

## ON A CONJECTURE BY KARLIN AND SZEGÖ

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ABSTRACT. In 1961, Karlin and Szegö conjectured : If  $\{P_n(x)\}_{n=0}^{\infty}$  is an orthogonal polynomial system and  $\{P'_n(x)\}_{n=1}^{\infty}$  is a Sturm sequence, then  $\{P_n(x)\}_{n=0}^{\infty}$  is essentially (that is, after a linear change of variable) a classical orthogonal polynomial system of Jacobi, Laguerre, or Hermite. Here, we prove that for any orthogonal polynomial system  $\{P_n(x)\}_{n=0}^{\infty}$ ,  $\{P'_n(x)\}_{n=1}^{\infty}$  is always a Sturm sequence. Thus, in particular, the above conjecture by Karlin and Szegö is false.

### 1. INTRODUCTION

At the end of their work [5, p.156], Karlin and Szegö made three conjectures for the characterization of classical orthogonal polynomials. The first and the third are answered by Al-Salam and Chihara [1] and Hahn [4] respectively. The second conjecture asks : if  $\{P_n(x)\}_{n=0}^{\infty}$  is an orthogonal polynomial system and  $\{P'_n(x)\}_{n=1}^{\infty}$  is a Sturm sequence, then is  $\{P_n(x)\}_{n=0}^{\infty}$  one of the three classical orthogonal polynomials of Jacobi  $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$  ( $\alpha, \beta > -1$ ), Laguerre  $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$  ( $\alpha > -1$ ), and Hermite  $\{H_n(x)\}_{n=0}^{\infty}$  (possibly after a suitable linear change of variable)?

Conversely, it is well known that any orthogonal polynomial system  $\{P_n(x)\}_{n=0}^{\infty}$  is a Sturm sequence (see Chihara [2], Chapter 1.5) and if  $\{P_n(x)\}_{n=0}^{\infty}$  is a classical orthogonal polynomial system, then  $\{P'_n(x)\}_{n=1}^{\infty}$  is also a classical orthogonal polynomial system (known as the Hahn-Sonine theorem) and so is also a Sturm sequence.

We will show that for any orthogonal polynomial system  $\{P_n(x)\}_{n=0}^{\infty}$ ,  $\{P'_n(x)\}_{n=1}^{\infty}$  is always a Sturm sequence (but is not necessarily orthogonal). In particular, the answer to the above question by Karlin and Szegö is no.

The orthogonality considered in Karlin and Szegö [5] is the one with respect to a positive Stieltjes measure  $d\mu(x)$ , where  $\mu(x)$  is a non-decreasing function. Here we consider a general sense of orthogonality with respect to a signed Stieltjes measure  $d\mu(x)$ , where  $\mu(x)$  is a function of bounded variation.

### 2. MAIN RESULTS

All polynomials in this work are assumed to be real polynomials in one variable. We use  $\deg(P)$  to denote the degree of a polynomial  $P(x)$  with the convention that  $\deg(0) = -1$ .

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**Definition 2.1** (Karlin and Szegö [5]). A sequence of polynomials  $\{P_n(x)\}_{n=0}^\infty$  with  $\deg(P_n) = n$ ,  $n \geq 0$ , is called a Sturm sequence on an open interval  $I = (a, b)$ ,  $-\infty \leq a < b \leq \infty$  if

- (i) each  $P_n(x)$  has exactly  $n$  simple real zeros in  $I$  ;
- (ii) for each  $n \geq 1$ , zeros of  $P_n(x)$  and  $P_{n+1}(x)$  strictly interlace.

If  $x_{n1} < x_{n2} < \dots < x_{nn}$  are zeros of  $P_n(x)$ ,  $n \geq 1$ , then the above condition (ii) means

$$(2.1) \quad x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad 1 \leq i \leq n.$$

In the following, we call a sequence of polynomials  $\{P_n(x)\}_{n=0}^\infty$  a polynomial system (PS) if  $\deg(P_n) = n$ ,  $n \geq 0$ .

