

SINGLE-POINT CONDENSATION AND LEAST-ENERGY SOLUTIONS

XIAOFENG REN AND JUNCHENG WEI

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ABSTRACT. We prove a conjecture raised in our earlier paper which says that the least-energy solutions to a two-dimensional semilinear problem exhibit single-point condensation phenomena as the nonlinear exponent gets large. Our method is based on a sharp form of a well-known borderline case of the Sobolev embedding theory. With the help of this embedding, we can use the Moser iteration scheme to carefully estimate the upper bound of the solutions. We can also determine the location of the condensation points.

1. INTRODUCTION

In this work, we shall confirm the conjecture raised in our earlier paper [8]. We study

$$(1.1) \quad \begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $p > 1$ and Ω is a smooth bounded domain in R^2 . We consider the so-called least-energy solutions of (1.1) obtained by minimizing the functional

$$(1.2) \quad J_p : \{v \in W_0^{1,2}(\Omega) : \|v\|_{p+1} = 1\} \rightarrow R$$

where $J_p(v) = \int_{\Omega} |\nabla v|^2$. Standard variational arguments show that the minimum of J_p is achieved by a positive function in $\{v \in W_0^{1,2}(\Omega) : \|v\|_{p+1} = 1\}$, and a positive scalar multiple of the function solves (1.1). Let us denote such a least-energy solution by u_p . In [8] we proved

Theorem 1.1. *There exist C_1, C_2 , independent of p , such that*

$$0 < C_1 < \|u_p\|_{L^\infty(\Omega)} < C_2 < \infty$$

for p large enough.

To understand the shape of u_p for large p , let

$$(1.3) \quad v_p = \frac{u_p}{\int_{\Omega} u_p^p}.$$

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For a sequence v_{p_n} of v_p we define the blow-up set S of v_{p_n} to be the subset of $\overline{\Omega}$ such that $x \in S$ if there exist a subsequence, still denoted by v_{p_n} , and a sequence x_n in Ω with

$$(1.4) \quad v_{p_n}(x_n) \rightarrow \infty \text{ and } x_n \rightarrow x.$$

We use $\#S$ to denote the cardinality of S . We also define a peak point P for u_p to be a point in $\overline{\Omega}$ such that u_p doesn't vanish in the L^∞ norm in any small neighborhood of P as $p \rightarrow \infty$. It turns out later that the set of the peak points of u_p is the same as the set of blow-up points of v_p . About the blow-up set of v_p we proved the following theorem in [8].

Theorem 1.2. *0 For any smooth bounded domain Ω and a sequence v_{p_n} of v_p with $p_n \rightarrow \infty$ there exists a subsequence of v_{p_n} such that the blow-up set S of that subsequence is contained in Ω and has the property $1 \leq \#S \leq 2$; in other words the subsequence must blow up, and it blows up at most at two points in Ω .*

(2) *If Ω meets condition (T), then the above subsequence must blow up at one point in Ω .*

Here condition (T) is a geometric condition on Ω . A domain Ω is said to satisfy condition (T) if (A) Ω is star-shaped with respect to some point $y \in \Omega$, i.e., $(x - y, n(x)) > 0$ for all $x \in \partial\Omega$, and (B)

$$\int_{\partial\Omega} \frac{ds}{(x - y, n(x))} < 2\pi e$$

where $n(x)$ denotes the outer normal of $\partial\Omega$ at x .

About the location of the blow-up points, we proved in [8]

Theorem 1.3. *Let Ω be a smooth bounded domain with condition (T). Then for any sequence v_{p_n} of v_p with $p_n \rightarrow \infty$ there exists a subsequence of v_{p_n} , still denoted by v_{p_n} , such that*

(1)

$$f_n := \frac{u_{p_n}^{p_n}}{\int_{\Omega} u_{p_n}^{p_n}} = \left(\int_{\Omega} u_{p_n}^{p_n} \right)^{p_n-1} v_{p_n}^{p_n} \rightarrow \delta(x_0)$$

in the sense of distribution where $\delta(x_0)$ is the δ function at point x_0 and $\{x_0\} = S$.

