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A CARTAN THEOREM FOR BANACH ALGEBRAS

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ABSTRACT. Let A be a semisimple Banach algebra, and let Ω_A be its spectral unit ball. We show that every holomorphic map $G\colon \Omega_A \to \Omega_A$ satisfying $G(0) = 0$ and $G'(0) = I$ fixes those elements of Ω_A which belong to the centre of A, but not necessarily any others. Using this, we deduce that the automorphisms of Ω_A all leave the centre invariant. As a further application, we give a new proof of Nagasawa's generalization of the Banach-Stone theorem.

1. STATEMENT OF RESULTS

Given a (complex, unital) Banach algebra A, its *spectral unit ball* is the set

$$\Omega_A := \{a \in A\colon \rho(a) < 1\},$$

where $\rho(a)$ denotes the spectral radius of a. It is always an open subset of A, though, unlike its norm counterpart, it need be neither convex nor bounded. It arises, for example, in the spectral Nevanlinna-Pick problem (see e.g. [3]), which is of importance in control theory.

A major problem is to identify the holomorphic automorphisms of Ω_A, these being the natural analogue of the Möbius automorphisms of the unit disc. In [5], M. C. White and the author studied these automorphisms in the important special case when $A = \mathcal{M}_n$, the algebra of $n \times n$ matrices. The main result was that, in contrast with the situation for the unit disc, the automorphism group is *not* transitive on $\Omega_{\mathcal{M}_n}$ if $n \geq 2$. The proof of this was rather intricate and depended on the special properties of matrices. In this paper we shall give a much simpler proof, which at the same time extends to more general Banach algebras. We shall also see that there is a link with a long-standing open problem in Banach-algebra theory.

Our principal tool is the following analogue of Cartan's theorem. We write I for the identity map, and Z_A for the centre of a Banach algebra A:

$$Z_A := \{c \in A\colon ac = ca \text{ for all } a \in A\}.$$

Theorem 1. *Let A be a semisimple Banach algebra.*

(a) *If $G\colon \Omega_A \to \Omega_A$ is a holomorphic map satisfying $G(0) = 0$ and $G'(0) = I$, then $G(c) = c$ for all $c \in \Omega_A \cap Z_A$.*

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(b) *Given $a \in \Omega_A \backslash Z_A$, there exists a holomorphic map $G\colon \Omega_A \to \Omega_A$ satisfying $G(0) = 0$ and $G'(0) = I$ such that $G(a) \neq a$.*

The reason why the classical Cartan theorem does not apply to give $G \equiv I$ on the whole of Ω_A is that, as remarked earlier, Ω_A may be unbounded. In fact, using [1, Corollary 5.2.3] one can easily show that Ω_A is bounded if and only if A is isomorphic to a uniform algebra.

Using Theorem 1, we deduce our result on non-transitivity.

Theorem 2. *Let A and B be semisimple Banach algebras. If $F\colon \Omega_A \to \Omega_B$ is a biholomorphic map, then $F(\Omega_A \cap Z_A) = \Omega_B \cap Z_B$.*

Thus, in particular, if $A = \mathcal{M}_n$ and F is an automorphism of $\Omega_{\mathcal{M}_n}$, then $F(0)$ is necessarily a multiple of the identity, thereby re-proving the main result of [5].

Returning to Theorem 1, even though we cannot conclude that G fixes points of $\Omega_A \backslash Z_A$, it is still possible to say something about its action there. This is a consequence of the following version of Schwarz's lemma. We write $\sigma(a)$ for the spectrum of a, and Γ_r for the circle in $\mathbb{C}$ with centre 0 and radius r.

