

## ON THE UNIQUE RANGE SET OF MEROMORPHIC FUNCTIONS

PING LI AND CHUNG-CHUN YANG

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**ABSTRACT.** This paper studies the unique range set of meromorphic functions and shows that there exists a finite set  $S$  such that for any two nonconstant meromorphic functions  $f$  and  $g$  the condition  $E_f(S) = E_g(S)$  implies  $f \equiv g$ . As a special case this also answers an open question posed by Gross (1977) about entire functions and improves some results obtained recently by Yi.

### 1. INTRODUCTION

Let  $f$  be a nonconstant meromorphic function on the complex plane  $C$  and  $S$  be a subset of distinct elements in  $C$ . Define

$$E_f(S) = \bigcup_{a \in S} \{z | f(z) - a = 0\},$$

here a zero of  $f(z) - a$  of multiplicity  $m$  appears  $m$  times in  $E_f(S)$ . In 1976 Gross proved [1] that there exist three finite sets  $S_j$  ( $j = 1, 2, 3$ ) such that for any two nonconstant entire functions  $f$  and  $g$  if  $E_f(S_j) = E_g(S_j)$  ( $j = 1, 2, 3$ ), then  $f \equiv g$ . In the same paper Gross posed the following problem: Can one find two (or possibly even one) finite sets  $S_j$  ( $j = 1, 2$ ) such that any two entire functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j)$  ( $j = 1, 2$ ) must be identical? In 1982, F. Gross and C. C. Yang proved the following result.

**Theorem A** ([2]). *Let  $T = \{z | e^z + z = 0\}$ . Let  $f$  and  $g$  be two nonconstant entire functions. If  $E_f(T) = E_g(T)$ , then  $f \equiv g$ .*

In [2] the set  $S$  such that for any two nonconstant entire functions  $f$  and  $g$  the condition  $E_f(S) = E_g(S)$  implies  $f \equiv g$  is called a unique range set (URS, in brief) of entire functions. A similar definition for meromorphic functions can be defined. Note that the set  $T = \{z | e^z + z = 0\}$  contains an infinite number of elements. Recently, Yi [6] exhibited a finite unique range set of entire functions which gave a positive answer to Gross's problem. He proved

**Theorem B.** *Let  $n \geq 15$ ,  $n > m \geq 5$  with  $n$  and  $m$  having no common factors. Let  $a$  and  $b$  be two nonzero constants such that the algebraic equation  $z^n + az^m + b = 0$  has no multiple roots. Then the set  $S = \{z | z^n + az^m + b = 0\}$  is a URS of entire functions.*

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In this paper, we shall exhibit, among other results, a finite URS of meromorphic functions with 19 elements and a URS of entire functions with nine elements.

**Theorem 1.** *Let  $m \geq 2$ ,  $n > 4m + 10$  with  $n$  and  $n - m$  having no common factors. Let  $a$  and  $b$  be two nonzero constants such that the algebraic equation  $z^n + az^{n-m} + b = 0$  has no multiple roots. Let  $S = \{z | z^n + az^{n-m} + b = 0\}$ . Then for any two nonconstant meromorphic functions  $f$  and  $g$ , the condition  $E_f(S) = E_g(S)$  implies  $f \equiv g$ .*

**Theorem 2.** *Let  $m \geq 2$ ,  $n > 4m + 6$  with  $n$  and  $n - m$  having no common factors. Let  $a, b$  and  $S$  be as in Theorem 1. Then for any two nonconstant meromorphic functions  $f$  and  $g$ , the conditions  $E_f(S) = E_g(S)$  and  $E_f\{\infty\} = E_g\{\infty\}$  imply  $f \equiv g$ .*

**Theorem 3.** *Let  $m \geq 1$ ,  $n > 4m + 4$  with  $n$  and  $n - m$  having no common factors. Let  $a, b$  and  $S$  be as in Theorem 1. Then for any two nonconstant entire functions  $f$  and  $g$ , the condition  $E_f(S) = E_g(S)$  implies  $f \equiv g$ .*

The main tool will be Nevanlinna's theory of meromorphic functions, and it is assumed that the reader is familiar with its basic notation and results (see Hayman [4]). In the sequel the letter  $E$  will be used to denote a set of  $r$  values of finite linear measure.