**Definition 2.2.** A PS  $\{P_n(x)\}_{n=0}^\infty$  is called a Tchebychev polynomial system (TPS) (respectively, an orthogonal polynomial system (OPS)) if there is a function  $\mu(x)$  of bounded variation (respectively, a non-decreasing function  $\mu(x)$ ) such that

$$(2.2) \quad \int_{-\infty}^\infty P_m(x)P_n(x)d\mu(x) = K_n\delta_{mn}, \quad m \text{ and } n \geq 0,$$

where  $K_n \neq 0$  (respectively,  $K_n > 0$ ),  $n \geq 0$ .

We first find sufficient conditions for any given PS  $\{P_n(x)\}_{n=0}^\infty$  under which both  $\{P_n(x)\}_{n=0}^\infty$  and  $\{P'_n(x)\}_{n=1}^\infty$  are Sturm sequences.

**Theorem 2.1.** Let  $\{P_n(x)\}_{n=0}^\infty$  be a PS such that all zeros of  $P_n(x)$ ,  $n \geq 1$ , are real and lie in  $I = (a, b)$ . Let

$$(2.3) \quad W_n(x) = P_n(x)P'_{n+1}(x) - P'_n(x)P_{n+1}(x), \quad n \geq 0,$$

be the Wronskian determinant of  $P_n(x)$  and  $P_{n+1}(x)$ . If  $W_n(x_0)W_n(x_1) > 0$  for any two zeros  $x_0$  and  $x_1$  of  $P_{n+1}(x)$ ,  $n \geq 1$  (respectively,  $W_n(y_0)W_n(y_1) > 0$  for any two zeros  $y_0$  and  $y_1$  of  $P'_{n+1}(x)$ ,  $n \geq 1$ ), then  $\{P_n(x)\}_{n=0}^\infty$  (respectively,  $\{P'_n(x)\}_{n=1}^\infty$ ) is a Sturm sequence in  $I$ .

*Proof.* Assume first that  $W_n(x_0)W_n(x_1) > 0$  for any two zeros  $x_0$  and  $x_1$  of  $P_{n+1}(x)$ ,  $n \geq 1$ . Since  $P_{n+1}(x_0) = 0$ ,  $W_n(x_0) = P_n(x_0)P'_{n+1}(x_0) \neq 0$  and so  $P'_{n+1}(x_0) \neq 0$ ,  $n \geq 1$ . Hence for all  $n \geq 1$ , zeros of  $P_n(x)$  are simple. Let  $a < x_{n1} < x_{n2} < \dots < x_{nn} < b$  be the zeros of  $P_n(x)$ . We may and shall assume that all  $P_n(x)$  are monic polynomials. Then

$$(2.4) \quad \text{sgn } P'_n(x_{nk}) = (-1)^{n-k}, \quad 1 \leq k \leq n.$$

On the other hand, we have by the assumption that

$$\begin{aligned} &W_n(x_{n+1,k})W_n(x_{n+1,k+1}) \\ &= P_n(x_{n+1,k})P'_{n+1}(x_{n+1,k})P_n(x_{n+1,k+1})P'_{n+1}(x_{n+1,k+1}) > 0, \quad 1 \leq k \leq n. \end{aligned}$$

Hence by (2.4)

$$P_n(x_{n+1,k})P_n(x_{n+1,k+1}) < 0, \quad 1 \leq k \leq n,$$

so that  $P_n(x)$  has one and only one zero in each interval  $(x_{n+1,k}, x_{n+1,k+1})$ ,  $1 \leq k \leq n$ . Hence,  $\{P_n(x)\}_{n=0}^\infty$  is a Sturm sequence in  $I$ .