Theorem 3. *Let A and B be Banach algebras, and let $F\colon \Omega_A \to \Omega_B$ be a holomorphic map such that $F(0) = 0$. Then for each $a \in \Omega_A$,*

$$\rho(F(a)) \leq \rho(a) \quad \textit{and} \quad \rho(F'(0)a) \leq \rho(a),$$

and

$$\sigma(F(a)) \cap \Gamma_{\rho(a)} = \sigma(F'(0)a) \cap \Gamma_{\rho(a)}.$$

Thus, under the hypotheses of Theorem 1, we may conclude that $G(a)$ always has the same peripheral spectrum as a, even if $a \in \Omega_A \backslash Z_A$. One is led naturally to ask if the same might be true of the whole spectrum.

Question 1. *Let A be a semisimple Banach algebra and let $G\colon \Omega_A \to \Omega_A$ be a holomorphic map satisfying $G(0) = 0$ and $G'(0) = I$. Does it follow that $\sigma(G(a)) = \sigma(a)$ for all $a \in \Omega_A$?*

If A is the algebra of $n \times n$ matrices, then the answer is yes: this was proved in [5, Theorem 3]. Even if the answer is no in general, perhaps it is still at least true that $\sigma(G(a))$ and $\sigma(a)$ have the same polynomial hull.

Combining the Schwarz lemma with the earlier Cartan theorem leads to our final result. We write e for the identity in a Banach algebra.

Theorem 4. *Let A and B be commutative semisimple Banach algebras, and let $F\colon \Omega_A \to \Omega_B$ be a biholomorphic map.*

(a) *If $F(0) = 0$, then F extends to a linear isomorphism $F\colon A \to B$.*
(b) *If also $F(e) = e$, then $F\colon A \to B$ is an algebra isomorphism.*

Part (b) of this result is originally due to M. Nagasawa [4] (see also [1, §4.1], which explains the connection with the Banach-Stone theorem). The proofs in both these references depend on convexity theory, whereas the complex-variable proof of Theorem 4 given below is quite different.

Once again, one is led to ask if this result has a non-commutative version. The most natural formulation seems to be the following, which has been an open question for about 25 years.

Question 2. *Let A and B be semisimple Banach algebras, and let $F\colon A \to B$ be a linear isomorphism such that $F(e) = e$ and $\sigma(F(a)) = \sigma(a)$ for all $a \in A$. Does it follow that F is a Jordan isomorphism, i.e. $F(a^2) = F(a)^2$ for all $a \in A$?*

The answer is known to be yes in a number of special cases (see [2], which also contains a brief history of the problem). It is possible that Question 2 might prove more tractable if we had a positive answer to Question 1.

2. Proofs

Proof of Theorem 1. (a) Fix $c \in \Omega_A \cap Z_A$, and consider the function $\lambda \mapsto G(\lambda c)$, which is holomorphic on $\{\lambda \in \mathbb{C}\colon |\lambda| < 1/\rho(c)\}$. Since $G(0) = 0$ and $G'(0) = I$, this function has a Taylor expansion about 0 of the form

$$G(\lambda c) = \lambda c + \sum_{j=2}^{\infty} \lambda^j a_j \qquad (|\lambda| < 1/\rho(c)), \tag{1}$$

where the a_j are elements of A. Our aim is to show that $a_j = 0$ for all j. Suppose, for a contradiction, that this is not the case, and let k be the smallest integer such that $a_k \neq 0$. Take $q \in A$ with $\rho(q) = 0$, and let $n \geq 1$. Then, writing G^n for the n-fold composition $G \circ \cdots \circ G$, we have

$$G^n(\lambda c + \lambda^k nq) = \lambda c + \lambda^k n(a_k + q) + O(\lambda^{k+1}) \quad \text{as } \lambda \to 0. \tag{2}$$

Now as c and q commute, it follows that $\rho(\lambda c + \lambda^k nq) \leq \rho(\lambda c) + \rho(\lambda^k nq) = |\lambda|\rho(c)$ (see e.g. [1, Corollary 3.2.10]), and so we can define a holomorphic function $g\colon \{0 < |\lambda| < 1/\rho(c)\} \to A$ by

$$g(\lambda) = \frac{G^n(\lambda c + \lambda^k nq) - \lambda c}{n\lambda^k} \qquad (0 < |\lambda| < 1/\rho(c)).$$

