WEIGHTED INEQUALITIES
FOR THE MAXIMAL GEOMETRIC MEAN OPERATOR

XIANGRONG YIN AND BENJAMIN MUCKENHOUPT

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Abstract. For nonnegative Borel measures \( \mu \) on \( \mathbb{R}^1 \) and for the maximal geometric mean operator \( G_f \), we characterize the weight pairs \((w,v)\) for which \( G_f \) is of weak type \((p,p)\) and of strong type \((p,p)\), \(0 < p < \infty\). No doubling conditions are needed. We also note that a previously published different characterization for the strong type inequality for \( G_f \) has an incorrect proof.

1. Introduction

Let \( \mu \) be a nonnegative Borel measure on \( \mathbb{R}^1 \). The maximal geometric mean operator is (see X. Shi [7])

\[
G_f(x) = \sup \exp \left[ \frac{1}{|I|} \int_{I} \log(|f(y)|) \, d\mu(y) \right],
\]

where the sup is taken over all intervals \( I \) in \( \mathbb{R}^1 \) containing \( x \) such that the integral is defined and \( 0 < |I| = \mu(I) < \infty \); if no such \( I \) exists we take \( G_f(x) = 0 \).

If \( 0 < p < \infty \), \( w = v \) and \( d\mu \) is Lebesgue measure on \( \mathbb{R}^1 \), then the inequality

\[
\int G_f(x)^p w(x) \, d\mu(x) \leq C_1 \int |f(x)|^p v(x) \, d\mu(x)
\]

for all \( f \) in \( L^p(vd\mu) \) is equivalent to the requirement that

\[
\left( \frac{1}{|I|} \int_{I} wd\mu \right) \exp \left( \frac{1}{|I|} \int_{I} \log(1/v) \, d\mu \right) \leq C_2
\]

for all intervals \( I \) (see [7]) and is also equivalent to the weak type inequality

\[
\int_{\{x: G_f(x) > \lambda\}} w \, d\mu \leq \frac{C_3}{\lambda^p} \int |f(x)|^p v(x) \, d\mu(x)
\]

for all \( f \) in \( L^p(vd\mu) \) (see [3]). In [3] it is stated that in spaces of homogeneous type condition (2) implies inequality (1) for general not necessarily equal \( v \) and \( w \) provided that \( \mu \) and \( vd\mu \) satisfy doubling conditions. The proof given in [3], however, is incorrect. They state on page 71 that “it is easy to see that” if \( f_0(x) = f(x) \) on the set where \( |f(x)| \leq \alpha/2 \) and 0 elsewhere and \( f_1(x) = f(x) - f_0(x) \), then \( G_f(x) \leq G_f_1(x) + \alpha/2 \). A simple counterexample is \( f(x) = \chi_{(0,1]}(x) + e^n\chi_{(1,\infty)}(x,x) \) and \( \alpha = 2 \); for \( 0 < x < 1 \) we have \( G_f(x) = e^n \) and \( G_f_1(x) = 0 \). We
will show in Theorem 1 that in general a stronger condition than (2) is needed to imply (1).

It might seem that since condition (2) implies (3) for all $p$, the Marcinkiewicz interpolation theorem would show that (2) implies (1). The Marcinkiewicz theorem cannot be applied here, however, because the operator is not quasilinear. This is easily seen in $\mathbb{R}^1$ with $d\mu = dx$ by taking $E$ to be a set such that $E \cap I$ and $E^c \cap I$ both have positive measure for every interval $I$. Then with $f = \chi_E$ and $g = \chi_{E^c}$ we have $G_f + g$ equal to 1 for all $x$ while $G_f$ and $G_g$ are 0 for all $x$.

2. The main theorems

**Theorem 1.** Let $\mu$ be a nonnegative Borel measure on $\mathbb{R}^1$, $0 < p < \infty$, and let $w$ and $v$ be two $\mu$ measurable weight functions. Then inequality (1) holds for all $f$ in $L^p(v d\mu)$ if and only if

\[
\int_I G_{(w-1\chi_I)}(x)w(x) d\mu(x) \leq C_4 |I|
\]

for every interval $I$.