(2) *$v_{p_n} \rightarrow G(x, x_0)$ in $W^{1,q}(\Omega)$ weakly for any $1 < q < 2$ where G is the Green's function of $-\Delta$ on Ω ; furthermore for any compact subset K of $\overline{\Omega} \setminus \{x_0\}$ we have $v_{p_n} \rightarrow G(x, x_0)$ in $C^{2,\alpha}(K)$.*

(3) *x_0 is a critical point of function ϕ where $\phi(x) = g(x, x)$ and*

$$g(x, y) = G(x, y) + \frac{1}{2\pi} \log|x - y|$$

is the regular part of the Green's function G .

At the end of [8], we conjectured that condition (T) is unnecessary for Theorem 1.2 (2) and Theorem 1.3, i.e., least-energy solutions must develop single peaks regardless of domains. In this paper we shall prove

Theorem 1.4. $1 \leq \liminf_{p \rightarrow \infty} \|u_p\|_{L^\infty(\Omega)} \leq \limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\Omega)} \leq \sqrt{e}$.

Theorem 1.5. *For any smooth bounded domain Ω and a sequence v_{p_n} of v_p with $p_n \rightarrow \infty$ there exists a subsequence of v_{p_n} such that the blow-up set S of that subsequence contains only one point in Ω ; in other words, the subsequence must blow up and it blows up at one point.*

Theorem 1.6. *The conclusions in Theorem 1.3 are true without condition (T).*

From the above results, we see that when p gets large, the least-energy solutions u_p look more and more like a single spike.

If we consider a convex domain Ω , then, using a result of L. Caffarelli and A. Friedman (Theorem 3.1 of [3]) which implies that ϕ is strictly convex, ϕ has strictly positive Hessian and the only critical point of ϕ is the global minimum, we have

Corollary 1.7. *For convex domain Ω the conclusions of Theorem 1.5 and Theorem 1.6 are true for the whole families $\{u_p\}$, $\{v_p\}$ and $\{f_p\}$ as $p \rightarrow \infty$, and x_0 is the unique critical point of ϕ .*

Remark 1.8. It was shown by C.-S. Lin [7] that for convex domain Ω the solution u_p of (1.1) minimizing (1.2) is unique.

Some techniques in this paper have been extended to the N -dimensional case where the Laplacian operator is replaced by the N -Laplacian operator. See [9] for details. The mixed boundary version of the two-dimensional problem is also considered by the authors in [10]. In this paper we shall prove Theorem 1.4 in section 2 and Theorem 1.5 in section 3. We omit the proof of Theorem 1.6 since it is identical to the proof of Theorem 1.3 [8].

2. PROOF OF THEOREM 1.4

Define

$$(2.1) \quad c_p := \inf \left\{ \int_{\Omega} |\nabla u|^2 \right\}^{1/2} : u \in W_0^{1,2}(\Omega), \|u\|_{p+1} = 1.$$

We collect some results from [8] that will lead to the asymptotic behavior of c_p .

Lemma 2.1. *For every $t \geq 2$ there is D_t such that*

$$\|u\|_{L^t} \leq D_t t^{1/2} \|\nabla u\|_{L^2}$$

for all $u \in W_0^{1,2}(\Omega)$ where Ω is a bounded domain in \mathbb{R}^2 ; furthermore

$$\lim_{t \rightarrow \infty} D_t = (8\pi e)^{-1/2}.$$

Lemma 2.2.

$$\lim_{p \rightarrow \infty} \frac{c_p}{p^{-1/2}} = (8\pi e)^{1/2}.$$

Corollary 2.3.

$$\lim_{p \rightarrow \infty} p \int_{\Omega} u_p^{p+1} = 8\pi e, \quad \lim_{p \rightarrow \infty} p \int_{\Omega} |\nabla u_p|^2 = 8\pi e.$$

We now define an important quantity

$$(2.2) \quad L_0 = \overline{\lim}_{p \rightarrow \infty} \frac{p\nu_p}{2\sqrt{e}}$$

where

$$(2.3) \quad \nu_p = \int_{\Omega} u_p^p.$$

Notice that the quantity L_0 defined here is different from the one in [8]. Indeed this difference will lead to Theorem 1.5.

