PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 1, January 1996

BEST POSSIBILITY OF THE FURUTA INEQUALITY

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $0 \leq p, q, r \in \mathbb{R}, p + 2r \leq (1 + 2r)q$, and $1 \leq q$. Furuta (1987) proved that if bounded linear operators $A, B \in B(H)$ on a Hilbert space H ($\dim(H) \geq 2$) satisfy $0 \leq B \leq A$, then $(A^r B^p A^r)^{1/q} \leq A^{(p+2r)/q}$. In this paper, we prove that the range $p + 2r \leq (1 + 2r)q$ and $1 \leq q$ is best possible with respect to the Furuta inequality, that is, if $(1 + 2r)q < p + 2r$ or $0 < q < 1$, then there exist $A, B \in B(\mathbb{R}^2)$ which satisfy $0 \leq B \leq A$ but $(A^r B^p A^r)^{1/q} \not\leq A^{(p+2r)/q}$.

Let A, B be bounded linear operators on a Hilbert space H with $\dim(H) \geq 2$. Furuta ([1]) proved the following interesting inequality.

Proposition 1 ([1]). *Let $0 \leq p, q, r \in \mathbb{R}$ and $A, B \in B(H)$ satisfy $0 \leq B \leq A$. If*

$$p + 2r \leq (1 + 2r)q \quad \text{and} \quad 1 \leq q, \tag{1}$$

then

$$(A^r B^p A^r)^{1/q} \leq A^{(p+2r)/q}. \tag{2}$$

This inequality (2) is an extension of the Löwner-Heinz inequality ([2], [3]), and many applications have been developed recently.

Proposition 2 ([2], [3]). *Let $A, B \in B(H)$ satisfy $0 \leq B \leq A$. If $0 < p < 1$, then*

$$B^p \leq A^p.$$

Furuta calculated many matrices, so the range (1) has been regarded as best possible. In this paper, we prove that the range (1) is indeed best with respect to the Furuta inequality, that is, if $(1 + 2r)q < p + 2r$ or $0 < q < 1$, then there exist $A, B \in B(\mathbb{R}^2)$ which satisfy $0 \leq B \leq A$ but $(A^r B^p A^r)^{1/q} \not\leq A^{(p+2r)/q}$.

We prove the following theorem to show the best possibility of the range (1).

Theorem. *Let $0 < p, q, r \in \mathbb{R}$. If $(1 + 2r)q < p + 2r$ or $0 < q < 1$, then there exist $A, B \in B(\mathbb{R}^2)$ with $0 \leq B \leq A$ which do not satisfy the inequality*

$$(A^r B^p A^r)^{1/q} \leq A^{(p+2r)/q}. \tag{2}$$

Proof. We will consider

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a - b - \delta)} \\ \sqrt{\varepsilon(a - b - \delta)} & b + \varepsilon + \delta \end{pmatrix} \tag{3}$$

Received by the editors February 25, 1994 and, in revised form, July 7, 1994.
1991 *Mathematics Subject Classification.* Primary 47B15.
Key words and phrases. The Löwner-Heinz inequality, the Furuta inequality, positive operator.

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \tag{4}$$

where

$$0 < b < 1 < a, \quad 0 < \varepsilon, \quad 0 < \delta, \quad \varepsilon(1-b) \le \delta(a-1+\varepsilon). \tag{5}$$

Since $0 \le B \le A$ is easy, we must prove that A and B do not satisfy the inequality (2) for some $a, b, \varepsilon, \delta$. We will define δ as a function of ε and prove that A and B do not satisfy the inequality (2) by letting $\varepsilon \to +0$.