## 2. SOME LEMMAS

The following lemmas will be needed in the proof of our theorems.

**Lemma 1** ([7]). *Let  $f$  and  $g$  be two nonconstant meromorphic functions, and  $c_1, c_2$ , and  $c_3$  be nonzero constants. If  $c_1f + c_2g \equiv c_3$ , then*

$$T(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, f).$$

Here and in the sequel  $S(r, f)$  denotes the quantity  $o(T(r, f))$ ,  $r \rightarrow \infty$ ,  $r \notin E$ .

**Lemma 2.** *Let  $f_1, f_2$ , and  $f_3$  be nonconstant meromorphic functions and  $f_1 + f_2 + f_3 \equiv 1$ . If  $f_1, f_2, f_3$  are linearly independent, then*

$$T(r, f_1) < 2 \sum_{i=1}^3 \bar{N}\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^3 \bar{N}(r, f_i) + o(T(r))$$

where  $T(r) = \max_{1 \leq i \leq 3} \{T(r, f_i)\}$  and  $r \notin E$ .

*Proof.* By the proof of a generalization of Borel's theorem (a generalization of Picard's theorem) by Nevanlinna [3] (page 70), we have

$$T(r, f_1) < \sum_{i=1}^3 N\left(r, \frac{1}{f_i}\right) - \sum_{i=2}^3 N(r, f_i) + N(r, D) - N\left(r, \frac{1}{D}\right) + o(T(r)),$$

where  $D$  is the Wronskian of  $f_1, f_2$ , and  $f_3$ , i.e.,

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}.$$

Since  $f_1 + f_2 + f_3 \equiv 1$ , we have

$$(1) \quad D = f_2'f_3'' - f_2''f_3' = -(f_1'f_3'' - f_1''f_3') = f_1'f_2'' - f_1''f_2'.$$

Write

$$N(r) = \sum_{i=1}^3 N\left(r, \frac{1}{f_i}\right) - \sum_{i=2}^3 N(r, f_i) + N(r, D) - N\left(r, \frac{1}{D}\right)$$

and

$$N^*(r) = 2 \sum_{i=1}^3 \overline{N}\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^3 \overline{N}(r, f_i).$$

Thus clearly Lemma 2 follows immediately from the inequality

$$(2) \quad N(r) \leq N^*(r),$$

which is to be shown next.

For a given meromorphic function  $f$  and a complex number  $a \in \overline{C}$ , we define

$$\mu_f^a(z) = \begin{cases} m, & z \text{ is an } a\text{-point of } f \text{ with multiplicity } m \geq 1, \\ 0, & z \text{ is not an } a\text{-point of } f \end{cases}$$

and

$$\overline{\mu}_f^a(z) = \begin{cases} 1, & z \text{ is an } a\text{-point of } f \text{ with multiplicity } m \geq 1, \\ 0, & z \text{ is not an } a\text{-point of } f. \end{cases}$$

Let

$$\mu = \mu_{f_1}^0 + \mu_{f_2}^0 + \mu_{f_3}^0 - \mu_{f_2}^\infty - \mu_{f_3}^\infty + \mu_D^\infty - \mu_D^0$$

and

$$\mu^* = 2\overline{\mu}_{f_1}^0 + 2\overline{\mu}_{f_2}^0 + 2\overline{\mu}_{f_3}^0 + \overline{\mu}_{f_1}^\infty + \overline{\mu}_{f_2}^\infty + \overline{\mu}_{f_3}^\infty.$$

Thus inequality (2) follows from  $\mu(z) \leq \mu^*(z)$  for any  $z$ . To prove this, we consider the following five cases for an arbitrary point  $z \in C$ .

*Case 1.*  $z$  is a zero-point of  $f_i$  ( $i = 1, 2, 3$ ) with multiplicity  $m_i \geq 0$ .

*Case 2.*  $z$  is a zero-point of  $f_1$  with multiplicity  $m \geq 1$  and a pole of  $f_2$  and  $f_3$  with multiplicity  $k \geq 1$ .