We have the following estimate for L_0 .

Corollary 2.4. *For any smooth bounded domain Ω of R^2 , $L_0 \leq 4\pi\sqrt{e}$.*

The proofs of these facts can be found in section 2 of [8].

Now let us start the proof of Theorem 1.4. A uniform lower bound actually exists for any positive solutions to (1.1). Let λ be the first eigenvalue of $-\Delta$ and φ be a corresponding positive eigenfunction. Then if u is any solution to (1.2) with the exponent equal to p , we have

$$0 = \int_{\Omega} u\Delta\varphi - \varphi\Delta u = -\lambda \int_{\Omega} u\varphi + \int_{\Omega} u^p\varphi.$$

That is, $\int_{\Omega} (u^p - \lambda u)\varphi = 0$. Hence $\|u\|_{L^\infty(\Omega)} \geq \lambda^{1/(p-1)} \rightarrow 1$ as $p \rightarrow \infty$ which yields a lower bound for large p .

To get an upper bound for $\{u_p\}$, we use an iteration argument. Fix positive α and ϵ that will be chosen small later. Letting $\nu = (1 + \alpha)(p + 1)$, from Lemma 2.1, we have

$$\left[\int_{\Omega} u_p^\nu \right]^{1/\nu} \leq (8\pi e)^{-1/2} E_{(1+\alpha)(p+1)} \nu^{1/2} \|\nabla u_p\|_{L^2(\Omega)}$$

where $\lim_{p \rightarrow \infty} E_{(1+\alpha)(p+1)} = 1$. But from Corollary 2.3, we know

$$\lim_{p \rightarrow \infty} p \int_{\Omega} |\nabla u_p|^2 = 8\pi e.$$

Hence there is P_0 such that for all $p > P_0$,

$$(2.4) \quad \left(\int_{\Omega} u_p^\nu \right)^{2/\nu} \leq 1 + \alpha + \epsilon, \text{ or } \int_{\Omega} u_p^\nu \leq (1 + \alpha + \epsilon)^{\nu/2}.$$

We henceforth only consider $p > P_0$.

Multiplying both sides of (1.1) by u_p^{2s-1} , we get, after integrating by parts,

$$(2.5) \quad \frac{2s-1}{s^2} \int_{\Omega} |\nabla u_p^s|^2 = \int_{\Omega} u_p^{p-1+2s}.$$

Using Lemma 2.1 again, we deduce

$$\left[\int_{\Omega} u_p^{\nu s} \right]^{1/\nu} \leq D_{\nu s} \nu^{1/2} \|\nabla u_p^s\|_{L^2(\Omega)};$$

$$\left[\int_{\Omega} u_p^{\nu s} \right]^{2/\nu} \leq C_0 \nu \frac{s^2}{2s-1} \int_{\Omega} u_p^{p-1+2s} \leq C_1 \nu s \int_{\Omega} u_p^{p-1+2s}$$

where $D_{\nu s}$ is defined in Lemma 2.1 and C_0 and C_1 are constants independent of $p > P_0$. Hence we have

$$(2.6) \quad \left[\int_{\Omega} u_p^{\nu s} \right]^{2/\nu} \leq C_1 \nu s \int_{\Omega} u_p^{p-1+2s}.$$

We now define two sequences $\{s_j\}$ and $\{M_j\}$ by

$$(2.7) \quad \begin{aligned} p-1+2s_0 &= \nu, & p-1+2s_{j+1} &= \nu s_j; \\ M_0 &= (1+\alpha+\epsilon)^{\nu/2}, & M_{j+1} &= [C_1 \nu s_j M_j]^{\nu/2} \end{aligned}$$

where C_1 is the constant in (2.6). From (2.4) and (2.6), we have

$$(2.8) \quad \int_{\Omega} u_p^{\nu s_{j-1}} \leq M_j.$$

Next we claim

$$(2.9) \quad M_j \leq \exp[m(\alpha, p, \epsilon)\nu s_{j-1}]$$

where $m(\alpha, p, \epsilon)$ is a constant depending on α, p and

$$\lim_{p \rightarrow \infty} m(\alpha, p, \epsilon) = \frac{1 + \alpha}{2\alpha} \log(1 + \alpha + \epsilon).$$