Let

$$\gamma = a - b + \varepsilon - \delta$$

and

$$U = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\sqrt{a-b-\delta} \end{pmatrix}.$$

Then U is unitary and

$$U^*AU = \begin{pmatrix} a+\varepsilon & 0 \\ 0 & b+\delta \end{pmatrix}.$$

Assume A, B satisfy (2). Then

$$(U^*A^rUU^*B^pUU^*A^rU)^{1/q} \le U^*A^{(p+2r)/q}U,$$

hence

$$\gamma^{-1/q} \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}^{1/q} \le \begin{pmatrix} (a+\varepsilon)^{(p+2r)/q} & 0 \\ 0 & (b+\delta)^{(p+2r)/q} \end{pmatrix} \tag{6}$$

where

$$\begin{aligned} A_1 =& (a+\varepsilon)^{2r}(a-b-\delta+\varepsilon b^p), \\ A_2 =& (b+\delta)^{2r}(\varepsilon + b^p(a-b-\delta)), \\ A_3 =& (a+\varepsilon)^r(b+\delta)^r(1-b^p)\sqrt{\varepsilon(a-b-\delta)}. \end{aligned}$$

Let

$$D = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}$$

and

$$V = \frac{1}{\sqrt{A_1 - A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{A_1 - A_2 + \varepsilon_1} & \sqrt{\varepsilon_1} \\ \sqrt{\varepsilon_1} & -\sqrt{A_1 - A_2 + \varepsilon_1} \end{pmatrix}$$

where

$$2\varepsilon_1 = -A_1 + A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}.$$

Then V is unitary and

$$V^*DV = \begin{pmatrix} A_1 + \varepsilon_1 & 0 \\ 0 & A_2 - \varepsilon_1 \end{pmatrix}.$$

Hence, by (6),

$$\gamma^{-1/q} \begin{pmatrix} (A_1+\varepsilon_1)^{1/q} & 0 \\ 0 & (A_2-\varepsilon_1)^{1/q} \end{pmatrix} \le \frac{1}{A_1 - A_2 + 2\varepsilon_1} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix}$$

where

$$\begin{aligned}
B_1 =&(a+\varepsilon)^{(p+2r)/q}(A_1-A_2+\varepsilon_1)+(b+\delta)^{(p+2r)/q}\varepsilon_1,\\
B_2 =&(a+\varepsilon)^{(p+2r)/q}\varepsilon_1+(b+\delta)^{(p+2r)/q}(A_1-A_2+\varepsilon_1),\\
B_3 =&((a+\varepsilon)^{(p+2r)/q}-(b+\delta)^{(p+2r)/q})\sqrt{\varepsilon_1(A_1-A_2+\varepsilon_1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
0 \le& \left|\gamma^{1/q}\begin{pmatrix} B_1 & B_3\\ B_3 & B_2\end{pmatrix} - (A_1-A_2+2\varepsilon_1)\begin{pmatrix}(A_1+\varepsilon_1)^{1/q} & 0\\ 0 & (A_2-\varepsilon_1)^{1/q}\end{pmatrix}\right|\\
=& (A_1-A_2+2\varepsilon_1)\{(a+\varepsilon)^{(p+2r)/q}(b+\delta)^{(p+2r)/q}(A_1-A_2+\varepsilon_1+\varepsilon_1)\gamma^{2/q}\\
&\quad - (a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}(A_1-A_2+\varepsilon_1)(A_2-\varepsilon_1)^{1/q}\\
&\quad - (b+\delta)^{(p+2r)/q}\gamma^{1/q}\varepsilon_1(A_2-\varepsilon_1)^{1/q}\\
&\quad - (a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}\varepsilon_1(A_1+\varepsilon_1)^{1/q}\\
&\quad - (b+\delta)^{(p+2r)/q}\gamma^{1/q}(A_1-A_2+\varepsilon_1)(A_1+\varepsilon_1)^{1/q}\\
&\quad + (A_1-A_2+\varepsilon_1+\varepsilon_1)(A_1+\varepsilon_1)^{1/q}(A_2-\varepsilon_1)^{1/q}\}\\
=& (A_1-A_2+2\varepsilon_1)\{(A_1-A_2+\varepsilon_1)((a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}-(A_1+\varepsilon_1)^{1/q})\\
&\quad \times((b+\delta)^{(p+2r)/q}\gamma^{1/q}-(A_2-\varepsilon_1)^{1/q})\\
&\quad + \varepsilon_1((a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}-(A_2-\varepsilon_1)^{1/q})\\
&\quad \times((b+\delta)^{(p+2r)/q}\gamma^{1/q}-(A_1+\varepsilon_1)^{1/q})\}.
\end{aligned}$$