*Case 3.*  $z$  is a zero-point of  $f_2$  with multiplicity  $m \geq 1$  and a pole of  $f_1$  and  $f_3$  with multiplicity  $k \geq 1$ .

*Case 4.*  $z$  is a zero-point of  $f_3$  with multiplicity  $m \geq 1$  and a pole of  $f_1$  and  $f_2$  with multiplicity  $k \geq 1$ .

*Case 5.*  $z$  is a pole of  $D$  but not a zero of  $f_1, f_2$ , and  $f_3$ .

In each case we can verify that the inequality  $\mu(z) \leq \mu^*(z)$  holds. For instance, take Case 2; then we have  $\mu_{f_1}^0(z) = m$ ,  $\mu_{f_2}^0 = \mu_{f_3}^0(z) = 0$ ,  $\mu_{f_2}^\infty(z) = \mu_{f_3}^\infty(z) = k$ . Thus  $\mu^*(z) = 4$ .

If  $k - m + 3 > 0$ , then from (1),  $z$  is a pole of  $D$  with multiplicity at most  $k - m + 3$ . This means that  $\mu_D^\infty \leq k - m + 3$ . It follows that

$$\mu(z) \leq m - 2k + (k - m + 3) = 3 - k \leq 2 < \mu^*(z).$$

If  $k - m + 3 \leq 0$ , then from (1)  $z$  is a zero of  $D$  with multiplicity at least  $m - k - 3$ . This means that  $\mu_D^\infty(z) = 0$  and  $\mu_D^0(z) \geq m - k - 3$ . Hence

$$\mu(z) \leq m - 2k - (m - k - 3) = 3 - k \leq 2 < \mu^*(z).$$

The remaining cases can be proved in a similar manner. This also completes the proof of the lemma.

**Lemma 3** ([5]). *Let  $f$  be a meromorphic function, and*

$$P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n$$

*be a polynomial in  $f$  of degree  $n$ , where  $a_0 (\neq 0), a_1, \dots, a_n$  are finite complex numbers. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

### 3. PROOF OF THEOREM 1

Let  $r_1, r_2, \dots, r_n$  be the roots of equation  $z^n + az^{n-m} + b = 0$ . Since  $E_f(S) = E_g(S)$ , we have from Nevanlinna's second fundamental theorem

$$\begin{aligned} (n-2)T(r, g) &< \sum_{k=1}^n \overline{N}\left(r, \frac{1}{g-r_k}\right) + S(r, g) \\ &= \sum_{k=1}^n \overline{N}\left(r, \frac{1}{f-r_k}\right) + S(r, g) \\ &\leq nT(r, f) + S(r, g). \end{aligned}$$

It follows that

$$(3) \quad T(r, g) \leq \frac{n}{n-2}T(r, f) + S(r, g).$$

Similarly the following inequality holds:

$$(4) \quad T(r, f) \leq \frac{n}{n-2}T(r, g) + S(r, f).$$

In the sequel we use  $S(r)$  to express either  $S(r, f)$  or  $S(r, g)$ .

Consider now the following meromorphic function

$$(5) \quad \psi = \frac{f^n + af^{n-m} + b}{g^n + ag^{n-m} + b}.$$

The condition  $E_f(S) = E_g(S)$  ensures that the zeros of  $\psi$  come from the poles of  $g$ , and the poles of  $\psi$  come from the poles of  $f$ . This means that the following inequalities hold:

$$(6) \quad \overline{N}\left(r, \frac{1}{\psi}\right) \leq \overline{N}(r, g)$$

and

$$(7) \quad \overline{N}(r, \psi) \leq \overline{N}(r, f).$$

Let

$$(8) \quad f_1 = -\frac{1}{b}f^{n-m}(f^m + a), \quad f_2 = \frac{1}{b}\psi g^{n-m}(g^m + a), \quad f_3 = \psi.$$

Then  $f_1, f_2$ , and  $f_3$  are meromorphic functions and  $f_1$  is not a constant. From (3), we have

$$(9) \quad f_1 + f_2 + f_3 \equiv 1.$$

Now we distinguish two cases.