In fact, we can write down $\{s_j\}$ explicitly as

$$(2.10) \quad s_j = \frac{1}{\nu - 2} \left\{ \left(\frac{\nu}{2}\right)^{j+1} (\nu - p - 1) + p - 1 \right\}.$$

Put

$$\sigma_j = \frac{\nu}{2} \log(C_1 \nu s_j), \quad \mu_j = \log M_j.$$

Hence

$$\mu_{j+1} = \frac{\mu_j}{2} + \sigma_j.$$

Therefore

$$\begin{aligned} \sigma_j &= \frac{\nu}{2} \left\{ \log\left[\frac{C_1 \nu}{\nu - 2}\right] + \log\left[\left(\frac{\nu}{2}\right)^{j+1} (\nu - p - 1) + p - 1 \right] \right\} \\ &\leq [\nu \log \sqrt{2C_1 \nu}](j + 1). \end{aligned}$$

Now we define $\{\tau_j\}$ by

$$(2.11) \quad \tau_0 = \mu_0, \quad \tau_{j+1} = \frac{1}{2}\nu\tau_j + (\nu \log \sqrt{2C_1 \nu})(j + 1).$$

Clearly $\mu_j \leq \tau_j$. Moreover we have

$$\begin{aligned} \tau_j &= \left(\frac{\nu}{2}\right)^j \left[\mu_0 + 2\nu \log(\sqrt{2C_1 \nu}) \frac{\nu}{(\nu - 2)^2} \right] - \frac{2}{\nu - 2} [\nu \log(\sqrt{2C_1 \nu})(j + \frac{\nu}{\nu - 2})] \\ &\leq \frac{\mu_0 + 2\nu \log(\sqrt{2C_1 \nu}) \frac{\nu}{(\nu - 2)^2}}{(\nu - 2)^{-1}(\nu - p - 1)} s_{j-1} \\ &\leq \frac{\mu_0 + 2\nu \log(\sqrt{2C_1 \nu}) \frac{\nu}{(\nu - 1)^2}}{\nu - p - 1} \frac{\nu - 2}{\nu} \nu s_{j-1} := m(\alpha, p, \epsilon) \nu s_{j-1} \end{aligned}$$

where

$$\lim_{p \rightarrow \infty} m(\alpha, p, \epsilon) = \frac{1 + \alpha}{2\alpha} \log(1 + \alpha + \epsilon).$$

Remember $\nu = (1 + \alpha)(p + 1)$. This proves (2.9). Therefore we get

$$\|u_p\|_{L^{\nu s_{j-1}}(\Omega)} \leq \exp[m(\alpha, p, \epsilon)].$$

Sending $j \rightarrow \infty$ and then $p \rightarrow \infty$, we deduce

$$\limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty} \leq (1 + \alpha + \epsilon)^{\frac{1 + \alpha}{2\alpha}}.$$

Sending $\epsilon, \alpha \rightarrow 0$, we deduce

$$\limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty} \leq \sqrt{e}.$$

We include a consequence of Theorem 1.4 here which will be used later.

Corollary 2.5. *There exist C_1 and C_2 such that*

$$\frac{C_1}{p} \leq \int_{\Omega} u_p^p \leq \frac{C_2}{p}.$$

Proof. The first inequality follows from Theorem 1.4 and the first limit of Corollary 2.3; the second inequality follows from the first limit of Corollary 2.3 through an interpolation argument. \square

3. PROOF OF THEOREM 1.5

The proof is similar to the proof of Theorem 1.2 in [8]. The major difference appears when we reach (3.7). We first state a boundary estimate lemma. The proof of the lemma is standard. One combines the moving plane method in [5] with a Kelvin transform. We refer to [4] and [5] for details.

