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ON A CONJECTURE BY KARLIN AND SZEGÖ

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ABSTRACT. In 1961, Karlin and Szegö conjectured : If $\{P_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial system and $\{P_n'(x)\}_{n=1}^{\infty}$ is a Sturm sequence, then $\{P_n(x)\}_{n=0}^{\infty}$ is essentially (that is, after a linear change of variable) a classical orthogonal polynomial system of Jacobi, Laguerre, or Hermite. Here, we prove that for any orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$, $\{P_n'(x)\}_{n=1}^{\infty}$ is always a Sturm sequence. Thus, in particular, the above conjecture by Karlin and Szegö is false.

1. Introduction

At the end of their work [5, p.156], Karlin and Szegö made three conjectures for the characterization of classical orthogonal polynomials. The first and the third are answered by Al-Salam and Chihara [1] and Hahn [4] respectively. The second conjecture asks : if $\{P_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial system and $\{P_n'(x)\}_{n=1}^{\infty}$ is a Sturm sequence, then is $\{P_n(x)\}_{n=0}^{\infty}$ one of the three classical orthogonal polynomials of Jacobi $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}(\alpha, \beta > -1)$, Laguerre $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}(\alpha > -1)$, and Hermite $\{H_n(x)\}_{n=0}^{\infty}$ (possibly after a suitable linear change of variable)?

Conversely, it is well known that any orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$ is a Sturm sequence (see Chihara [2], Chapter 1.5) and if $\{P_n(x)\}_{n=0}^{\infty}$ is a classical orthogonal polynomial system, then $\{P_n'(x)\}_{n=1}^{\infty}$ is also a classical orthogonal polynomial system (known as the Hahn-Sonine theorem) and so is also a Sturm sequence.

We will show that for any orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$, $\{P_n'(x)\}_{n=1}^{\infty}$ is always a Sturm sequence (but is not necessarily orthogonal). In particular, the answer to the above question by Karlin and Szegö is no.

The orthogonality considered in Karlin and Szegö [5] is the one with respect to a positive Stieltjes measure $d\mu(x)$, where $\mu(x)$ is a non-decreasing function. Here we consider a general sense of orthogonality with respect to a signed Stieltjes measure $d\mu(x)$, where $\mu(x)$ is a function of bounded variation.

2. Main results

All polynomials in this work are assumed to be real polynomials in one variable. We use $deg(P)$ to denote the degree of a polynomial $P(x)$ with the convention that $deg(0) = -1$.

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Definition 2.1 (Karlin and Szegö [5])**.** A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $deg(P_n) = n,\ n \geqslant 0$, is called a Sturm sequence on an open interval $I = (a, b),\ -\infty \leq a < b \leq \infty$ if

(i) each $P_n(x)$ has exactly n simple real zeros in I ;

(ii) for each $n \geqslant 1$, zeros of $P_n(x)$ and $P_{n+1}(x)$ strictly interlace.

If $x_{n1} < x_{n2} < \cdots < x_{nn}$ are zeros of $P_n(x), n \geqslant 1$, then the above condition (ii) means

$$x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad 1 \leq i \leq n. \tag{2.1}$$

In the following, we call a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ a polynomial system (PS) if $deg(P_n) = n,\ n \geqslant 0$.

Definition 2.2. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called a Tchebychev polynomial system (TPS) (respectively, an orthogonal polynomial system (OPS)) if there is a function $\mu(x)$ of bounded variation (respectively, a non-decreasing function $\mu(x)$) such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)d\mu(x) = K_n\delta_{mn}, \quad m \text{ and } n \geqslant 0, \tag{2.2}$$

where $K_n \neq 0$ (respectively, $K_n > 0$), $n \geqslant 0$.

We first find sufficient conditions for any given PS $\{P_n(x)\}_{n=0}^{\infty}$ under which both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P_n'(x)\}_{n=1}^{\infty}$ are Sturm sequences.

Theorem 2.1. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS such that all zeros of $P_n(x), n \geqslant 1$, are real and lie in $I = (a, b)$. Let*

$$W_n(x) = P_n(x)P_{n+1}'(x) - P_n'(x)P_{n+1}(x), \quad n \geqslant 0, \tag{2.3}$$

be the Wronskian determinant of $P_n(x)$ and $P_{n+1}(x)$. If $W_n(x_0)W_n(x_1) > 0$ for any two zeros x_0 and x_1 of $P_{n+1}(x),\ n \geqslant 1$ (respectively, $W_n(y_0)W_n(y_1) > 0$ for any two zeros y_0 and y_1 of $P_{n+1}'(x),\ n \geqslant 1$), then $\{P_n(x)\}_{n=0}^{\infty}$ (respectively, $\{P_n'(x)\}_{n=1}^{\infty}$) is a Sturm sequence in I.