Since $0 < A_1 - A_2 + 2\varepsilon_1$, we have the following key inequality

$$\begin{aligned}
&\varepsilon_1((a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}-(A_2-\varepsilon_1)^{1/q})((A_1+\varepsilon_1)^{1/q}-(b+\delta)^{(p+2r)/q}\gamma^{1/q})\\
&\quad \le (A_1-A_2+\varepsilon_1)((a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}-(A_1+\varepsilon_1)^{1/q})\\
&\quad\quad \times((b+\delta)^{(p+2r)/q}\gamma^{1/q}-(A_2-\varepsilon_1)^{1/q}).
\end{aligned} \tag{7}$$

Now we estimate each term of the inequality (7) as far as order of ε and δ. o means $o(\varepsilon)$ or $o(\delta)$, i.e., $\frac{o}{\varepsilon}, \frac{o}{\delta} \to 0$ $(\varepsilon, \delta \to +0)$.

Then

$$\begin{aligned}
A_1 &= a^{2r}(a-b)\left(1+\left(\frac{2r}{a}+\frac{b^p}{a-b}\right)\varepsilon+\frac{-1}{a-b}\delta+o\right),\\
A_2 &= b^{p+2r}(a-b)\left(1+\frac{1}{b^p(a-b)}\varepsilon+\left(\frac{2r}{b}-\frac{1}{a-b}\right)\delta+o\right),\\
A_3^2 &= a^{2r}b^{2r}(a-b)(1-b^p)^2\varepsilon\left(1+\frac{2r}{a}\varepsilon+\left(\frac{2r}{b}-\frac{1}{a-b}\right)\delta+o\right),
\end{aligned}$$

$$\begin{aligned}
\varepsilon_1 =&\frac{1}{2}(A_1-A_2)\left(-1+\sqrt{1+\frac{4A_3^2}{(A_1-A_2)^2}}\right)\\
=&\frac{a^{2r}b^{2r}(1-b^p)^2\varepsilon}{a^{2r}-b^{p+2r}}\left(1+\frac{o}{\varepsilon}\right),
\end{aligned}$$

$$\begin{aligned}(b+\delta)^{(p+2r)/q}\gamma^{1/q} =&(a-b)^{1/q}b^{(p+2r)/q}\\ &\times\left(1+\frac{1}{q(a-b)}\varepsilon+\frac{1}{q}\left(\frac{p+2r}{b}-\frac{1}{a-b}\right)\delta+o\right),\\ (A_2-\varepsilon_1)^{1/q} =&(a-b)^{1/q}b^{(p+2r)/q}\\ &\times\left(1+\frac{2a^{2r}-a^{2r}b^p-b^{2r}}{q(a-b)(a^{2r}-b^{p+2r})}\varepsilon+\frac{1}{q}\left(\frac{2r}{b}-\frac{1}{a-b}\right)\delta+o\right),\end{aligned}$$

$$\begin{aligned}&(b+\delta)^{(p+2r)/q}\gamma^{1/q}-(A_2-\varepsilon_1)^{1/q}\\ &=(a-b)^{1/q}b^{(p+2r)/q}\varepsilon\left(\frac{-(1-b^p)(a^{2r}-b^{2r})}{q(a-b)(a^{2r}-b^{p+2r})}+\frac{p}{qb}\frac{\delta}{\varepsilon}+\frac{o}{\varepsilon}\right),\end{aligned}$$

