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ON LINEAR SERIES ON GENERAL k-GONAL PROJECTIVE CURVES

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ABSTRACT. Let X be a general k-gonal curve of genus g. Here we prove a strong upper bound for the dimension of linear series on X, i.e. we prove that $\dim(W^r_d(X)) \le \rho(g,r,d) + (g-2k+2) := g-(r+1)(r+g-d)+(g-2k+2)$.

In recent years several papers were devoted to the study of linear series on a general k-gonal curve. This note contains the proof of the following result.

Theorem 0.1. *Fix integers g, r, d, k with $k \ge 2$, $2k-2 \le g \le 4k-4$, $2 \le r \le d \le g-1$. Assume $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > 2g$. Let X be a general k-gonal curve of genus g. Then*

$$\dim(W^r_d(X)) \le \rho(g,r,d) + (g-2k+2) := g-(r+1)(r+g-d)+(g-2k+2).$$

Here $\rho(g,r,d) = g-(r+1)(r+d-g) = (r+1)(d+r)-rg$ is the Brill-Noether number. Note that the bound in the statement of 0.1 is just $\rho(2k-2,r,d)-(r-1)(g-2k+2) = (r+1)(d+r)-r(2k-2)-(r-1)(g-2k-2)$ and this is a hint of the way in which we will prove the theorem in section 1 by induction on g starting from the case $g = 2k-2$ (which is known to be true because a general curve of genus $2k-2$ is a k-gonal curve). The tools are the theory of limit linear series of Eisenbud and Harris (see [EH2]) and the theory of admissible coverings of Harris-Mumford (see [HM]). The reader is assumed to have a working knowledge of these theories. In this paper we work always over an algebraically closed base field $\mathbf{K}$ with either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > 2g$.

We believe that Theorem 0.1 is quite good for high r, but not sharp. For instance, it says nothing if $r = 1$, while much is known about the pencils on X. The upper bound for $\dim(W^r_d(X))$ in its statement just comes from part (d) of its proof.

Proof of 0.1. The proof of Theorem 0.1 is divided into 4 steps.

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(a) The assumption on char($\mathbf{K}$) allows us to use freely in the range of integers that we will consider the results on admissible coverings in [HM] and limit linear series in [EH1], [EH2], [EH3] and [EH4].

(b) Set $m := 2k-2$, $s := g-m$, and let D be a general curve of genus m. By the generality of D, D satisfies the Brill-Noether conditions, has the expected number of g^1_k and each of them has $6k-6$ simple ramification points. Fix one g^1_k on D and s of its ramification points, $P_1, \dots, P_s$. Let A be the stable genus g curve of compact type union of D and s disjoint elliptic curves $E_1, \dots, E_s$, with each E_i intersecting D at P_i.

(c) First we will check that A is a limit in $\overline{M}_g$ of smooth k-gonal curves of genus g. Indeed this curve is the stable reduction of an admissible degree k covering $f\colon\ X' \to U'$ (in the sense of Harris-Mumford [HM]) (or see [EH2], sect. 5) with the following description. U' is a connected nodal curve with $p_a(U') = 0$. X' is the union of D and $s(k-2)$ $\mathbf{P}^1$'s. The morphism f sends D to one of the $\mathbf{P}^1$'s, say J, of U' as the chosen g^1_k; each of the s families of $k-2$ $\mathbf{P}^1$'s of X' are linked to D at the other points of the g^1_k over the image of $D \cap E_i$ and mapped by f to a different component, say J_i, of U' with $u(\mathrm{Sing}(X')) = \mathrm{Sing}(U')$.

(d) To prove Theorem 0.1 it is sufficient to prove that the dimension of limit linear series (in the sense of Eisenbud-Harris) is bounded by $\rho(g,r,d)+s$ (and that the same is true for any curve which is obtained from A by inserting chains of $\mathbf{P}^1$'s at each point of $\mathrm{Sing}(A)$). This (and all the Brill-Noether-Gieseker package) would hold if we had chosen general points of D as $\mathrm{Sing}(A)$ instead of the points P_i; this was remarked (more or less explicitly) several times (see [EH3], Theorem 1.1, or [EH4], Proposition 5.2 and Corollary 5.3, or [W], Remarks 1.12 and 2.9). But of course, since D has only finitely many g^1_k, the set $\{P_i\}_{1 \le i \le s}$ is not a general set of s points of D. It is a standard calculation in the theory of limit linear series that each general point of D to which an elliptic tail is linked "drops" the dimension of the set of all limits g^r_d on D (which have dimension $\rho(m,r,d)$) by r. We want to check that each E_i imposes at least $r-1$ independent conditions to the set of all limits g^r_d on D, proving 0.1. Indeed we claim that this holds for any choice of the points P_i. To prove the claim we degenerate D to the union of a general curve Z of genus $m-s$ (which by assumption is ≥ 0) and s elliptic curves F_i with E_i linked to F_i. By the theory of admissible coverings of Harris-Mumford, to preserve the condition that the arithmetic genus g curve W obtained in this way is still a limit of smooth k-gonal curve, it is sufficient that the singular points of each F_i, say R_i and S_i (with R_i on Z and S_i on E_i), are such that $2R_i$ is linearly equivalent to $2S_i$ on F_i and that the points R_i are ramification points of a g^1_k on Z. It is possible to satisfy the last condition by Riemann-Hurwitz because $4k-4 \ge g$. It is the torsion condition on $R_i - S_i$ which assures us that there is a 2:1 morphism $F_i \in \mathbf{P}^1$ ramified over R_i and S_i. It is this torsion condition which makes (in general) false the Brill-Noether bound for W (see [W], Remarks 1.12 and 2.9, [EH1], Lemma 7.3, [EH3], Theorem 1.1, [EH4], Proposition 5.2 and Corollary 5.3). To prove our claim (and hence 0.1), look at the proofs in [EH1] in which the case of a curve with general moduli is proved. To get the upper bound $\rho(g,r,d)$ for $\dim(W^r_d)$ for the genus g starting from the bound $\rho(2k-2,r,d)$ for the genus $2k-2$, it would be sufficient to perform the numerical calculations in the first half of page 134 of [EH1]. To make those calculations the only missing step in our situation is part (iii) of Lemma 7 of [EH1]. To have our weaker upper bound it is sufficient to have, for each of the s elliptic tails, instead of part (iii) of Lemma 7 of [EH1] which says

"... for more than one value of the index called i in [EH1] only if there are two or more independent sections of V_{F_i} vanishing only at the two points R_i and S_i", the weaker statement "... for more than two values of i only if there are three or more independent sections of V_{F_i} vanishing only at the two points R_i and S_i". To prove this weaker statement it is sufficient to note that on each F_i there is no g^2_d with a positive divisor supported on $\{R_i, S_i\}$ because no degree 0 line bundle on F_i has two linearly independent sections. $\square$

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