Proof. Assume first that $W_n(x_0)W_n(x_1) > 0$ for any two zeros x_0 and x_1 of $P_{n+1}(x),\ n \geqslant 1$. Since $P_{n+1}(x_0) = 0, W_n(x_0) = P_n(x_0)P_{n+1}'(x_0) \neq 0$ and so $P_{n+1}'(x_0) \neq 0,\ n \geqslant 1$. Hence for all $n \geqslant 1$, zeros of $P_n(x)$ are simple. Let $a < x_{n1} < x_{n2} < \cdots < x_{nn} < b$ be the zeros of $P_n(x)$. We may and shall assume that all $P_n(x)$ are monic polynomials. Then

$$sgn\, P_n'(x_{nk}) = (-1)^{n-k}, \quad 1 \leqslant k \leqslant n. \tag{2.4}$$

On the other hand, we have by the assumption that

$$\begin{aligned} &W_n(x_{n+1,k})W_n(x_{n+1,k+1}) \\ &= P_n(x_{n+1,k})P_{n+1}'(x_{n+1,k})P_n(x_{n+1,k+1})P_{n+1}'(x_{n+1,k+1}) > 0, \quad 1 \leqslant k \leqslant n. \end{aligned}$$

Hence by (2.4)

$$P_n(x_{n+1,k})P_n(x_{n+1,k+1}) < 0, \quad 1 \leqslant k \leqslant n,$$

so that $P_n(x)$ has one and only one zero in each interval $(x_{n+1,k}, x_{n+1,k+1})$, $1 \leqslant k \leqslant n$. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is a Sturm sequence in I.

We now assume that $W_n(y_0)W_n(y_1) > 0$ for any two zeros y_0 and y_1 of $P'_{n+1}(x)$, $n \geqslant 1$. Since $P'_{n+1}(y_0) = 0, W_n(y_0) = -P'_n(y_0)P_{n+1}(y_0) \neq 0$ and so $P_{n+1}(y_0) \neq 0, n \geqslant 1$. Hence for all n $\geqslant 1$, zeros of $P_n(x)$ are simple.

Let $a < x_{n1} < x_{n2} < \cdots < x_{nn} < b$ be the zeros of $P_n(x)$. Then by Rolle's theorem, $P'_n(x)$ has one and only one zero y_{nk} in each interval $(x_{nk}, x_{n,k+1})$, $1 \leqslant k \leqslant n-1$.

Assuming all $P_n(x)$ are monic, we have

$$sgn\, P_n(y_{nk}) = (-1)^{n-k}, \quad 1 \leqslant k \leqslant n-1. \tag{2.5}$$

On the other hand, we have by the assumption that

$$\begin{aligned} &W_n(y_{n+1,k})W_n(y_{n+1,k+1}) \\ &= P'_n(y_{n+1,k})P_{n+1}(y_{n+1,k})P'_n(y_{n+1,k+1})P_{n+1}(y_{n+1,k+1}) > 0, \quad 1 \leqslant k \leqslant n-1. \end{aligned}$$

Hence by (2.5)

$$P'_n(y_{n+1,k})P'_n(y_{n+1,k+1}) < 0, \quad 1 \leqslant k \leqslant n-1,$$

so that $P'_n(x)$ has one and only one zero in each interval $(y_{n+1,k}, y_{n+1,k+1}), 1 \leqslant k \leqslant n-1$. Hence, $\{P'_n(x)\}_{n=1}^{\infty}$ is also a Sturm sequence in I. □

Corollary 2.2. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a PS such that all zeros of $P_n(x), n \geqslant 1$, are real and lie in $I = (a,b)$. If $W_n(x) > 0$, $n \geqslant 1$, for all real x, then both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P'_n(x)\}_{n=1}^{\infty}$ are Sturm sequences in I.*

Remark. Since $W_n(x)$ is a monic polynomial of degree $2n$, $W_n(x) > 0$ for all real x if and only if $W_n(x) \neq 0$ for all real x.

It is well known that if $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS, then $W_n(x) > 0$, $n \geqslant 1$, for all real x, which follows immediately from the Christoffel-Darboux identity satisfied by any TPS (see Chihara [2], Chapter 1.4). Therefore we have the following as a special case of Corollary 2.2.

