ASYMPTOTIC BEHAVIOR OF NONEXPANSIVE SEQUENCES AND MEAN POINTS

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Abstract. Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $\{x_n\}_{n \geq 0}$ be a nonexpansive sequence in $E$ (i.e., $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i, j \geq 0$). Let $K = \bigcap_{n=1}^{\infty} \text{co}\{x_i - x_{i-1}\}_{i \geq n}$. We deal with the mean point of $\{x_n\}$ concerning a Banach limit. We show that if $E$ is reflexive and $d = d(0, K)$, then $d = d(0, \text{co}\{x_n - x_0\})$ and there exists a unique point $z_0$ with $\|z_0\| = d$ such that $z_0 \in \text{co}\{x_n - x_0\}$. This result is applied to obtain the weak and strong convergence of $\{x_n\}$.

1. Introduction

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $\{x_n\}_{n \geq 0}$ be a nonexpansive sequence in $E$ (i.e., $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i, j \geq 0$). Recently, Djafari Rouhani [2] obtained an interesting result on the weak convergence of $\{x_n\}$ under the assumption that $E$ is reflexive and strictly convex.

In this paper, we deal with his result without the assumption of strict convexity of $E$. That is, instead of the weak limit of $\{x_n\}$, we deal with the mean point of $\{\frac{x_n}{n}\}$ concerning a Banach limit under the assumption that $E$ is reflexive. Using the mean point, we obtain the weak convergence of $\{\frac{x_n}{n}\}$, which case $E^*$ has a Fréchet differentiable norm. Our results improve and extend the corresponding results in [5, 6, 7, 8, 9, 10] as in [2].

2. Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$ and dual $(E^*, \| \cdot \|)$. The duality pairing between $E$ and $E^*$ will be denoted by $(\cdot, \cdot)$. The duality mapping $J$ from $E$ into the family of nonempty closed convex subsets of $E^*$ is defined by

$$J(x) = \{x^* \in E^*: (x, x^*) = \|x\|^2 = \|x^*\|^2\}.$$
Note that we have for \( x, y \in E \) and \( j \in J(x) \),
\[
(x - y, j) = \|x\|^2 - (y, j) \geq \|x\|^2 - \frac{1}{2} (\|y\|^2 + \|j\|^2) = \frac{1}{2} (\|x\|^2 - \|y\|^2).
\]

We recall that if \( E \) is reflexive and strictly convex and \( K \) is a nonempty closed convex subset of \( E \), the nearest point projection mapping \( P_K \) of \( E \) onto \( K \) is well defined, i.e., \( K \) is a \( \text{Chebyshev set} \) (see [1, 4]).

We say that the sequence \( \{x_n\}_{n \geq 0} \) is nonexpansive if \( \|x_{n+1} - x_{j+1}\| \leq \|x_i - x_j\| \) for all \( i, j \geq 0 \).

Let \( \mu \) be a mean on integers \( N \), i.e., a continuous linear functional on \( \ell^\infty \) satisfying \( \|\mu\| = 1 = \mu(1) \). Then we know that \( \mu \) is a mean on \( N \) if and only if
\[
\inf\{a_n : n \in N\} \leq \mu(a) \leq \sup\{a_n : n \in N\}
\]
for every \( a = (a_1, a_2, \cdots) \in \ell^\infty \). According to time and circumstances, we use \( \mu_n(a_n) \) instead of \( \mu(a) \). A mean \( \mu \) on \( N \) is called a Banach limit if
\[
\mu_n(a_n) = \mu_n(a_{n+1})
\]
for every \( a = (a_1, a_2, \cdots) \in \ell^\infty \). Using the Hahn–Banach theorem, we can prove the existence of a Banach limit. We know that if \( \mu \) is a Banach limit, then
\[
\lim_{n \to \infty} \inf a_n \leq \mu_n(a_n) \leq \lim_{n \to \infty} \sup a_n
\]
for every \( a = (a_1, a_2, \cdots) \in \ell^\infty \).

Let \( E \) be a reflexive Banach space and let \( \{x_n\} \) be a bounded sequence in \( E \). Then, for a Banach limit \( \mu \), we can obtain a point \( x_0 \) in \( E \) such that
\[
\mu_n(x_n, x^*) = (x_0, x^*)
\]
for all \( x^* \in E^* \). In fact, the function \( \mu_n(x_n, x^*) \) is linear in \( x^* \). Further, since
\[
\|\mu_n(x_n, x^*)\| \leq \sup_n \|x_n\| \cdot \|x^*\|,
\]
the function \( \mu_n(x_n, x^*) \) is also bounded in \( x^* \). So, we have \( x_0^* \in E^{**} \) such that \( \mu_n(x_n, x^*) = (x_0^*, x^*) \) for every \( x^* \in E^* \). Since \( E \) is reflexive, we obtain \( x_0 \in E \) such that \( \mu_n(x_n, x^*) = (x_0^*, x^*) \) for all \( x^* \in E^* \). This point \( x_0 \) is called a mean point of \( \{x_n\} \) concerning \( \mu \). We also know that this mean point \( x_0 \) is contained in \( \bigcap_{n \geq 1} \text{co}\{x_n\} \). In fact, if not, there exists \( n_0 \in N \) such that \( x_0 \notin \text{co}\{x_n : n \geq n_0\} \).

