CLASS NUMBERS AND IWASAWA INVARIANTS
OF QUADRATIC FIELDS

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(Communicated by William Adams)

Abstract. Let \( \mathbf{Q}(\sqrt{-d}) \) and \( \mathbf{Q}(\sqrt{3d}) \) be quadratic fields with \( d \equiv 2 \) (mod 3) a positive integer. Let \( \lambda^-, \lambda^+ \) be the respective Iwasawa \( \lambda \)-invariants of the cyclotomic \( \mathbb{Z}_3 \)-extension of these fields. We show that if \( \lambda^- = 1 \), then 3 does not divide the class number of \( \mathbf{Q}(\sqrt{3d}) \) and \( \lambda^+ = 0 \).

Introduction

Let \( k^- = \mathbf{Q}(\sqrt{-d}) \) and \( k^+ = \mathbf{Q}(\sqrt{3d}) \) with \( d \) a positive integer. In [5], Washington showed that constraints on the 3-Sylow subgroup and the fundamental unit of \( k^+ \) force \( \lambda^- \) to be 1, where \( \lambda^- \) is the Iwasawa \( \lambda \)-invariant associated to the cyclotomic \( \mathbb{Z}_3 \)-extension of \( k^- \). Here, using similar methods, we show that if \( \lambda^- = 1 \) and 3 splits in \( k^- \), then 3 does not divide the class number of \( k^+ \). Since recent results of Jochnowitz [3, 4] imply that there are infinitely many imaginary quadratic fields in which 3 splits and \( \lambda^- = 1 \), we obtain as a corollary that there are infinitely many real quadratic fields \( \mathbf{Q}(\sqrt{3d}) \) with 3 splitting in \( \mathbf{Q}(\sqrt{-d}) \) such that the class number of \( \mathbf{Q}(\sqrt{3d}) \) is relatively prime to 3. (We note that Horie [1] has proven a similar result concerning real quadratic fields by requiring that 3 neither divides the class number of nor splits in \( \mathbf{Q}(\sqrt{-d}) \).)

It then follows that there are infinitely many real quadratic fields with \( \lambda^+ = 0 \), where \( \lambda^+ \) is the Iwasawa \( \lambda \)-invariant associated to the \( \mathbb{Z}_3 \)-extension of \( k^+ \).

The author wishes to thank Lawrence Washington for many informative conversations.

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We begin with a brief review of \( p \)-adic \( L \)-functions. For more details, see [6]. Let \( p \) be an odd prime and let \( \mathbb{Z}_p \), \( \mathbf{Q}_p \) and \( \mathbf{C}_p \) denote the \( p \)-adic integers, the \( p \)-adic rationals and the completion of the algebraic closure of \( \mathbf{Q}_p \) respectively. Let \( \omega \) denote the Teichmüller character and let \( \psi \) be a primitive Dirichlet character of conductor \( f \), with \( p^2 \) not dividing \( f \). We let \( d = f \) if \( p \) does not divide \( f \) and \( d = \frac{f}{p} \) if \( p \) does divide \( f \). The generalized Bernoulli number \( B_{n,\psi} \) is defined by

\[
\sum_{a=1}^{f} \frac{\psi(a)e^{at}}{e^{at} - 1} = \sum_{n=0}^{\infty} B_{n,\psi} \frac{t^n}{n!}.
\]
The \( p \)-adic \( L \)-function \( L_p(s, \psi) \) is the unique meromorphic \( p \)-adic function \( \mathbb{Z}_p \rightarrow \mathbb{C}_p \) which for \( n \geq 1 \) satisfies

\[
L_p(1 - n, \psi) = -(1 - \psi \omega^{-n})(p)n^{-1} B_{n,\psi\omega^{-n}}. 
\]

In order to ensure that \( L_p(s, \psi) \) is not identically zero, we now assume that \( \psi \) is a non-trivial even character. If \( O_\psi = \mathbb{Z}_p[\psi(1), \psi(2), \ldots] \), Iwasawa has shown that there is a power series \( F(T, \psi) \in \mathcal{O}_\psi[[T]] \) such that

\[
L_p(s, \psi) = F((1 + pd)^s - 1, \psi). 
\]

From the \( p \)-adic Weierstrass Preparation Theorem [6] we see that \( F(T, \psi) = G(T) U(T) \) where \( U(T) \) is a unit of \( \mathcal{O}_\psi[[T]] \) and \( G(T) \) is a distinguished polynomial. Then, \( G(T) = a_0 + a_1 T + \cdots + a_{\lambda - 1} T^{\lambda - 1} + T^\lambda \) and if \( \pi \) generates the ideal of \( O_\psi \) lying over \( p \), then \( \pi \) divides \( a_i \), \( 0 \leq i \leq \lambda - 1 \). We note that if \( \psi \) is an even quadratic character and \( p = 3 \), then \( \lambda \) is related to the class group of certain number fields. That is, we let \( k \) be the imaginary quadratic field associated to \( \psi \omega^{-1} \) with \( k_\infty \) its cyclotomic \( \mathbb{Z}_3 \)-extension. Also, let \( k_n \) be the unique subfield of \( k_\infty \) of degree \( 3^n \) over \( k \) and let \( A_n \) be the \( 3 \)-Sylow subgroup of \( k_n \). Then, via the natural injection \( A_n \rightarrow A_{n+1} \) for all \( n \geq 0 \),

\[
\bigcup_{n \geq 0} A_n \cong (\mathbb{Q}_3/\mathbb{Z}_3)^\lambda. 
\]