From (2) we see that the isolated singularity at $\lambda = 0$ can be removed by setting $g(0) = a_k + q$. By Vesentini's theorem (see e.g. [1, Theorem 3.4.7]), the composition $\rho \circ g$ is a subharmonic function on $\{|\lambda| < 1/\rho(c)\}$, and so by the maximum principle

$$\rho(g(0)) \leq \max_{|\lambda|=1} \rho(g(\lambda)).$$

Making use of [1, Corollary 3.2.10] again to estimate the right-hand side, it follows that

$$\rho(a_k + q) \leq 2/n.$$

As this is true for each n, we can let $n \to \infty$ deduce that $\rho(a_k + q) = 0$. And as this holds for each $q \in A$ with $\rho(q) = 0$, Zemánek's characterization of the radical [1, Theorem 5.3.1] implies that a_k belongs to the radical of A, which is zero since A is semisimple. Thus $a_k = 0$, and we have arrived at a contradiction. We conclude that indeed $a_j = 0$ for all $j \geq 2$, and hence from (1) that $G(c) = c$.

(b) Let $a \in \Omega_A \backslash Z_A$. Then there exists $u \in A$ such that $au \neq ua$. We can suppose that $\|u\| < 1$. Then $v := \log(e - u)$ satisfies $e^{-v}ae^v \neq a$. Choose a continuous linear functional ϕ on A such that $\phi(a) = 1$, and define $G\colon \Omega_A \to \Omega_A$ by

$$G(x) = e^{-\phi(x)v} x e^{\phi(x)v} \qquad (x \in \Omega_A).$$

Then G is holomorphic, $G(0) = 0$ and $G'(0) = I$, but $G(a) = e^{-v}ae^v \neq a$. $\square$

Proof of Theorem 2. Fix $c \in \Omega_A \cap Z_A$, and suppose also, for the moment, that $c \neq 0$. Then $F(c) \neq F(0)$, so there exists a continuous linear functional ψ on B such that $\psi(F(c) - F(0)) = 1$. Take $b \in B$, and define $C\colon \Omega_B \to \Omega_B$ by

$$C(y) = e^{-\psi(y-F(0))^2 b} y e^{\psi(y-F(0))^2 b} \qquad (y \in \Omega_B).$$

Then C is holomorphic, $C(F(0)) = F(0)$ and $C'(F(0)) = I$. Hence if we set $G = F^{-1} \circ C \circ F$, then G is a holomorphic self-map of Ω_A satisfying $G(0) = 0$ and $G'(0) = I$. By Theorem 1, it follows that $G(c) = c$, so that $C(F(c)) = F(c)$ or, in other words $e^{-b}F(c)e^b = F(c)$. This holds for every $b \in B$, so as in the proof of part (b) of Theorem 1, we deduce that $F(c) \in Z_B$. This has been proved under the assumption that $c \neq 0$, but by continuity it remains true if $c = 0$. Thus $F(\Omega_A \cap Z_A) \subset \Omega_B \cap Z_B$, and the reverse inclusion follows by applying the same argument to F^{-1}. □

Proof of Theorem 3. Fix $a \in \Omega_A$, and define $f\colon \{0 < |\lambda| < 1/\rho(a)\} \to B$ by

$$f(\lambda) = F(\lambda a)/\lambda \quad (0 < |\lambda| < 1/\rho(a)).$$

Then f is holomorphic, and since $F(0) = 0$ we can remove the isolated singularity at $\lambda = 0$ by setting $f(0) = F'(0)a$. Now using Vesentini's theorem once again, the function $\rho \circ f$ is subharmonic on $\{|\lambda| < 1/\rho(a)\}$, and so for each $r < 1/\rho(a)$ it follows from the maximum principle that

$$\rho(f(\lambda)) \leq \max_{|\mu|=r} \rho(f(\mu)) \leq 1/r \qquad (|\lambda| \leq r).$$