By separation theorem, we obtain a point \( x^* \in E^* \) such that
\[
(x_0, x^*) < \inf\{(z, x^*) : z \in \text{co}\{x_n : n \geq n_0\}\}.
\]

So, we have
\[
\mu_n(x_n, x^*) = (x_0, x^*) < \inf\{(x_n, x^*) : n \geq n_0\}
\leq \mu_n\{(x_n, x^*) : n \geq n_0\} = \mu_n(x_n, x^*).
\]

This is a contradiction. For these facts, see [11].

Let \( S = \{x \in E : \|x\| = 1\} \). Then the norm of \( E \) is Fréchet differentiable if for each \( x \in S \), the limit
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]
exesists uniformly for each \( y \in S \). The following lemma is well known (cf. [3]).
Lemma 2.1. \( E^* \) has a Fréchet differentiable norm if and only if \( E \) is reflexive and strictly convex, and has the following property: if the weak limit \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} \|x_n\| = \|x\| \) for a sequence \( \{x_n\} \) in \( E \), then \( \{x_n\} \) converges strongly to \( x \).

Finally, let \( D \) be a subset of \( E \). Then we denote the closure of \( D \) by \( \overline{D} \) and the closed convex hull of \( D \) by \( \text{co}D \), respectively. We also denote its distance from a point \( x \) in \( E \) by \( d(x,D) = \inf_{y \in D} \|x - y\| \).

3. Main result

In this section, for a nonexpansive sequence \( \{x_n\} \) in \( E \), we study the mean point of \( \{x_n\} \) concerning a Banach limit.

We begin with the known result which will play a crucial role in our result.

Lemma 3.1 [2]. Let \( E \) be a Banach space and let \( \{x_n\} \) be a nonexpansive sequence in \( E \). Then

\[
\lim_{n \to \infty} \frac{x_n}{n} \quad \text{exists and} \quad \lim_{n \to \infty} \frac{x_n}{n} = \inf_{n \geq 1} \frac{x_n - x_0}{n}.
\]

The following result is essentially in spirit of Djafari Rouhani [2]. For completeness, we give the proof.

Lemma 3.2. Let \( E \) be a reflexive Banach space and let \( \{x_n\} \) be a nonexpansive sequence in \( E \). Let

\[
K = \bigcap_{n=1}^{\infty} \text{co}(\{x_i - x_{i-1}\}_{i \geq n}).
\]

Then \( \lim_{n \to \infty} \frac{\|x_n\|}{n} = d(0,K) = \inf_{n \geq 1} \|x_n - x_k\| \).

Proof. Let \( k \geq 1 \) be fixed and \( j_n \in J(x_n - x_{k-1}) \) for \( n \geq k \). Then we have for \( n \geq k \geq 1 \)

\[
(x_k - x_{k-1}, j_n) \geq \frac{1}{2} \|x_n - x_{k-1}\|^2 - \frac{1}{2} \|x_n - x_k\|^2 \geq \frac{1}{2} \|x_n - x_{k-1}\|^2 - \frac{1}{2} \|x_{n-1} - x_{k-1}\|^2.
\]

Hence we obtain

\[
(3.1) \quad \frac{1}{2} \left( \frac{2}{n^2} \sum_{i=k}^{n} j_i, \frac{x_n - x_{k-1}}{n} \right) \geq \frac{x_n - x_{k-1}}{n}.
\]

Let \( S_n = \frac{1}{n^2} \sum_{i=k}^{n} j_i \) for \( n \geq k \). Then we have

\[
\|S_n\| \leq \frac{2}{n^2} \sum_{i=k}^{n} \|x_i - x_{k-1}\| = \frac{2}{n^2} \sum_{i=k}^{n} i \|x_i - x_{k-1}\| i
\]