Let \( K \) be a real quadratic field with character \( \chi \), fundamental unit \( \epsilon \), discriminant \( D \) and class number \( h^+ \). Leopoldt’s \( p \)-adic class number formula says that

\[
\frac{2h^+ \log_p(\epsilon)}{\sqrt{D}} \left( 1 - \frac{\chi(p)}{p} \right) = L_p(1, \chi) 
\]

where \( \log_p \) denotes the \( p \)-adic logarithm.

We now assume that \( p = 3 \) and let \( \lambda^- \) (resp. \( \lambda^+ \)) be the Iwasawa \( \lambda \)-invariant associated to the cyclotomic \( \mathbb{Z}_3 \)-extension of \( \mathbb{Q}(\sqrt{-d}) \) (resp. \( \mathbb{Q}(\sqrt{3d}) \)) for the prime 3.

**Theorem.** Assume \( d \equiv 2 \pmod{3} \) and \( \lambda^- = 1 \). Then 3 does not divide the class number of \( \mathbb{Q}(\sqrt{3d}) \). In particular, \( \lambda^+ = 0 \).

**Proof.** Let \( \chi \) be the non-trivial even quadratic character of conductor \( 3d \). Since 3 splits in \( \mathbb{Q}(\sqrt{-d}) \), we have that \( L_3(0, \chi) = 0 \) from (1). Furthermore, since \( \chi^\lambda = 1 \), \( F(T, \chi) = (b_0 + b_1 T) U(T) \) where \( U(T) \) is a unit of \( \mathbb{Z}_3[[T]] \) and \( b_1 \) is a 3-adic unit. Because \( L_3(0, \chi) = F(0, \chi) \), we have \( F(T, \chi) = (b_1 T) U(T) \). Since 3 does not divide \( b_1 \), \( L_3(1, \chi) \neq 0 \pmod{9} \). (See Lemma 1 of [5].) Then,

\[
\frac{2h^+ \log_3(\epsilon)}{\sqrt{D}} \neq 0 \pmod{9} 
\]

from (2). Thus, in order to prove that 3 does not divide \( h^+ \), it suffices to show that \( \log_3(\epsilon) \equiv 0 \pmod{3\sqrt{3d}} \). In order to prove this congruence, note that the
3-integrality of $L_3(1, \chi)$ together with the fact that in this situation there is a $\sqrt{3d}$ in the denominator of (2) imply that $\log_3(\epsilon)$ must have half-integral (non-integral) 3-adic valuation. Thus, it is sufficient to show that $\log_3(\epsilon) \equiv 0 \mod 3$. Let

$$\epsilon = a + b\sqrt{3d} \text{ or } a + b\sqrt{3d}/2.$$ 

Then

$$\epsilon^2 - 1 \equiv 2ab\sqrt{3d} \pmod{3}.$$ 

Since

$$\log_3(\epsilon^2) = \log_3(\epsilon^2 - 1 + 1) \equiv (\epsilon^2 - 1) - \frac{(\epsilon^2 - 1)^2}{2} + \frac{(\epsilon^2 - 1)^3}{3} \pmod{3},$$

we see that

$$\log_3(\epsilon^2) \equiv 2ab\sqrt{3d} + 8a^3b^3d\sqrt{3d} \pmod{3}.$$ 

Since $d \equiv 2 \pmod{3}$, we see that $\log_3(\epsilon^2) \equiv 0 \pmod{3}$. Thus, $\log_3(\epsilon) \equiv 0 \pmod{3}$ as well.

Finally, a theorem of Iwasawa [2] says that if 3 totally ramifies in the cyclotomic $\mathbb{Z}_3$-extension of $\mathbb{Q}(\sqrt{3d})$ and $h^+$ is not divisible by 3, then $\lambda^+ = 0$.

In [4], Jochnowitz proves that given an arbitrary odd prime $p$, there are infinitely many imaginary quadratic fields in which $p$ splits and whose Iwasawa $\lambda$-invariant associated to $p$ equals 1. This immediately implies the following.

**Corollary.** There are infinitely many real quadratic fields which have class number not divisible by 3 and whose Iwasawa $\lambda$-invariant associated to 3 equals zero.

**Examples.** We now give several examples. The first illustrates our theorem, the next two show that if $d \not\equiv 2 \pmod{3}$ and $\lambda^- = 1$, then it is possible to have 3 dividing $h^+$, and the final one shows that if $d \equiv 2 \pmod{3}$ and $\lambda^- \neq 1$, then it is also possible to have 3 dividing $h^+$.

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**References**

4. ______, An alternative approach to non-vanishing theorems for coefficients of half integral weight forms mod \( p \) and implications for Iwasawa’s \( \lambda \)-invariant for quadratic fields (to appear).