Lemma 3.1. *Let u be a solution of*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ u|_{\partial\Omega} = 0, \quad u > 0 & \text{in } \Omega \end{cases}$$

where Ω is bounded, smooth and f is a smooth function. Then there exist a neighborhood ω of $\partial\Omega$ and a constant C both depending on the geometry of Ω only such that

$$\|u\|_{L^\infty(\omega)} \leq C\|u\|_{L^1(\Omega)}.$$

Applying this lemma to

$$v_p = \frac{u_p}{\int_{\Omega} u_p^p}$$

we have the following uniform boundary estimate. In particular, it implies that $\{v_p\}$ doesn't blow up on the boundary of Ω ; hence by Corollary 2.5 $\{u_p\}$ has no peak on the boundary of Ω .

Lemma 3.2. *There exist a constant C and a neighborhood ω of $\partial\Omega$ both depending on the geometry of Ω only such that $v_p \leq C$ in ω .*

Proof. Because $\|\Delta v_p\|_{L^1} = 1$, combining the elliptic L^p estimate with the duality argument (see [2] for details), we have that v_p is bounded uniformly in $W^{1,q}(\Omega)$ for $1 \leq q < 2$; hence v_p is uniformly bounded in L^1 . Then using Lemma 3.1, we obtain the desired result. \square

Lemma 3.3. *Let S be the blow-up set defined in (1.4) of a subsequence v_n of v_p . Then S is nonempty and there is a small neighborhood ω of $\partial\Omega$ which depends on the geometry of Ω only such that $S \cap \omega = \emptyset$.*

Proof. The second assertion follows immediately from Lemma 3.2. For the first one, observe

$$\max v_n(x) \geq \frac{C}{v_{p_n}} \rightarrow \infty$$

from Theorem 1.4, Corollary 2.5 and the definition of v_n , (1.3). \square

We quote an interesting L^1 estimate from [1].

Lemma 3.4. *Let u be a solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 . We have for $0 < \epsilon < 4\pi$

$$\int_{\Omega} \exp\left[\frac{(4\pi - \epsilon)|u(x)|}{\|f\|_{L^1}}\right] dx \leq \frac{4\pi^2}{\epsilon} (\text{diam } \Omega)^2.$$

Now recall

$$L_0 = \overline{\lim}_{p \rightarrow \infty} \frac{p\nu_p}{2\sqrt{e}} \leq 4\pi\sqrt{e}$$

where $\nu_p = \int_{\Omega} u_p^p$.

We denote any sequence u_{p_n} of u_p by u_n . Because u_n has property

$$\int_{\Omega} \frac{u_n^{p_n}}{\int_{\Omega} u_n^{p_n}} = 1,$$

we can extract a subsequence of u_n , still denoted by u_n , so that there is a positive bounded measure μ in $M(\Omega)$, the set of all real bounded Borel measures on Ω , such that

$$(3.1) \quad \int_{\Omega} f_n \varphi \rightarrow \int_{\Omega} \varphi d\mu$$

for all $\varphi \in C_0(\Omega)$ where

$$f_n = \nu_{p_n}^{p_n-1} u_n^{p_n} \text{ and } v_n = \frac{u_n}{\nu_{p_n}}.$$

For any $\delta > 0$, we call x_0 a δ -regular point if there is a function φ in $C_0(\Omega)$, $0 \leq \varphi \leq 1$, with $\varphi = 1$ in a neighborhood of x_0 such that

$$(3.2) \quad \int_{\Omega} \varphi d\mu < \frac{4\pi}{L_0 + 2\delta}.$$

We define $\Sigma(\delta) = \{x_0 \in \Omega : x_0 \text{ is not a } \delta\text{-regular point}\}$. Clearly

$$(3.3) \quad \mu(x_0) \geq \frac{4\pi}{L_0 + 2\delta}$$

for all $x_0 \in \Sigma(\delta)$.

Our next lemma plays a central role in the proof of Theorem 1.5. It says that smallness of μ at a point x_0 implies boundedness of v_n near x_0 .