**Theorem 2.** Let $w, v, \mu$ and $p$ be as in Theorem 1. Then inequality (3) holds for all $f$ in $L^p(v d\mu)$ if and only if (2) holds for every interval $I$.

From the proof of X. Shi [7] we can easily see that (4) is equivalent to (2) if $w = v$. This is also a simple consequence of Theorem 1 and well-known facts by the following reasoning. If $w = v$ satisfies (2), then by Theorem 1, page 254 of [2], $w$ satisfies the condition $A_\infty$. By the Theorem on page 104 of [4] there is a $p > 1$ such that $w$ satisfies $A_p$, and since $G_f(x) \leq Mf(x)$, where $M$ denotes the Hardy-Littlewood maximal operator, we have (1) by Theorem 2, page 216 of [5]. Theorem 1 then shows that $w$ satisfies (4).

Without the assumption $w = v$ conditions (2) and (4) are not equivalent; in section 6 we give an example of a pair of functions $v, w$ that satisfy (2) but do not satisfy (4).

These theorems can be viewed as limiting cases of known results about the Hardy-Littlewood maximal operator $M$. This is based on the equality

\[
\lim_{p \to \infty} \left( \frac{1}{|I|} \int_I v^{-1/p} d\mu \right)^p = \exp \left( \frac{1}{|I|} \int_I \log(1/v) d\mu \right),
\]

which is valid if the left side is finite (see [1], [2]), and the consequence that $G_f(x) = \lim_{p \to \infty} [M(f^{1/p})]^p$ for suitably restricted $f$. Theorem 1 is a limiting case of the fact from Theorem B, p. 7 of [6] that $\int_M [M(f^{1/p})]^p w d\mu \leq C \int f v d\mu$ if and only if for every interval $I \int_M [M(\chi_I v^{-1/(p-1)})]^p w d\mu \leq C \int v^{-1/(p-1)} d\mu$. Theorem 2 is a limiting case of the fact from Theorem 6, p. 219 of [5] that

\[
\int_{\{x\mid M(f^{1/p})^p > \lambda\}} w d\mu \leq \frac{C}{\lambda} \int f v d\mu
\]

if and only if for every interval $I$

\[
\left( \frac{1}{|I|} \int_I w d\mu \right) \left( \frac{1}{|I|} \int_I v^{-1/(p-1)} d\mu \right)^{p-1} \leq C.
\]
3. Covering lemmas

Our proofs of the theorems are based on the following lemmas. Lemma 1 is the covering idea used in the proof of Theorem (a), p. 1231 of [8].

**Lemma 1.** If \( \nu \) is a Borel measure, \( E \) is a subset of \( \mathbb{R}^1 \), \( B \) is a collection of intervals that cover \( E \) and there is a constant \( C \) such that \( 0 < \nu(I) \leq C \) for every \( I \) in \( B \), then there is a disjoint sequence \( \{ I_j \} \) of intervals in \( B \) such that \( \nu(E) \leq \sum 5\nu(I_j) \).

To prove this each \( I_j \) is chosen to be disjoint from \( I_1, \ldots, I_{j-1} \) and with \( \nu(I_j) > (1/2) \sup \nu(I) \), where the sup is taken over all \( I \)'s in \( B \) disjoint from \( I_1, \ldots, I_{j-1} \). If \( \sum \nu(I_j) = \infty \), there is nothing more to prove. Therefore, assume that \( \sum \nu(I_j) < \infty \). For each \( I_j \) define \( I_j^* \) to be the union of all intervals \( H \) in \( B \) that intersect \( I_j \) and have \( \nu(H) < 2\nu(I_j) \). Then \( \nu(I_j^*) \leq 5\nu(I_j) \) is immediate. Furthermore, if \( x \) is in \( E \), let \( J \) be a member of \( B \) that contains \( x \). Since \( \lim_{j \to \infty} \nu(I_j) = 0 \), there is a first member \( I_j \) of the sequence that intersects \( J \). Then by the selection procedure \( \nu(I_j) > (1/2) \nu(J) \) and by its definition \( J \subset I_j^* \). From this \( E \subset \bigcup I_j^* \). Therefore, \( \nu(E) \leq \sum \nu(I_j^*) \leq \sum 5\nu(I_j) \). This completes the proof of Lemma 1.