Theorem 2.3. *If $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS, then both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P'_n(x)\}_{n=1}^{\infty}$ are Sturm sequences. Moreover, if $[a,b]$ is the true interval of orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$, then $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P'_n(x)\}_{n=1}^{\infty}$ are Sturm sequences in (a,b).*

Proof. Since any OPS is a Sturm sequence and the true interval of orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$ is the smallest closed interval containing all zeros of $P_n(x), n \geqslant 1$, the conclusion follows immediately from Corollary 2.2. □

It is well known that for a PS $\{P_n(x)\}_{n=0}^{\infty}$, both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{P'_n(x)\}_{n=1}^{\infty}$ are TPS's (respectively, OPS's) if and only if $\{P_n(x)\}_{n=0}^{\infty}$ is a classical TPS (respectively, a classical OPS). This fact was first proved for OPS's by Sonine [9] and Hahn [3] and was later extended to TPS's (see [6]). For a simple unified proof of this fact and some other characterizations of classical orthogonal polynomials, we refer to [8].

Hence, Theorem 2.3 answers negatively the question by Karlin and Szegö [5] : if $\{P_n(x)\}_{n=0}^\infty$ is an OPS and $\{P'_n(x)\}_{n=1}^\infty$ is a Sturm sequence, is $\{P_n(x)\}_{n=0}^\infty$ a classical OPS?

We finally ask the following question: if $\{P_n(x)\}_{n=0}^\infty$ is a TPS but not an OPS, can $\{P_n(x)\}_{n=0}^\infty$ be a Sturm sequence?

Theorem 2.4. *Let $\{P_n(x)\}_{n=0}^\infty$ be a TPS. Then $\{P_n(x)\}_{n=0}^\infty$ is a Sturm sequence if and only if $\{P_n(x)\}_{n=0}^\infty$ is an OPS.*

Proof. The sufficiency follows from Theorem 2.3. We now assume that $\{P_n(x)\}_{n=0}^\infty$ is a Sturm sequence. By Farvard's theorem, $\{P_n(x)\}_{n=0}^\infty$ satisfies a three-term recurrence relation:

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_nP_{n-1}(x), \quad n \geqslant 0 \quad (P_{-1}(x) \equiv 0) \tag{2.6}$$

(assuming all $P_n(x)$ are monic) where $c_n \neq 0,\ n \geqslant 1$. If we let $x_{n1} < x_{n2} < \cdots < x_{nn}$ be the zeros of $P_n(x)$, then

$$b_n = \sum_{k=1}^{n+1} x_{n+1,k} - \sum_{k=1}^{n} x_{nk} = x_{n+1,1} + \sum_{k=1}^{n}(x_{n+1,k+1} - x_{nk}).$$

Hence

$$x_{n+1,1} - b_n = \sum_{k=1}^{n}(x_{nk} - x_{n+1,k+1}) \,<\, 0$$

since $x_{n+1,k} < x_{nk} < x_{n+1,k+1},\ 1 \leqslant k \leqslant n$. From (2.6), we also have

$$c_n = \frac{(x_{n+1,1} - b_n)P_n(x_{n+1,1})}{P_{n-1}(x_{n+1,1})}, \quad n \geqslant 1.$$

Since $P_n(x_{n+1,1})P_{n-1}(x_{n+1,1}) < 0,\ c_n > 0,\ n \geqslant 1$, so that $\{P_n(x)\}_{n=0}^\infty$ is an OPS (see Chihara [2], Chapter 1.4). □

In particular, Theorem 2.4 implies that if $\{P_n(x)\}_{n=0}^\infty$ is a TPS but not an OPS, then $\{P_n(x)\}_{n=0}^\infty$ cannot be a Sturm sequence. However, $\{P'_n(x)\}_{n=1}^\infty$ may be a Sturm sequence as the next example shows.

Example. Consider Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ satisfying the second-order differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x).$$

It is well known that $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ is a TPS if and only if $\alpha \neq -1, -2, \ldots$ and is an OPS if and only if $\alpha > -1$. We also have (see Chihara [2], p.149)

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \qquad n \geqslant 1.$$

Hence if $-2 < \alpha < -1$, then $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ is a TPS (but not an OPS) but $\{\frac{d}{dx}L_n^{(\alpha)}(x)\}_{n=1}^\infty$ is an OPS and so is a Sturm sequence.

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