$$A_1-A_2+\varepsilon_1=(a-b)(a^{2r}-b^{p+2r})\left(1+\frac{o}{\varepsilon}\right),$$

$$(a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}-(A_2-\varepsilon_1)^{1/q}=(a-b)^{1/q}(a^{(p+2r)/q}-b^{(p+2r)/q})\left(1+\frac{o}{\varepsilon}\right),$$

$$(A_1+\varepsilon_1)^{1/q}-(b+\delta)^{(p+2r)/q}\gamma^{1/q}=(a-b)^{1/q}(a^{2r/q}-b^{(p+2r)/q})\left(1+\frac{o}{\varepsilon}\right),$$

and

$$(a+\varepsilon)^{(p+2r)/q}\gamma^{1/q}-(A_1+\varepsilon_1)^{1/q}=(a-b)^{1/q}a^{2r/q}(a^{p/q}-1)\left(1+\frac{o}{\varepsilon}\right).$$

Then, by (7),

$$\begin{aligned}&a^{2r}b^{2r}(1-b^p)^2(a^{(p+2r)/q}-b^{(p+2r)/q})(a^{2r/q}-b^{(p+2r)/q})\left(1+\frac{o}{\varepsilon}\right)\\ &\le a^{2r/q}b^{(p+2r)/q}(a-b)(a^{2r}-b^{p+2r})^2(a^{p/q}-1)\\ &\quad\times\left(\frac{-(1-b^p)(a^{2r}-b^{2r})}{q(a-b)(a^{2r}-b^{p+2r})}+\frac{p}{qb}\frac{\delta}{\varepsilon}+\frac{o}{\varepsilon}\right).\end{aligned}\tag{8}$$

We remark that

$$\liminf_{\varepsilon,\delta\to+0}\frac{\delta}{\varepsilon}\ge\liminf_{\varepsilon,\delta\to+0}\frac{1-b}{a-1+\varepsilon}=\frac{1-b}{a-1},$$

and the minimum of the right term of inequality (8) in which $\varepsilon,\delta\to+0$ will be realized if $\frac{\delta}{\varepsilon}=\frac{1-b}{a-1}$.

Define

$$\delta=\frac{1-b}{a-1}\varepsilon.$$

Then, by letting $\varepsilon\to+0$, (8) becomes

$$\begin{aligned}&q(1-a^{-1})(1-b^p)^2(1-a^{(p+2r)/q}b^{(p+2r)/q})(1-a^{-2r/q}b^{2r/q})\\ &\le a^{2r(q-1)/q}b^{(p+2r)/q-2r-1}(1-a^{-2r}b^{p+2r})(1-a^{-p/q})\\ &\quad\times\{p(1-b)(1-a^{-1}b)(1-a^{-2r}b^{p+2r})\\ &\quad-b(1-b^p)(1-a^{-1})(1-a^{-2r}b^{2r})\}.\end{aligned}$$

If $0<q<1$, by letting $a\to\infty$, we have

$$0<q(1-b^p)^2\le 0.$$

This is a contradiction.

Also if $(1+2r)q < p+2r$, by letting $b \to +0$, we have

$$0 < q(1-a^{-1}) \leq 0.$$

This is a contradiction. $\square$

Appendix

There are more simple examples $A, B \in B(\mathbb{C}^2)$ in case of $(1+2r)q < p+2r$. First of all, we make the following remark:

Let $a, b, d, \theta \in \mathbb{R}$ satisfy $0 < a+b, ab = d^2$, and

$$S = \begin{pmatrix} a & de^{-i\theta} \\ de^{i\theta} & b \end{pmatrix}.$$

Then

$$S^p = (a+b)^{p-1} S \quad \text{for } 0 < p.$$

As a matter of fact, if we put a unitary

$$U = \frac{1}{\sqrt{b^2+d^2}} \begin{pmatrix} de^{-i\theta} & b \\ b & -de^{i\theta} \end{pmatrix},$$

then we have

$$U^* S U = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix},$$

and so

$$S^p = U(U^*SU)^p U^* = U \begin{pmatrix} (a+b)^p & 0 \\ 0 & 0 \end{pmatrix} U^* = (a+b)^{p-1} S.$$