**Lemma 2.** If \( E \) is a subset of \( \mathbb{R}^1 \) and \( B \) is a collection of closed intervals of positive length that cover \( E \), then \( E \) is covered by a countable subcollection of \( B \).

To prove this let \( A \) be the set of points in \( E \) that occur only as left ends of intervals in \( B \). For every \( x \) in \( A \) there is an interval in \( B \) that contains it, and this interval contains no other points of \( A \) in its interior. Therefore, \( A \) is countable and is consequently covered by a countable subcollection from \( B \). The same reasoning applies to the points that occur only as right end points. The other points of \( E \) are covered by the open interiors of the intervals in \( B \), and there is a countable subcollection that covers them by Lindelöf’s theorem.

4. Proof of Theorem 1

The proof that (1) implies (4) is trivial by taking \( f(x) = v(x)^{-1/p} \chi_I(x) \).

For the proof that (4) implies (1) we need only consider the case \( p = 2 \) since \( G_f(x)^p = G_{f^p}(x) \). It is enough to prove the result for bounded functions \( f \) with support \( E \) having finite \( \mu \) measure and \( \int |f|^2 \,d\mu < \infty \). For these \( f \)'s it is enough to consider \( v \)'s with positive lower bound by the following reasoning. Given arbitrary \( v(x) \), let \( v_n(x) \) be the larger of \( 1/n \) and \( v(x) \). Then \( v_n \) will still satisfy (4) with the same value of \( C_4 \). It will appear in the proof that the constant \( C_4 \) depends only on \( C_4 \). Therefore, (1) will be valid for all \( v_n \)'s with fixed \( C_4 \), and the restriction on \( f \) insures that the right side of (1) converges properly as \( n \to \infty \).

Having fixed such an \( f \) and \( v \), we define a function \( F \) on intervals \( I \) in \( \mathbb{R}^1 \) by

\[
F(I) = \exp \left[ \frac{1}{|I|} \int_I \log(|f(y)|) \,d\mu(y) \right]
\]

for intervals with \( 0 < |I| < \infty \) for which the integral is defined and \( F(I) = 0 \) on other intervals. For each \( x \) for which \( G_f(x) > 0 \) we will now construct a closed interval \( I_x \) containing \( x \) such that \( F(I_x) > G_f(x)/2 \) and so that the end points of \( I_x \) are measurable functions of \( x \).

To do this let \( E_k \) be the set of \( x \) for which \( 2^k < G_f(x) \leq 2^{k+1} \), and let \( C_k \) consist of all closed intervals \( I \) with \( 2^k < F(I) \leq 2^{k+1} \). Let \( A \) be the points \( a \) in \( E_k \) with \( [a, a] \in C_k \). The set \( A \) is countable since if \( a \) is in \( A \), \( [a, a] > 0 \) and \( a \) lies in the
support of $f$ which has been assumed to have finite measure. Therefore, there is a countable subcollection of $C_k$ that covers $A$. If $x \in E_k \cap A^c$ there is an interval $I$ containing $x$ with $F(I) > 2^k$ and $I$ must have positive length since $[x, x]$ is not in $C_k$. If one or both ends of $I$ are open, a closed interval $J$ of positive length can be found inside $I$ that contains $x$ and has $F(J) > 2^k$. Therefore, the intervals of positive length in $C_k$ cover $E_k \cap A^c$, and by Lemma 2 there is a countable subcollection that covers this set. Let $\{J_n\}$ be a sequence of closed intervals in $C_k$ that cover $E_k$. For each $x$ in $E_k$ define $I_x$ to be the first $J_n$ that contains $x$. Then for $x$ in $E_k$ we have $F(I_x) > 2^k \geq G_f(x)/2$ and the ends of $I_x$ are measurable functions because the set where $I_x = J_n$ is the measurable set $J_n \cap E_k \cap \bigcap_{m=1}^{n-1} J_m$.