*Case 1:  $f_3$  is not a constant.* If  $f_1$  and  $f_2$  are linearly dependent, then  $f_2 = cf_1$ ,  $c \neq -1$ . From (9) we have

$$(1+c)f_1 + f_3 \equiv 1.$$

By using Lemma 1 and Lemma 3 together with the inequalities (3) and (6), we deduce

$$\begin{aligned} nT(r, f) &= T(r, f_1) + S(r) \\ &< \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_1) + S(r) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^m+a}\right) + \overline{N}(r, g) + \overline{N}(r, f) + S(r) \\ &\leq (m+2)T(r, f) + T(r, g) + S(r) \\ &\leq \left(m+2 + \frac{n}{n-2}\right)T(r, f) + S(r) \\ &= \left(m+3 + \frac{2}{n-2}\right)T(r, f) + S(r), \end{aligned}$$

which is contradictory to  $n > 4m + 10$ . Hence  $f_1$  and  $f_2$  must be linearly independent.

If  $f_1, f_2$ , and  $f_3$  are linearly independent and  $f_2$  is not a constant, then by using Lemma 2 we have

$$\begin{aligned} T(r, f_1) &< 2\overline{N}\left(r, \frac{1}{f_1}\right) + 2\overline{N}\left(r, \frac{1}{f_2}\right) + 2\overline{N}\left(r, \frac{1}{f_3}\right) \\ &\quad + \overline{N}(r, f_1) + \overline{N}(r, f_2) + \overline{N}(r, f_3) + S(r). \end{aligned}$$

From the identities (5) and (8), we can easily see that the zeros of  $f_2$  cannot come from the zeros of  $\psi$ , and the poles of  $f_2$  must come from the poles of  $f$ . By the above inequality and Lemma 3 together with (6), (7) and (8), we deduce that

$$\begin{aligned} nT(r, f) &< 2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{f^m+a}\right) + 2\overline{N}\left(r, \frac{1}{g}\right) + 2\overline{N}\left(r, \frac{1}{g^m+a}\right) \\ &\quad + 2\overline{N}(r, g) + \overline{N}(r, f) + \overline{N}(r, f) + \overline{N}(r, f) + S(r) \\ &\leq (2m+5)T(r, f) + 2(m+2)T(r, g) + S(r) \\ &\leq \left[(2m+5) + 2(m+2)\frac{n}{n-2}\right]T(r, f) + S(r) \\ &= \left(4m+9 + \frac{4m+8}{n-2}\right)T(r, f) + S(r). \end{aligned}$$

This contradicts the assumption  $n > 4m + 10$ . It follows that when  $f_1, f_2$ , and  $f_3$  are linearly independent,  $f_2$  must be constant and  $f_2 \neq -1$ , i.e.  $f_1 + f_3 = 1 - f_2$  is a nonzero constant. By Lemma 1,

$$T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_1) + S(r).$$

This leads to

$$nT(r, f) \leq \left(m+3 + \frac{2}{n-2}\right)T(r, f) + S(r),$$

which is a contradiction to  $n > 4m + 10$ .

If  $f_1, f_2$ , and  $f_3$  are linearly dependent, then there exist three constants  $c_1, c_2$ , and  $c_3$ , at least one of them is not zero, such that

$$(10) \quad c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0.$$

This and the fact that  $f_1, f_2$  are linearly independent imply  $c_3 \neq 0$ . So

$$(11) \quad c_1 \frac{f_1}{\psi} + c_2 \frac{f_2}{\psi} = -c_3.$$

If  $c_1 = 0$ , then  $g^{n-m}(g^m + a)$ ; hence  $g$  is a constant. This is impossible.