Lemma 3.5. *If x_0 is a δ -regular point, then $\{v_n\}$ is bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0 > 0$.*

Proof. Let x_0 be a δ -regular point. From the definition of δ -regular points, there exists $R_1 > 0$ such that

$$\int_{B_{R_1}(x_0)} f_n < \frac{4\pi}{L_0 + \delta}.$$

Split v_n into two parts

$$v_n = v_{1n} + v_{2n}$$

where v_{1n} is the solution of

$$(3.4) \quad \begin{cases} \Delta v_{1n} + f_n = 0 \text{ in } B_{R_1}(x_0), \\ v_{1n}|_{\partial B_{R_1}(x_0)} = 0 \end{cases}$$

and v_{2n} solves

$$(3.5) \quad \begin{cases} \Delta v_{2n} = 0 \text{ in } B_{R_1}(x_0), \\ v_{2n}|_{\partial B_{R_1}(x_0)} = v_n|_{\partial B_{R_1}(x_0)}. \end{cases}$$

From the maximum principle, $v_{1n}, v_{2n} > 0$. By the mean value theorem for harmonic functions, we have

$$\|v_{2n}\|_{L^\infty(B_{R_1/2})} \leq C\|v_{2n}\|_{L^1(B_{R_1})} \leq C\|v_n\|_{L^1(\Omega)} \leq C$$

where the last inequality follows as in the proof of Lemma 3.2. So we need only to consider v_{1n} .

We first claim that when n is large enough

$$(3.6) \quad f_n(x) \leq \exp(L_0 + \delta/2)v_n(x)$$

for all $x \in \Omega$. Observe that

$$\limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\omega)} \leq \sqrt{e}, \quad \lim_{p \rightarrow \infty} \left(\int_{\Omega} u_p \right)^{1/p} = 1$$

from Theorem 1.4 and Corollary 2.5. Therefore

$$\limsup_{p \rightarrow \infty} \frac{\|u_p\|_{L^\infty(\omega)}}{\left(\int_{\Omega} u_p \right)^{1/p}} \leq \sqrt{e} < e.$$

Let

$$(3.7) \quad \alpha_n = \frac{\|u_{p_n}\|_{L^\infty(\omega)}}{\left(\int_{\Omega} u_{p_n} \right)^{1/p_n}}.$$

Then

$$\limsup_{n \rightarrow \infty} \alpha_n < e.$$

Consider the function $\log x/x$ which is increasing if $x < e$. Since for every $x \in \Omega$

$$\frac{u_n(x)}{\nu_n^{1/p_n}} \leq \alpha_n,$$

fixing small ϵ , we have for large n ,

$$\frac{\log \frac{u_{p_n}(x)}{\nu_n^{1/p_n}}}{\frac{u_{p_n}(x)}{\nu_n^{1/p_n}}} \leq \frac{\log \alpha_n}{\alpha_n} \leq \frac{1}{2\sqrt{e}} + \epsilon.$$

Therefore

$$\begin{aligned} f_n(x) &\leq \exp\left[\frac{p_n u_{p_n}}{\nu_{p_n}^{1/p_n}} \left(\frac{1}{2\sqrt{e}} + \epsilon\right)\right] = \exp\left[\left(\frac{1}{2\sqrt{e}} + \epsilon\right) p_n \nu_{p_n}^{1 - \frac{1}{p_n}} v_n\right] \\ &\leq \exp\left[\left(\frac{1}{2\sqrt{e}} + 2\epsilon\right) (\limsup_{n \rightarrow \infty} p_n \nu_{p_n}) v_n\right] = \exp[(L_0 + 4\sqrt{e}L_0\epsilon)v_n] \end{aligned}$$

for large n . If we choose ϵ small enough, we have

$$f_n(x) \leq \exp\left[\left(L_0 + \frac{\delta}{2}\right)v_n(x)\right]$$

for large n .