Letting $r \to 1/\rho(a)$, we obtain

$$\rho(f(\lambda)) \leq \rho(a) \qquad (|\lambda| < 1/\rho(a)).$$

In particular, taking $\lambda = 1$ and $\lambda = 0$, we deduce that $\rho(F(a)) \leq \rho(a)$ and $\rho(F'(0)a) \leq \rho(a)$, which proves the two inequalities in the theorem. (This half of the proof was essentially the same as in [5, Theorem 2].)

Now fix $\zeta \in \mathbb{C}$ with $|\zeta| = \rho(a)$. Then we have $\rho(f(\lambda) + \zeta e) \leq 2\rho(a)$ for all λ, with equality if and only if $\zeta \in \sigma(f(\lambda))$. However, by Vesentini's theorem once more, the function $\lambda \mapsto \rho(f(\lambda) + \zeta e)$ is subharmonic, so if it attains a maximum, then it must be constant. Therefore if $\zeta \in \sigma(f(\lambda))$ for *one* value of λ, then the same holds for *all* λ. The conclusion is that the set $\sigma(f(\lambda)) \cap \Gamma_{\rho(a)}$ is independent of λ. In particular, taking $\lambda = 1$ and $\lambda = 0$ we obtain $\sigma(F(a)) \cap \Gamma_{\rho(a)} = \sigma(F'(0)a) \cap \Gamma_{\rho(a)}$, which proves the final statement of the theorem. □

Proof of Theorem 4. (a) By the second inequality of Theorem 3, applied both to F and F^{-1}, we see that $F'(0)|_{\Omega_A}$ is a biholomorphic map of Ω_A onto Ω_B. Hence if we set $G = F^{-1} \circ F'(0)|_{\Omega_A}$, then G is a holomorphic self-map of Ω_A satisfying $G(0) = 0$ and $G'(0) = I$. From Theorem 1, together with the fact that A is commutative, it follows that $G(a) = a$ for all $a \in \Omega_A$, and so $F \equiv F'(0)$ on Ω_A. Thus $F'(0)$ is a linear isomorphism of A onto B which extends F.

(b) Assume now that F has been so extended, and that $F(e) = e$. For $t \in (-1, 1)$, define $M_t\colon \Omega_A \to \Omega_A$ and $N_t\colon \Omega_B \to \Omega_B$ by

$$M_t(x) = (x - te)(e - tx)^{-1} \qquad (x \in \Omega_A),$$
$$N_t(y) = (y - te)(e - ty)^{-1} \qquad (y \in \Omega_B).$$

Then M_t is biholomorphic, with $M_t(0) = -te$ and $M_t'(0) = (1 - t^2)I$, and similarly for N_t. Also, since F is linear and maps e to e, we have $F(-te) = -te$. Hence if we define $G = F^{-1} \circ N_t^{-1} \circ F \circ M_t$, then G is a holomorphic self-map of Ω_A satisfying $G(0) = 0$ and $G'(0) = I$. Applying Theorem 1 once more, we deduce that $G(a) = a$ for all $a \in \Omega_A$, or in other words that $F(M_t(a)) = N_t(F(a))$ for all $a \in \Omega_A$. Now fix $a \in \Omega_A$, and expand both sides of this last equation in powers of t. This gives

$$F(a) + tF(a^2 - e) + O(t^2) = F(a) + t(F(a)^2 - e) + O(t^2) \quad \text{as } t \to 0,$$

and equating the coefficients of t we obtain $F(a^2) = F(a)^2$. Applying this with $a = \frac{1}{2}(a_1 \pm a_2)$, we deduce that $F(a_1 a_2) = F(a_1)F(a_2)$ for all $a_1, a_2 \in \Omega_A$, and hence, by homogeneity, for all $a_1, a_2 \in A$. This completes the proof. □

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