alpha\beta}$, indexed by the elements α, β of a certain set Λ_k, and such that for all $\alpha, \beta, \gamma, \delta \in \Lambda_k$

(i) *either $G_{\alpha\beta}$ is a linear group, or $G_{\alpha\beta}^2 \subseteq I_{k-1}$;*

(ii) $G_{\alpha\beta} M_n(F) G_{\gamma\delta} \subseteq G_{\alpha\delta} \cup I_{k-1}$;

(iii) $G_{\alpha *} \cup I_{k-1}$ *is a right ideal of $M_n(F)$, where $G_{\alpha *} = \bigcup_{\lambda\in\Lambda_k} G_{\alpha\lambda}$;*

(iv) $G_{\alpha\beta} \cup I_{k-1}$ *is a left ideal of $G_{\alpha *} \cup I_{k-1}$.*

These facts are well known. They are contained, for example, in Lemma 1.4 and Theorems 1.3, 1.6 of [11] which use the terms "completely 0-simple semigroup" and "Rees matrix semigroup" (see also [12, Lemma 2] and [18]).

Lemma 6. *Let S be a cancellative semigroup, R an S-graded algebra, and $I \neq 0$ an ideal of R. Let $P = P(I)$ be the set of all elements of the minimal positive length in I, and let $\mathrm{Min}(I)$ be the linear span of $H(P)$ in R. Then $\mathrm{Min}(I)$ is an ideal of R.*

Proof. Take any $x \in H(P)$ and $y \in H(R)$ such that $xy \neq 0$. Clearly, $x = r_s$ for some $r \in P, s \in S$. Given that S is cancellative and $xy \neq 0$, we get $ry \in P$. Hence $xy \in H(P)$. It follows that $\mathrm{Min}(I)$ is a right ideal of R. Similarly, it is a left ideal of R.

Lemma 7. *If G is a linear group and R is a G-graded algebra, then $\mathcal{J}(R)$ is homogeneous.*

Proof. For any subgroup T of G, it is known that $\mathcal{J}(R_T) \supseteq R_T \cap \mathcal{J}(R)$ ([7, Proposition 6.18]). Let $\mathcal{F}$ be the set of all finitely generated subgroups of G. Then $R = \bigcup_{T\in\mathcal{F}} R_T$ and all $\mathcal{J}(R_T) \supseteq R_T \cap \mathcal{J}(R)$ imply $\bigcup_{T\in\mathcal{F}} \mathcal{J}(R_T) \supseteq \mathcal{J}(R)$. Hence it suffices to show that $\mathcal{J}(R_T)$ is homogeneous for every $T \in \mathcal{F}$. So we may assume that G is finitely generated itself. Then G has a normal subgroup N of finite index such that N is residually finite ([7, Lemma 49.8]). Therefore G itself is a residually finite group.

For any $0 \neq r \in \mathcal{J}(R_G)$, there exists a normal subgroup K of finite index in G such that $gK \neq hK$ for every $g, h \in \mathrm{supp}(R)$, $g \neq h$. By Lemma 3 R is G/K-graded, and Lemma 4 shows that $\mathcal{J}(R)$ is homogeneous in this gradation. Since $r_{gK} = r_g$ for every $g \in \mathrm{supp}(r)$, we get $H(r) \subseteq \mathcal{J}(R_G)$. Therefore $\mathcal{J}(R_G)$ is homogeneous.

Note that $M_n(F)$ may have cancellative subsemigroups which are not contained in a subgroup of $M_n(F)$ ([11, Example 1.10]). Therefore Lemma 7 alone is not sufficient to imply Theorem 1. However, the proof follows from Lemmas 5 and 7.

Proof of Theorem 1. Let S be a cancellative subsemigroup of $M_n(F)$. Suppose to the contrary that there exists an S-graded algebra R such that $\mathcal{J}(R)$ is not homogeneous. Factoring out the largest homogeneous ideal of $\mathcal{J}(R)$ we may assume that $\mathcal{J}(R) \neq 0$ has no nonzero homogeneous elements.

Let $L = \mathrm{Min}(\mathcal{J}(R))$ be the homogeneous ideal defined in Lemma 6. For $k = 1, \ldots, n$, let I_k be the set of all matrices of rank $\leq k$ in $M_n(F)$, and let $R_k = R_{I_k}$. Consider the minimal positive integer k such that L has a nonzero intersection with ideal R_k introduced in Lemma 5. Then $K = R_k \cap L$ is a homogeneous ideal of R.

Using the sets $G_{\alpha\beta}$, $\alpha, \beta \in \Lambda_k$, defined in Lemma 5, we put $R_{\alpha\beta} = R_{G_{\alpha\beta}}, K_{\alpha\beta} = K \cap R_{\alpha\beta}$, and $K_{\alpha*} = K \cap R_{G_{\alpha*}}$. By the choice of k we get $K_{I_{k-1}} = 0$. Therefore Lemma 5 gives the following:

(i) either $G_{\alpha\beta}$ is a linear group and $K_{\alpha\beta}$ is a $G_{\alpha\beta}$-graded algebra, or $K^2_{\alpha\beta} = 0$;
(ii) $R_{\alpha\beta} K R_{\alpha\beta} \subseteq K_{\alpha\beta}$;
(iii) $K_{\alpha*}$ is a right ideal of R;
(iv) $K_{\alpha\beta}$ is a left ideal of $K_{\alpha*}$.