If  $c_2 = 0$ , then

$$(12) \quad \frac{c_1}{b} f^{n-m}(f^m + a) = c_3 \psi.$$

Let  $s_0 = 0, s_1, \dots, s_m$  be the distinct roots of the equation  $z^n + az^{n-m} = 0$ . Then (12) shows that any  $s_j$ -point of  $f$  must be a zero of  $\psi$  and hence a pole of  $g$ . But from (5) and (12) one can see that the multiplicity of any zero of  $\psi$  is at least  $n$ , so the multiplicity of an  $s_j$ -point ( $j \neq 0$ ) of  $f$  is at least  $n$  and at least  $m$  for an  $s_0$ -point of  $f$ . Hence, we have

$$\Theta(s_j, f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}(r, \frac{1}{f-s_j})}{T(r, f)} \geq 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}(r, \frac{1}{f-s_j})}{N(r, \frac{1}{f-s_j})} \geq 1 - \frac{1}{n},$$

$j = 1, 2, \dots, m$ , and

$$\Theta(s_0, f) \geq 1 - \frac{1}{m}.$$

Again by the second fundamental theorem about the deficiencies of meromorphic functions, we have

$$1 - \frac{1}{m} + m \left(1 - \frac{1}{n}\right) \leq \sum_{j=0}^m \Theta(s_j, f) \leq 2.$$

This is impossible because  $m \geq 2, n > 4m + 10$ .

Now that we have obtained  $c_1 \neq 0, c_2 \neq 0, c_3 \neq 0$ , by Lemma 1 and (11)

$$\begin{aligned} T\left(r, \frac{f_2}{\psi}\right) &< \overline{N}\left(r, \frac{\psi}{f_2}\right) + \overline{N}\left(r, \frac{\psi}{f_1}\right) + \overline{N}\left(r, \frac{f_2}{\psi}\right) + S(r) \\ &< \overline{N}\left(r, \frac{\psi}{f_2}\right) + \overline{N}(r, \psi) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{f_2}{\psi}\right) + S(r). \end{aligned}$$

Hence from Lemma 3 and (7) and (8), we have

$$\begin{aligned} nT(r, g) &< \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^m + a}\right) + \overline{N}(r, f) \\ &\quad + \overline{N}\left(r, \frac{1}{f^m + a}\right) + \overline{N}(r, g) + S(r) \\ &\leq (m+2)T(r, g) + (m+2)T(r, f) + S(r) \\ &\leq (m+2) \left(1 + \frac{n}{n-2}\right) T(r, f) + S(r), \end{aligned}$$

which is a contradiction to  $n > 4m + 10$ .

We can rule out Case 1.

Case 2:  $f_3$  is a constant. In this case,  $f_2$  cannot be a constant. From (5), we have

$$(13) \quad T(r, f) = T(r, g) + S(r).$$

If  $f_3 \neq 1$ , then  $f_1 + f_2 = 1 - f_3 \neq 0$ . By Lemma 1

$$T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r).$$

That is,

$$\begin{aligned} nT(r, f) &< \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f^m + a}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g^m + a}\right) + \overline{N}(r, f) + S(r) \\ &< (m + 2)T(r, f) + (m + 1)T(r, g) + S(r) \\ &= (2m + 3)T(r, f) + S(r). \end{aligned}$$

This contradicts the assumption that  $n > 4m + 10$ .

If  $f_3 = 1$ , then from (5) we get

$$(14) \quad g^m(h^n - 1) = -a(h^{n-m} - 1)$$

where  $h = f/g$  is a meromorphic function. Further (14) can be rewritten as

$$(15) \quad g^m(h - u_1)(h - u_2) \cdots (h - u_n) = -a(h^{n-m} - 1)$$

where  $u_j = e^{i2j\pi/n}$ ,  $j = 1, 2, \dots, n$ . Since  $n$  and  $n - m$  have no common factors, we see that  $u_j^{n-m} - 1 \neq 0$ ,  $j = 1, \dots, n - 1$ . Hence from (15) the multiplicity of a  $u_j$ -point of  $h$  is at least  $m$ . Suppose that  $h$  is not a constant, then we have

$$\begin{aligned} \Theta(u_j, h) &= 1 - \lim_{r \rightarrow \infty} \frac{\overline{N}(r, \frac{1}{h-u_j})}{T(r, h)} \\ &\geq 1 - \lim_{r \rightarrow \infty} \frac{\overline{N}(r, \frac{1}{h-u_j})}{N(r, \frac{1}{h-u_j})} \geq 1 - \frac{1}{m}, \quad j = 1, \dots, n - 1. \end{aligned}$$