We now assume that  $W_n(y_0)W_n(y_1) > 0$  for any two zeros  $y_0$  and  $y_1$  of  $P'_{n+1}(x)$ ,  $n \geq 1$ . Since  $P'_{n+1}(y_0) = 0, W_n(y_0) = -P'_n(y_0)P_{n+1}(y_0) \neq 0$  and so  $P_{n+1}(y_0) \neq 0, n \geq 1$ . Hence for all  $n \geq 1$ , zeros of  $P_n(x)$  are simple.

Let  $a < x_{n1} < x_{n2} < \dots < x_{nn} < b$  be the zeros of  $P_n(x)$ . Then by Rolle's theorem,  $P'_n(x)$  has one and only one zero  $y_{nk}$  in each interval  $(x_{nk}, x_{n,k+1})$ ,  $1 \leq k \leq n - 1$ .

Assuming all  $P_n(x)$  are monic, we have

$$(2.5) \quad \text{sgn } P_n(y_{nk}) = (-1)^{n-k}, \quad 1 \leq k \leq n - 1.$$

On the other hand, we have by the assumption that

$$\begin{aligned} &W_n(y_{n+1,k})W_n(y_{n+1,k+1}) \\ &= P'_n(y_{n+1,k})P_{n+1}(y_{n+1,k})P'_n(y_{n+1,k+1})P_{n+1}(y_{n+1,k+1}) > 0, \quad 1 \leq k \leq n - 1. \end{aligned}$$

Hence by (2.5)

$$P'_n(y_{n+1,k})P'_n(y_{n+1,k+1}) < 0, \quad 1 \leq k \leq n - 1,$$

so that  $P'_n(x)$  has one and only one zero in each interval  $(y_{n+1,k}, y_{n+1,k+1})$ ,  $1 \leq k \leq n - 1$ . Hence,  $\{P'_n(x)\}_{n=1}^\infty$  is also a Sturm sequence in  $I$ .  $\square$

**Corollary 2.2.** *Let  $\{P_n(x)\}_{n=0}^\infty$  be a PS such that all zeros of  $P_n(x), n \geq 1$ , are real and lie in  $I = (a, b)$ . If  $W_n(x) > 0, n \geq 1$ , for all real  $x$ , then both  $\{P_n(x)\}_{n=0}^\infty$  and  $\{P'_n(x)\}_{n=1}^\infty$  are Sturm sequences in  $I$ .*

*Remark.* Since  $W_n(x)$  is a monic polynomial of degree  $2n$ ,  $W_n(x) > 0$  for all real  $x$  if and only if  $W_n(x) \neq 0$  for all real  $x$ .

It is well known that if  $\{P_n(x)\}_{n=0}^\infty$  is an OPS, then  $W_n(x) > 0, n \geq 1$ , for all real  $x$ , which follows immediately from the Christoffel-Darboux identity satisfied by any TPS (see Chihara [2], Chapter 1.4). Therefore we have the following as a special case of Corollary 2.2.

**Theorem 2.3.** *If  $\{P_n(x)\}_{n=0}^\infty$  is an OPS, then both  $\{P_n(x)\}_{n=0}^\infty$  and  $\{P'_n(x)\}_{n=1}^\infty$  are Sturm sequences. Moreover, if  $[a, b]$  is the true interval of orthogonality of  $\{P_n(x)\}_{n=0}^\infty$ , then  $\{P_n(x)\}_{n=0}^\infty$  and  $\{P'_n(x)\}_{n=1}^\infty$  are Sturm sequences in  $(a, b)$ .*

*Proof.* Since any OPS is a Sturm sequence and the true interval of orthogonality of  $\{P_n(x)\}_{n=0}^\infty$  is the smallest closed interval containing all zeros of  $P_n(x), n \geq 1$ , the conclusion follows immediately from Corollary 2.2.  $\square$

It is well known that for a PS  $\{P_n(x)\}_{n=0}^\infty$ , both  $\{P_n(x)\}_{n=0}^\infty$  and  $\{P'_n(x)\}_{n=1}^\infty$  are TPS's (respectively, OPS's) if and only if  $\{P_n(x)\}_{n=0}^\infty$  is a classical TPS (respectively, a classical OPS). This fact was first proved for OPS's by Sonine [9] and Hahn [3] and was later extended to TPS's (see [6]). For a simple unified proof of

this fact and some other characterizations of classical orthogonal polynomials, we refer to [8].