If $G_{\alpha\beta}$ is not a group, then $K^2_{\alpha\beta} = 0$, and so $K_{\alpha\beta}$ is quasi-regular. Next suppose that $G_{\alpha\beta}$ is a group. Let $P = P(\mathcal{J}(R))$ be the set of all elements of the minimal positive length in $\mathcal{J}(R)$, and let $Q = H(P) \cap R_{\alpha\beta}$. Then $K_{\alpha\beta}$ is the linear span of Q (see Lemma 6). Take any $q \in Q$. There exist $r \in P$ and $g \in G_{\alpha\beta}$ such that $q = r_g$. For any $a, b \in H(R_{\alpha\beta})$, it follows from (ii) that $arb \in K_{\alpha\beta}$. By (iii) and (iv) we get $arb \in \mathcal{J}(K_{\alpha\beta})$. Lemma 7 shows that $\mathcal{J}(K_{\alpha\beta})$ is homogeneous. Since S is cancellative, $aqb = (arb)_h$ for some $h \in G_{\alpha\beta}$. Therefore $aqb \in \mathcal{J}(K_{\alpha\beta})$. It follows that $K^3_{\alpha\beta} \subseteq \mathcal{J}(K_{\alpha\beta})$, and so $K_{\alpha\beta}$ is quasi-regular, again.

Since $K_{\alpha *}$ is the sum of quasi-regular left ideals $K_{\alpha\beta}$, and K is the sum of right ideals $K_{\alpha *}$, evidently K is quasi-regular. Therefore $\mathcal{J}(R)$ contains a nonzero homogeneous ideal K. This contradiction completes the proof.

A ring R is said to be *left T-nilpotent* if, for every sequence of elements $x_1, x_2, \ldots$ of R, there exists n such that $x_1 x_2 \cdots x_n = 0$. A ring R is *semilocal* if $R/\mathcal{J}(R)$ is Artinian. Further, R is *semiprimary* (*left perfect*) if R is semilocal and $\mathcal{J}(R)$ is nilpotent (left T-nilpotent). If R is S-graded and $R_s \subseteq \mathcal{J}(R)$ for all but finitely many $s \in S$, then we say that $R/\mathcal{J}(R)$ is finitely graded.

Camillo and Fuller proved that a $\mathbb{Z}$-graded ring R is semilocal (left perfect) if and only if R_1 is semilocal (left perfect) and $R/\mathcal{J}(R)$ is finitely graded ([2, Propositions 8 and 10]). For semigroup graded analogs of this theorem we need the following

Lemma 8 ([9])**.** *Let a ring R be the direct sum of a finite number of its additive semigroups R_i, $i = 1, \ldots, n$, and let the union of the R_i be closed under multiplication. Then R is semilocal* (*right or left perfect*; *semiprimary*; *nilpotent*; *locally nilpotent*; *right or left T-nilpotent*; *Baer radical*; *quasiregular*; *P. I.*) *if and only if all subrings among the R_i satisfy the same property.*

This fact and [4, Proposition 4] (see [6]) give us

Lemma 9. *Let S be a semigroup without infinite periodic subgroups and R an S-graded algebra with homogeneous radical $\mathcal{J}(R)$. Then R is semilocal if and only if $R/\mathcal{J}(R)$ is finitely graded and R_e is semilocal for every idempotent e of S.*

Therefore results on homogeneity of the Jacobson radical can be applied to derive corollaries concerning finiteness conditions. For example, combining Theorem 1 with Lemma 9, we obtain the following corollary. (Note that if e is an idempotent of a cancellative semigroup S, then e is the identity of S).

Corollary 10. *Let S be a linear cancellative semigroup with identity e and without infinite periodic subgroups, and let R be a S-graded algebra over a field of characteristic zero. Then R is semilocal if and only if $R/\mathcal{J}(R)$ is finitely graded and R_e is semilocal.*

Saorín [20] considered left perfect strongly graded rings. An S-graded ring R is said to be *strongly graded* if $R_s R_t = R_{st}$ for all s, t. For a strongly group graded algebra with homogeneous $\mathcal{J}(R)$ it is routine to verify that $\mathcal{J}(R)$ is nilpotent (left T-nilpotent) if and only if $\mathcal{J}(R_e)$ is nilpotent (left T-nilpotent), see [7, Corollary 27.4]. Therefore, Corollary 10 yields the following

Corollary 11. *Let G be a linear group with identity e and without infinite periodic subgroups, and let R be a strongly G-graded algebra over a field of characteristic zero. Then R is left perfect* (*semiprimary*) *if and only if $R/\mathcal{J}(R)$ is finitely graded and R_e is left perfect* (*semiprimary*).

Passman's example of a field which is a twisted group algebra of an infinite periodic group ([15, Proposition 4.3]) shows that the restriction on periodic subgroups cannot be removed from Corollaries 10 and 11.

In conclusion we record a related problem of interest. Amitsur [1] proved that if F is a field of characteristic zero and F is not algebraic over $\mathbb{Q}$, then FG is semiprimitive for every group G.

Problem 12. *Let F be a field of characteristic zero which is not algebraic over the field of rational numbers. Let S be a cancellative semigroup, and let R be an S-graded F-algebra. Is it true that the Jacobson radical of R is homogeneous*?

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