Thus

$$(n - 1) \left(1 - \frac{1}{m}\right) \leq \sum_{j=1}^{n-1} \Theta(u_j, h) \leq 2,$$

which contradicts  $m \geq 2$  and  $n > 4m + 10$ . This shows that  $h$  must be a constant. Furthermore from (14) we can see that  $h$  must be equal to 1. Otherwise we will deduce that  $g$  is a constant. Hence  $f \equiv g$ . This completes the proof of Theorem 1.

Note that the function  $\psi$  in (5) will assume the form  $e^\alpha$  with  $\alpha$  being an entire function under the assumptions of Theorem 2 and Theorem 3. Furthermore under the assumption of Theorem 3 the inequalities (3) and (4) will be replaced by

$$T(r, g) \leq \frac{n}{n-1}T(r, f) + S(r, g)$$

and

$$T(r, f) \leq \frac{n}{n-1}T(r, g) + S(r, f)$$

respectively, and we can then prove these two theorems immediately following the same procedure of the proof of Theorem 1.

**Example 1.** The set  $S = \{z|z^{19} - z^{17} + 1 = 0\}$  is a URS of meromorphic functions with 19 elements.

**Example 2.** The set  $S = \{z|z^9 - z^8 + 1 = 0\}$  is a URS of entire functions with nine elements.

#### 4. CONCLUDING REMARKS

We would like to pose the following problems about the unique range set of meromorphic functions and entire functions for further investigations.

**Problem 1.** Can one find a URS of entire functions with fewer than nine elements? What is the smallest cardinality for a URS of entire functions?

**Problem 2.** Can one find a URS of meromorphic functions with fewer than 19 elements? What is the smallest cardinality for a URS of meromorphic functions?

Now we introduce the following notation:

$$\begin{aligned} U_M &= \{S|S \text{ is a URS of meromorphic functions}\}, \\ U_E &= \{S|S \text{ is a URS of entire functions}\}, \\ \lambda_M &= \min\{n(S)|S \in U_M\}, \\ \lambda_E &= \min\{n(S)|S \in U_E\}, \end{aligned}$$

where  $n(S)$  denotes the cardinal number of the set  $S$ . Obviously,

$$\lambda_E \leq \lambda_M.$$

Example 1 and Example 2 show that  $\lambda_E \leq 9$  and  $\lambda_M \leq 19$ , respectively. We claim

**Theorem 4.**  $\lambda_E \geq 4$ .

*Proof.* Let us consider the two entire functions

$$f = \frac{\omega_2 e^h}{\omega_2 - \omega_1} + \frac{t\omega_1 e^{-h}}{\omega_2 - \omega_1} + \frac{1}{3}(a_1 + a_2 + a_3)$$

and

$$g = \frac{e^h}{\omega_2 - \omega_1} + \frac{te^{-h}}{\omega_2 - \omega_1} + \frac{1}{3}(a_1 + a_2 + a_3)$$

where  $h$  is any nonconstant entire function,  $a_1, a_2$ , and  $a_3$  are three finite distinct complex numbers, and

$$\begin{aligned} t &= a_1 a_2 + a_1 a_3 + a_2 a_3 - \frac{1}{3}(a_1 + a_2 + a_3)^2, \\ \omega_1 &= e^{i2\pi/3}, \quad \omega_2 = e^{i4\pi/3}. \end{aligned}$$

It is easy to verify that

$$(f - a_1)(f - a_2)(f - a_3) \equiv (g - a_1)(g - a_2)(g - a_3),$$

which shows  $E_f\{a_1, a_2, a_3\} = E_g\{a_1, a_2, a_3\}$ , but obviously  $f$  is not identically equal to  $g$ . Hence  $\lambda_E \geq 4$ .

We conjecture that  $\lambda_E = 4$  is the answer to Problem 1.

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DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY,  
CLEAR WATER BAY, KOWLOON, HONG KONG  
*E-mail address*: mayang@usthk.bitnet