Then if $B$ is the set where $G_f(x) > 0$,

$$\int_B G_f(x)^2 w(x) d\mu(x) < \int_B 4 \left[ \exp \left( \frac{1}{|I_x|} \int_{I_x} \log(|f|) d\mu \right) \right]^2 w(x) d\mu(x) \quad \text{(5)}$$

$$= 4 \int_B \left[ \exp \left( \frac{1}{|I_x|} \int_{I_x} \log(|f|) d\mu \right) \right]^2 \times \exp \left( \frac{1}{|I_x|} \int_{I_x} \log \left( \frac{1}{v} \right) d\mu \right) w(x) d\mu(x).$$

Some justification is needed for this equality since in general an equality of the form

$$\exp \left[ \int_I \log(g) d\mu \right] = \exp \left[ \int_I \log(gh) d\mu \right] \exp \left[ \int_I \log(1/h) d\mu \right]$$

can fail if the right side has the form $0 \cdot \infty$. This does not happen here because the restrictions on $f$ and $v$ insure that $\int_{I_x} \log(|f|) d\mu$, $\int_{I_x} \log(|f|/2) d\mu$ and $\int_{I_x} \log(1/v) d\mu$ are all finite. By Jensen’s inequality (5) is bounded by

$$4 \int_B \left[ \frac{1}{|I_x|} \int_{I_x} |f|^{1/2} d\mu \right]^2 \exp \left[ \frac{1}{|I_x|} \int_{I_x} \log \left( \frac{1}{v} \right) d\mu \right] w(x) d\mu(x). \quad \text{(6)}$$

Define the measure $U$ by $dU(x) = w(x) \exp \left( \frac{1}{|I_x|} \int_{I_x} \log \left( \frac{1}{v} \right) d\mu \right) d\mu(x)$, and let $T$ be the linear operator defined by $Tg(x) = \frac{1}{|I_x|} \int_{I_x} g d\mu$. Then $T$ is clearly bounded from $L^\infty(d\mu)$ to $L^\infty(dU)$. We shall show that $T$ is bounded from $L^1(d\mu)$ to weak $L^1(dU)$. This will be sufficient since these imply that $T$ is bounded from $L^2(d\mu)$ to $L^2(dU)$ and, therefore, that (6) equals

$$4 \int_B T \left( |f|^{1/2} \right)^2 dU \leq C \int |f|^{1/2}^2 d\mu = C \int |f|^2 v d\mu.$$

To prove the weak type assertion for $T$ fix $g$ in $L^1(d\mu)$ and $\lambda > 0$, let $D = \{Tg(x) > \lambda\}$, and let $E = \bigcup_{x \in D} I_x$. Since the set of all $I_x$’s is countable, $E$ can be written as the countable union of its connected components $\{J_n\}$. Note that if $x$ is in $D \cap J_n$, then $I_x \subset J_n$. We have, therefore,

$$\int_D dU = \sum_n \int_{D \cap J_n} \exp \left[ \frac{1}{|I_x|} \int_{I_x} \log \left( \frac{1}{v} \right) d\mu \right] w(x) d\mu(x)$$

$$\leq \sum_n \int_{J_n} \exp \left[ \frac{1}{|I_x|} \int_{I_x} \log \left( \frac{1}{v} \right) \chi_{J_n} d\mu \right] w(x) d\mu(x).$$
By condition (4) this is bounded by
\[ C_4 \sum_n |J_n|. \]

Now each \( J_n \) is the union of \( I_x \)'s with \( x \) in \( D \). From the fact that \( x \) is in \( D \) we have
\[ 0 < |I_x| < \frac{1}{\lambda} \int_{I_x} g \, d\mu \leq \frac{1}{\lambda} \|g\|_1. \]
We can, therefore, apply Lemma 1 to each \( J_n \). This will produce disjoint sequences of intervals \( I_{n,k} \subset J_n \), and since the \( I_{n,k} \)'s are \( I_x \)'s with \( x \in D \),
\[ \lambda < \frac{1}{|I_{n,k}|} \int_{I_{n,k}} |g| \, d\mu. \]
Furthermore, since the \( J_n \)'s are disjoint, all the intervals \( I_{n,k} \) are disjoint. Using these two facts and Lemma 1 shows that (7) has the bound
\[ 5C_4 \sum_{n,k} |I_{n,k}| < \sum_{n,k} \frac{5}{\lambda} C_4 \int_{I_{n,k}} |g| \, d\mu \leq \frac{5}{\lambda} C_4 \|g\|_1. \]

This completes the proof of Theorem 1.

5. Proof of Theorem 2

The proof that (3) implies (2) is trivial by taking \( f(x) = v(x)^{-1/p} \chi_I(x) \) and \( \lambda = (1/2) \exp \left( \frac{1}{|I|} \int_I \log(v^{-1/p}) \, d\mu \right) \).

For the proof that (2) implies (3) we need only consider the case \( p = 1 \). As in the case of the proof of Theorem 1 we need only consider functions \( f \) that are bounded with support of finite \( \mu \) measure and \( \int |f| v \, d\mu < \infty \) and weights \( v \) with a positive lower bound. Given such functions \( f \) and \( v \) and a \( \lambda > 0 \), define an operator \( M_f \) by
\[ M_f(x) = \sup_I \left( \int_I \frac{|f(y)|v(y) \, d\mu(y)}{\int_I w(t) \, d\mu(t)} \right) \]
with the sup taken over all intervals \( I \) containing \( x \) with the quotient taken as 0 in ambiguous cases. By (2), Jensen's inequality and the restrictions on \( f \) and \( v \)
\[ G_f(x) = \sup_I \exp \left( \left[ \frac{1}{|I|} \int_I \log(|f|v) \, d\mu \right] \left[ \frac{1}{|I|} \int_I \log \left( \frac{1}{v} \right) \, d\mu \right] \right) \]
\[ \leq \sup_I C_2 \frac{1}{|I|} \int_I |f| v \, d\mu = C_2 M_f(x). \]

Let \( E \) be the set where \( M_f(x) > \lambda/C_2 \), let \( D \) be the set where \( M_f(x) = \infty \) and let \( C \) be the collection of intervals \( I \) such that \( \int_I |f| v \, d\mu > 0 \) and \( \int_I w \, d\mu = 0 \). Let \( A \) be the sets of points \( a \) such that \([a, a] \in C \). The assumption that \( \int |f| v \, d\mu < \infty \) implies that \( A \) is countable. The set \( D \cap A^c \) is covered by intervals in \( C \) with positive length. Therefore, by Lemma 2 and the countability of \( A \), \( D \) has a countable covering \( \{I_k\} \) of intervals in \( C \). From this \( \int_D w \, d\mu \leq \sum_k \int_{I_k} w \, d\mu = 0 \). This and (9) show that
\[ \int_{\{G_f(x) > \lambda\}} w \, d\mu \leq \int_E w \, d\mu = \int_{E \cap D^c} w \, d\mu. \]

Now let \( B \) consist of all intervals \( I \) with \( \int_I |f| v \, d\mu > \frac{\lambda}{C_2} \int_I v \, d\mu > 0 \). Then \( B \) covers \( E \cap D^c \) and we can apply Lemma 1 with \( d\nu = w \, d\mu \). The result is a disjoint
sequence \( \{J_n\} \) from \( B \) and

\[
\int_{E \cap D^c} w \, d\mu \leq 5 \sum_n \int_{J_n} w \, d\mu < \frac{5C_2}{\lambda} \sum_n \int_{J_n} |f| v \, d\mu \leq \frac{5C_2}{\lambda} \int |f| v \, d\mu.
\]

Combining (10) and (11) completes the proof of Theorem 2.

6. An Example

In this section we derive a pair \((w, v)\) of functions that satisfy (2) but do not satisfy (1). It is also easily seen that they do not satisfy (4).