Next we claim that $\{f_n\}$ is uniformly bounded in $L^{1+\delta_0}(B_{R_1}/2)$ for δ_0 sufficiently small. Because $\{v_{2n}\}$ is uniformly bounded in $B_{R_1/2}(x_0)$, we see from the previous claim

$$\begin{aligned} \int_{B_{R_1}/2} f_n^{1+\delta_0} &\leq \int_{B_{R_1}/2} \exp[(1 + \delta_0)(L_0 + 0.5\delta)v_n] \\ &\leq C \int_{B_{R_1}/2} \exp[(1 + \delta_0)(L_0 + 0.5\delta)v_{1n}] \\ &\leq C \int_{B_{R_1}/2} \exp \frac{4\pi(1 + \delta_0) \frac{L_0 + 0.5\delta}{L_0 + \delta} v_{1n}}{\int_{B_{R_1/2}(x_0)} f_n} \leq C' \end{aligned}$$

with the aid of Lemma 3.3 if we choose δ_0 sufficiently small. So we have proved the claim.

Now take $B_{R_1/4}(x_0)$. We conclude from the weak Harnack inequality (Theorem 8.17, [6]),

$$\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \leq C[\|v_n\|_{L^2(B_{R_1/2}(x_0))} + \|f_n\|_{L^{1+\delta_0}(B_{R_1/2}(x_0))}] \leq C.$$

Here the boundedness of $\{v_n\}$ in $L^2(B_{R_1/2}(x_0))$ follows again from Lemma 3.3. \square

We now start to prove Theorem 1.5. We first claim $S = \Sigma(\delta)$ for any $\delta > 0$. Clearly $S \subset \Sigma$. In fact, let $x_0 \notin \Sigma$; then x_0 is a δ -regular point. Hence by Lemma 3.5, $\{v_n\}$ is bounded in $L^\infty(B_R(x_0))$ for some R , i.e., $x_0 \notin S$. Conversely suppose $x_0 \in S$. Then we have for every $R > 0$, passing to a subsequence of $\{v_n\}$ if necessary,

$$(3.8) \quad \lim_{n \rightarrow \infty} \|v_n\|_{L^\infty(B_R(x_0))} = \infty.$$

Otherwise there would be some $R_0 > 0$ such that $\|v_n\|_{L^\infty(B_{R_0}(x_0))} < C$ for some C independent of n . Then

$$f_n = \nu_n^{p_n-1} v_n^{p_n} \leq \left(\frac{M}{p_n}\right)^{p_n-1} C^{p_n} \rightarrow 0$$

uniformly as $n \rightarrow \infty$ on $B_{R_0}(x_0)$. Then

$$\int_{B_{R_0}(x_0)} f_n = \int_{B_{R_0}(x_0)} \nu_n^{p_n-1} v_n^{p_n} \leq \epsilon_0 < \frac{4\pi}{L_0 + 2\delta},$$

which implies that x_0 is a δ -regular point, i.e., $x_0 \notin \Sigma$. This contradiction implies (3.8); hence by definition (1.4) of S , $x_0 \in S$. This completes the proof of our claim.

Next we go back to measure μ defined in (3.1). Clearly by (3.3)

$$1 \geq \mu(\Omega) \geq \frac{4\pi}{L_0 + 2\delta} \#\Sigma(\delta) = \frac{4\pi}{L_0 + 2\delta} \#S.$$

Hence combining this with Lemma 3.4

$$1 \leq \#S \leq \frac{L_0 + 2\delta}{4\pi}.$$

Applying Corollary 2.4, we finally conclude that, by choosing δ small, $\#S = 1$.

Remark 3.6. The fact that the peak set of $\{u_n\}$ is included in the blow-up set S of $\{v_n\}$ follows easily from the fact that $\nu_n \rightarrow 0$ as $n \rightarrow \infty$. Because the blow-up set contains only one point and the peak set is nonempty from Theorem 1.1, we conclude that the peak set is identical with the blow-up set.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

Current address, X. Ren: Institute for Mathematics & Applications, University of Minnesota, Minneapolis, Minnesota 55455

E-mail address, X. Ren: ren@ima.umn.edu

Current address, J. Wei: Department of Mathematics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong