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CLASS NUMBERS AND IWASAWA INVARIANTS OF QUADRATIC FIELDS

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Abstract. Let $\mathbf{Q}(\sqrt{-d}\,)$ and $\mathbf{Q}(\sqrt{3d}\,)$ be quadratic fields with $d \equiv 2 \pmod 3$ a positive integer. Let λ^-, λ^+ be the respective Iwasawa λ-invariants of the cyclotomic $\mathbf{Z}_3$-extension of these fields. We show that if $\lambda^- = 1$, then 3 does not divide the class number of $\mathbf{Q}(\sqrt{3d}\,)$ and $\lambda^+ = 0$.

Introduction

Let $k^- = \mathbf{Q}(\sqrt{-d}\,)$ and $k^+ = \mathbf{Q}(\sqrt{3d}\,)$ with d a positive integer. In [5], Washington showed that constraints on the 3-Sylow subgroup and the fundamental unit of k^+ force λ^- to be 1, where λ^- is the Iwasawa λ-invariant associated to the cyclotomic $\mathbf{Z}_3$-extension of k^-. Here, using similar methods, we show that if $\lambda^- = 1$ and 3 splits in k^-, then 3 does not divide the class number of k^+. Since recent results of Jochnowitz [3, 4] imply that there are infinitely many imaginary quadratic fields in which 3 splits and $\lambda^- = 1$, we obtain as a corollary that there are infinitely many real quadratic fields $\mathbf{Q}(\sqrt{3d}\,)$ with 3 splitting in $\mathbf{Q}(\sqrt{-d}\,)$ such that the class number of $\mathbf{Q}(\sqrt{3d}\,)$ is relatively prime to 3. (We note that Horie [1] has proven a similar result concerning real quadratic fields by requiring that 3 neither divides the class number of nor splits in $\mathbf{Q}(\sqrt{-d})$.)

It then follows that there are infinitely many real quadratic fields with $\lambda^+ = 0$, where λ^+ is the Iwasawa λ-invariant associated to the $\mathbf{Z}_3$-extension of k^+.

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We begin with a brief review of p-adic L-functions. For more details, see [6]. Let p be an odd prime and let $\mathbf{Z}_p$, $\mathbf{Q}_p$ and $\mathbf{C}_p$ denote the p-adic integers, the p-adic rationals and the completion of the algebraic closure of $\mathbf{Q}_p$ respectively. Let ω denote the Teichmüller character and let ψ be a primitive Dirichlet character of conductor f, with p^2 not dividing f. We let $d = f$ if p does not divide f and $d = \dfrac{f}{p}$ if p does divide f. The generalized Bernoulli number $B_{n,\psi}$ is defined by

$$\sum_{a=1}^{f} \frac{\psi(a)e^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\psi}\frac{t^n}{n!}.$$

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The p-adic L-function $L_p(s,\psi)$ is the unique meromorphic p-adic function $\mathbf{Z}_p \to \mathbf{C}_p$ which for $n \geq 1$ satisfies

$$L_p(1-n,\psi) = -(1-\psi\omega^{-n}(p)p^{n-1})\frac{B_{n,\psi\omega^{-n}}}{n}. \tag{1}$$

In order to ensure that $L_p(s,\psi)$ is not identically zero, we now assume that ψ is a non-trivial even character. If $O_\psi = \mathbf{Z}_p[\psi(1),\psi(2),\ldots]$, Iwasawa has shown that there is a power series $F(T,\psi) \in O_\psi[[T]]$ such that

$$L_p(s,\psi) = F((1+pd)^s - 1, \psi).$$

From the p-adic Weierstrass Preparation Theorem [6] we see that $F(T,\psi) = G(T)U(T)$ where $U(T)$ is a unit of $O_\psi[[T]]$ and $G(T)$ is a distinguished polynomial. Then, $G(T) = a_0 + a_1T + \cdots + a_{n-1}T^{\lambda-1} + T^\lambda$ and if π generates the ideal of O_ψ lying over p, then π divides a_i, $0 \leq i \leq \lambda - 1$. We note that if ψ is an even quadratic character and $p = 3$, then λ is related to the class group of certain number fields. That is, we let k be the imaginary quadratic field associated to $\psi\omega^{-1}$ with k_∞ its cyclotomic $\mathbf{Z}_3$-extension. Also, let k_n be the unique subfield of k_∞ of degree 3^n over k and let A_n be the 3-Sylow subgroup of k_n. Then, via the natural injection $A_n \to A_{n+1}$ for all $n \geq 0$,

$$\bigcup_{n\geq 0} A_n \cong (\mathbf{Q}_3/\mathbf{Z}_3)^\lambda.$$

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Let K be a real quadratic field with character χ, fundamental unit ϵ, discriminant D and class number h^+. Leopoldt's p-adic class number formula says that

$$\frac{2h^+ log_p(\epsilon)}{\sqrt{D}}\left(1 - \frac{\chi(p)}{p}\right) = L_p(1,\chi) \tag{2}$$

where $\log_p$ denotes the p-adic logarithm.

We now assume that $p = 3$ and let λ^- (resp. λ^+) be the Iwasawa λ-invariant associated to the cyclotomic $\mathbf{Z}_3$-extension of $\mathbf{Q}(\sqrt{-d}\,)$ (resp. $\mathbf{Q}(\sqrt{3d}\,)$) for the prime 3.

Theorem. *Assume* $d \equiv 2 \pmod 3$ *and* $\lambda^- = 1$. *Then* 3 *does not divide the class number of* $\mathbf{Q}(\sqrt{3d}\,)$. *In particular,* $\lambda^+ = 0$.

Proof. Let χ be the non-trivial even quadratic character of conductor $3d$. Since 3 splits in $\mathbf{Q}(\sqrt{-d})$, we have that $L_3(0,\chi) = 0$ from (1). Furthermore, since $\lambda^- = 1, F(T,\chi) = (b_0 + b_1T)U(T)$ where $U(T)$ is a unit of $\mathbf{Z}_3[[T]]$ and b_1 is a 3-adic unit. Because $L_3(0,\chi) = F(0,\chi)$, we have $F(T,\chi) = (b_1T)U(T)$. Since 3 does not divide b_1, $L_3(1,\chi) \not\equiv 0 \pmod 9$. (See Lemma 1 of [5].) Then,

$$\frac{2h^+ log_3(\epsilon)}{\sqrt{D}} \not\equiv 0 \pmod 9$$

from (2). Thus, in order to prove that 3 does not divide h^+, it suffices to show that $\log_3(\epsilon) \equiv 0 \pmod{3\sqrt{3d}\,}$. In order to prove this congruence, note that the

3-integrality of $L_3(1,\chi)$ together with the fact that in this situation there is a $\sqrt{3d}$ in the denominator of (2) imply that $\log_3(\epsilon)$ must have half-integral (non-integral) 3-adic valuation. Thus, it is sufficient to show that $\log_3(\epsilon) \equiv 0 \bmod 3$. Let

$$\epsilon = a + b\sqrt{3d} \text{ or } \frac{a+b\sqrt{3d}}{2}.$$

Then

$$\epsilon^2 - 1 \equiv 2ab\sqrt{3d} \pmod 3.$$

Since

$$\log_3(\epsilon^2) = \log_3(\epsilon^2 - 1 + 1) \equiv (\epsilon^2 - 1) - \frac{(\epsilon^2-1)^2}{2} + \frac{(\epsilon^2-1)^3}{3} \pmod 3,$$

we see that

$$\log_3(\epsilon^2) \equiv 2ab\sqrt{3d} + 8a^3b^3d\sqrt{3d} \pmod 3.$$

Since $d \equiv 2 \pmod 3$, we see that $\log_3(\epsilon^2) \equiv 0 \pmod 3$. Thus, $\log_3(\epsilon) \equiv 0 \pmod 3$ as well.

Finally, a theorem of Iwasawa [2] says that if 3 totally ramifies in the cyclotomic $\mathbf{Z}_3$-extension of $\mathbf{Q}(\sqrt{3d})$ and h^+ is not divisible by 3, then $\lambda^+ = 0$.

In [4], Jochnowitz proves that given an arbitrary odd prime p, there are infinitely many imaginary quadratic fields in which p splits and whose Iwasawa λ-invariant associated to p equals 1. This immediately implies the following.

Corollary. *There are infinitely many real quadratic fields which have class number not divisible by* 3 *and whose Iwasawa λ-invariant associated to* 3 *equals zero.*

Examples. We now give several examples. The first illustrates our theorem, the next two show that if $d \not\equiv 2 \pmod 3$ and $\lambda^- = 1$, then it is possible to have 3 dividing h^+, and the final one shows that if $d \equiv 2 \pmod 3$ and $\lambda^- \neq 1$, then it is also possible to have 3 dividing h^+.

d	λ^-	h^+
23	1	1
237	1	3
262	1	6
107	2	3

References

1. K. Horie, *A note on basic Iwasawa λ-invariants of imaginary quadratic fields*, Invent. Math. **88** (1987), 31–38. MR **88i:**11073
2. K. Iwasawa, *A note on class numbers of algebraic number fields*, Abh. Math. Sem. Univ. Hamburg **20** (1956), 257–258. MR **18:**644d
3. N. Jochnowitz, *A p-adic conjecture about derivatives of L-series attatched to modular forms*, Proceedings of the Boston University Conference on p-Adic Monodromy and the p-Adic Birch and Swinnerton-Dyer Conjecture (to appear). CMP 94:13

4. ———, *An alternative approach to non-vanishing theorems for coefficients of half integral weight forms mod p and implications for Iwasawa's λ-invariant for quadratic fields* (to appear).
5. L. Washington, *Zeroes of p-adic L-functions*, Sém Delange-Pisot- Poitou, Théorie des Nombres, 1980/1981, Birkhäuser, Boston, Basel, and Stuttgart, 1982. MR **84f:**12008
6. ———, *Introduction to cyclotomic fields*, Graduate Texts in Math., Springer-Verlag, New York, 1982. MR **85g:**11001

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