Let \( d\mu(x) = dx \),

\[
v(x) = \begin{cases} 
\exp \left[ -\frac{1}{x(\log x)^2} \right], & x \in (0, e^{-2}], \\
\infty, & \text{elsewhere},
\end{cases}
\]

and

\[
w(x) = \chi_{[0, e^{-2}]}(x) \frac{d}{dx} \left[ x \exp \left( \frac{1}{x \log x} \right) \right].
\]

Note that

\[
w(x) = \chi_{[0, e^{-2}]}(x) \left( 1 - \frac{1}{x \log x} - \frac{1}{x(\log x)^2} \right) \exp \left( \frac{1}{x \log x} \right)
\]

so that for \( x \in (0, e^{-2}] \)

\[
w(x) \leq -\frac{2}{x \log x} \exp \left( \frac{1}{x \log x} \right)
\]

and

\[
w(x) \geq -\frac{1}{2x \log x} \exp \left( \frac{1}{x \log x} \right).
\]

To prove (2) we need only consider intervals \( I \subset [0, e^{-2}] \) since the second factor in (2) is 0 if \( I \not\subset [0, e^{-2}] \). We will prove that (2) holds for \( I = (a, b) \) in two cases: \( a < 2b/3 \) and \( a \geq 2b/3 \).

For \( a < 2b/3 \)

\[
\frac{1}{|I|} \int_I w(x) \, dx \leq \frac{3}{b} \int_0^b w(x) \, dx = 3 \exp \left( \frac{1}{b \log b} \right),
\]

and since \( \log \left( \frac{1}{x} \right) = \frac{1}{x(\log x)^2} \) is decreasing on \((0, e^{-2}],\)

\[
\frac{1}{|I|} \int_I \log \left( \frac{1}{v(x)} \right) \, dx < \frac{1}{b} \int_0^b \log \left( \frac{1}{v(x)} \right) \, dx = \frac{-1}{b \log b}.
\]

Therefore,

\[
\exp \left[ \frac{1}{|I|} \int_I \log \left( \frac{1}{v(x)} \right) \, dx \right] < \exp \left( \frac{-1}{b \log b} \right).
\]

Combining (14) and (15) gives (2) with \( C_2 = 3 \) for this case.

For \( a \geq 2b/3 \) use (12) and the fact that the right side of (12) is increasing on \((0, e^{-2}]\) to see that

\[
\frac{1}{|I|} \int_I w(x) \, dx < -\frac{2}{b \log b} \exp \left( \frac{1}{b \log b} \right).
\]
Next, since \( \frac{1}{x \log x} \) decreases on \((0, e^{-2})\),

\[
(17) \quad \frac{1}{|I|} \int_I \log \left( \frac{1}{v(x)} \right) \, dx = \frac{1}{b-a} \int_a^b \frac{dx}{x \log x^2} < \frac{1}{a \log a^2}.
\]

Now for \( 2b/3 \leq a < b < e^{-2} \)

\[
\frac{1}{a \log a^2} < -\frac{1}{2a \log a} < -\frac{1}{(4/3)b \log b}.
\]

Using this, (16) and (17) gives

\[
\left[ \frac{1}{|I|} \int_I w(x) \, dx \right] \left[ \exp \left( \frac{1}{|I|} \int_I \log \left( \frac{1}{v(x)} \right) \, dx \right) \right] < -\frac{2}{b \log b} \exp \left( \frac{1}{4b \log b} \right).
\]

Since \( 8y \exp(-y) \leq 8/e \) for \( y \) in \((0, \infty)\), the right side is bounded by \( 8/e \). This completes the proof of (2) with \( C_2 = 3 \) for this pair.

To show that (1) is false for this \( v \) and \( w \) take \( f(x) = v(x)^{-1/p} \chi_{(0,e^{-2})}(x) \). The integral on the right side of (1) is \( e^{-2} \). Now

\[
G_f(x) \geq \exp \left( \frac{1}{x} \int_0^x \log f(t) \, dt \right) = \exp \left( \frac{-1}{px \log x} \right).
\]

This and (13) show that the left side of (1) is bounded below by \( \int_0^{e^{-2}} \frac{-dx}{2x \log x} = \infty \). This computation with \( p = 1 \) also shows directly that (4) fails for this pair.

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DEPARTMENT OF MATHEMATICS, HANGZHOU UNIVERSITY, ZHEJIANG 310028, PEOPLE’S REPUBLIC OF CHINA

Current address: Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

E-mail address: xyin@icarus.math.mcmaster.ca

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903