Now we explain simple examples A and B. Let $0 < c < 1, \theta \in \mathbb{R}$,

$$A = \begin{pmatrix} 2 & 2\sqrt{c(1-c)}e^{i\theta} \\ 2\sqrt{c(1-c)}e^{-i\theta} & 4c \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we have $0 \leq B \leq A$. We prove that A and B do not satisfy (2). Let

$$V = \begin{pmatrix} \sqrt{1-c}e^{i\theta} & \sqrt{c} \\ \sqrt{c} & -\sqrt{1-c}e^{-i\theta} \end{pmatrix}.$$

Then V is unitary and

$$V^* A V = \begin{pmatrix} 2+2c & 0 \\ 0 & 2c \end{pmatrix}.$$

Note that (2) holds for A and B if and only if

$$((V^*AV)^r V^* B^p V (V^*AV)^r)^{1/q} \leq (V^*AV)^{(p+2r)/q}.$$

By the preceding remark, the latter is rephrased as

$$\begin{aligned}
&\begin{pmatrix} (2+2c)^{2r}(1-c) & -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{-i\theta} \\ -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{i\theta} & (2c)^{2r}c \end{pmatrix}^{1/q} \\
&\quad = \delta \begin{pmatrix} (2+2c)^{2r}(1-c) & -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{-i\theta} \\ -(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{i\theta} & (2c)^{2r}c \end{pmatrix} \\
&\quad \leq \begin{pmatrix} (2+2c)^{(p+2r)/q} & 0 \\ 0 & (2c)^{(p+2r)/q} \end{pmatrix}
\end{aligned}$$

where

$$\delta = ((2+2c)^{2r}(1-c) + (2c)^{2r}c)^{1/q-1}.$$

Hence

$$0 \leq \begin{pmatrix} (2+2c)^{(p+2r)/q} - \delta(2+2c)^{2r}(1-c) & \delta(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{-i\theta} \\ \delta(2+2c)^r(2c)^r\sqrt{c(1-c)}e^{i\theta} & (2c)^{(p+2r)/q} - \delta(2c)^{2r}c \end{pmatrix}.$$

By taking the determinant of the right matrix,

$$\begin{aligned}
0 \leq &((2+2c)^{(p+2r)/q} - \delta(2+2c)^{2r}(1-c))((2c)^{(p+2r)/q} - \delta(2c)^{2r}c) \\
&- \delta^2(2+2c)^{2r}(2c)^{2r}c(1-c).
\end{aligned}$$

Hence

$$\begin{aligned}
&\delta(2+2c)^{(p+2r)/q}(2c)^{2r}c + \delta(2+2c)^{2r}(2c)^{(p+2r)/q}(1-c) \\
&\quad \leq (2+2c)^{(p+2r)/q}(2c)^{(p+2r)/q},
\end{aligned}$$

and

$$\begin{aligned}
&\delta(2+2c)^{(p+2r)/q}2^{2r} + \delta(2+2c)^{2r}2^{(p+2r)/q}c^{(p+2r)/q-2r-1}(1-c) \\
&\quad \leq (2+2c)^{(p+2r)/q}2^{(p+2r)/q}c^{(p+2r)/q-2r-1}.
\end{aligned}$$

Since

$$0 < \frac{p+2r}{q} - 2r - 1,$$

by letting $c \to +0$, we have

$$0 < (2^{2r})^{1/q-1}2^{(p+2r)/q}2^{2r} \leq 0.$$

This is a contradiction.

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