Hence, Theorem 2.3 answers negatively the question by Karlin and Szegő [5] : if  $\{P_n(x)\}_{n=0}^\infty$  is an OPS and  $\{P'_n(x)\}_{n=1}^\infty$  is a Sturm sequence, is  $\{P_n(x)\}_{n=0}^\infty$  a classical OPS?

We finally ask the following question: if  $\{P_n(x)\}_{n=0}^\infty$  is a TPS but not an OPS, can  $\{P_n(x)\}_{n=0}^\infty$  be a Sturm sequence?

**Theorem 2.4.** *Let  $\{P_n(x)\}_{n=0}^\infty$  be a TPS. Then  $\{P_n(x)\}_{n=0}^\infty$  is a Sturm sequence if and only if  $\{P_n(x)\}_{n=0}^\infty$  is an OPS.*

*Proof.* The sufficiency follows from Theorem 2.3. We now assume that  $\{P_n(x)\}_{n=0}^\infty$  is a Sturm sequence. By Farvard's theorem,  $\{P_n(x)\}_{n=0}^\infty$  satisfies a three-term recurrence relation:

$$(2.6) \quad P_{n+1}(x) = (x - b_n)P_n(x) - c_nP_{n-1}(x), \quad n \geq 0 \quad (P_{-1}(x) \equiv 0)$$

(assuming all  $P_n(x)$  are monic) where  $c_n \neq 0$ ,  $n \geq 1$ . If we let  $x_{n1} < x_{n2} < \dots < x_{nn}$  be the zeros of  $P_n(x)$ , then

$$b_n = \sum_{k=1}^{n+1} x_{n+1,k} - \sum_{k=1}^n x_{nk} = x_{n+1,1} + \sum_{k=1}^n (x_{n+1,k+1} - x_{nk}).$$

Hence

$$x_{n+1,1} - b_n = \sum_{k=1}^n (x_{nk} - x_{n+1,k+1}) < 0$$

since  $x_{n+1,k} < x_{nk} < x_{n+1,k+1}$ ,  $1 \leq k \leq n$ . From (2.6), we also have

$$c_n = \frac{(x_{n+1,1} - b_n)P_n(x_{n+1,1})}{P_{n-1}(x_{n+1,1})}, \quad n \geq 1.$$

Since  $P_n(x_{n+1,1})P_{n-1}(x_{n+1,1}) < 0$ ,  $c_n > 0$ ,  $n \geq 1$ , so that  $\{P_n(x)\}_{n=0}^\infty$  is an OPS (see Chihara [2], Chapter 1.4). □

In particular, Theorem 2.4 implies that if  $\{P_n(x)\}_{n=0}^\infty$  is a TPS but not an OPS, then  $\{P_n(x)\}_{n=0}^\infty$  cannot be a Sturm sequence. However,  $\{P'_n(x)\}_{n=1}^\infty$  may be a Sturm sequence as the next example shows.

**Example.** Consider Laguerre polynomials  $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$  satisfying the second-order differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x).$$

It is well known that  $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$  is a TPS if and only if  $\alpha \neq -1, -2, \dots$  and is an OPS if and only if  $\alpha > -1$ . We also have (see Chihara [2], p.149)

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \quad n \geq 1.$$

Hence if  $-2 < \alpha < -1$ , then  $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$  is a TPS (but not an OPS) but  $\{\frac{d}{dx}L_n^{(\alpha)}(x)\}_{n=1}^\infty$  is an OPS and so is a Sturm sequence.

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