and so \( \{S_n\} \) is bounded since \( \{\frac{x_n}{n}\} \) is bounded by Lemma 3.1. Thus from the weak–star compactness of the closed unit ball of \( E^* \), it follows that the sequence
\{S_n\} has a weak-star cluster point \( j \in E^* \) (obviously independent of \( k \geq 1 \)). Now by Lemma 3.1 and (3.1), we obtain
\[
(x_k - x_{k-1}, j) \geq \lim_{n \to \infty} \frac{x_n}{n}^2
\]
for \( k \geq 1 \) and hence
\[
(\frac{x_n - x_0}{n}, j) \geq \lim_{n \to \infty} \frac{x_n}{n}^2
\]
for all \( n \geq 1 \). We also have
\[
\| j \| \leq \liminf_{n \to \infty} \| S_n \|
\]
\[
\leq \limsup_{n \to \infty} \frac{2}{n^2} \sum_{i=k}^{n} i\| x_i - x_{k-1} \| = \lim_{n \to \infty} \frac{x_n}{n}^2
\]
and so
\[
(x_k - x_{k-1}, j) \geq \lim_{n \to \infty} \frac{x_n}{n}^2 \geq \| j \|^2
\]
for all \( k \geq 1 \). So, for any \( z \in \overline{w} \{x_{i+1} - x_i \}_{i \geq 0} \),
\[
\frac{1}{2} \lim_{l \to \infty} \frac{\| x_l \|^2}{n} + \frac{1}{2} \| z \|^2 \geq \frac{1}{2} \| j \|^2 + \frac{1}{2} \| z \|^2
\]
\[
\geq (z, j) \geq \lim_{n \to \infty} \frac{x_n}{n}^2 \geq \| j \|^2.
\]
Since \( K \subset \overline{w} \{x_{i+1} - x_i \}_{i \geq 0} \), it follows from (3.3) that
\[
\| j \| \leq \lim_{n \to \infty} \frac{x_n}{n} \leq \inf_{z \in K} \| z \| = d(0, K).
\]
On the other hand, since \( \{ \frac{x_n}{n} \} \) is bounded and \( E \) is reflexive, \( \{ \frac{x_n - x_0}{n} \} \) contains a weakly convergent subsequence \( \{ \frac{x_{n_l} - x_0}{n_l} \} \). Let \( \{ \frac{x_{n_l} - x_0}{n_l} \} \) converge weakly to \( q \in K \). Then we have
\[
\| q \| \leq \liminf_{l \to \infty} \frac{x_{n_l} - x_0}{n_l} = \lim_{n \to \infty} \frac{x_n}{n}
\]
and hence \( \lim_{n \to \infty} \frac{x_n}{n} = d(0, K) \). This completes the proof.

Now, using Lemmas 3.1 and 3.2, we obtain the main result.

**Theorem 3.3.** Let \( E \) be a reflexive Banach space and let \( \{x_n\} \) be a nonexpansive sequence in \( E \). Let
\[
K = \bigcap_{n=1}^{\infty} \overline{w} \{x_i - x_{i-1} \}_{i \geq n}
\]
and \( d = d(0, K) \). Then \( d = d(0, \overline{w} \{x_n - x_0\}) \) and there exists a point \( z_0 \) with \( \| z_0 \| = d \) such that \( z_0 \in \overline{w} \{x_n - x_0\} \).

**Proof.** Since \( \lim_{n \to \infty} \frac{x_n}{n} = d(0, K) \) and \( \frac{x_n - x_0}{n} = d \) by Lemma 3.2, we may assume that \( \{ \frac{x_n - x_0}{n} \} \) is bounded. So, it follows from reflexivity of \( E \) that for a Banach limit \( \mu \), there exists \( z_0 \in \overline{w} \{x_n - x_0\} \) such that
\[
\mu_n \left( \frac{x_n - x_0}{n}, x^* \right) = (z_0, x^*)
\]
for every $x^* \in E^*$. For $j_0 \in J(z_0)$, where $J$ is the duality mapping of $E$, we have

$$
\|z_0\|^2 = (z_0, j_0) = \mu_n\left(\frac{x_n - x_0}{n}, j_0\right)
\leq \mu_n\left(\|\frac{x_n - x_0}{n}\|\right) \cdot \|j_0\| = d \cdot \|j_0\| = d \cdot \|z_0\|,
$$

and hence $\|z_0\| \leq d$. On the other hand, by the proof of Lemma 3.2 and (3.2), there exists a functional $j \in E^*$ with $\|j\| \leq d$ such that

$$(3.5) \quad (\frac{x_n - x_0}{n}, j) \geq d^2
$$

for all $n \geq 1$. Hence we have $(z_0, j) \geq d^2$. Since $\|j\| \leq d$, we obtain

$$
d^2 \geq \|z_0\| \cdot \|j\| \geq (z_0, j) \geq d^2
$$

and hence $\|z_0\| = \|j\| = d$. It also follows from (3.5) that $(z, j) \geq d^2$ for every $z \in \overline{co}\left(\frac{x_n - x_0}{n}\right)$ and so

$$
\|z\| \cdot d = \|z\| \cdot \|j\| \geq (z, j) \geq d^2.
$$

Hence we have $\|z\| \geq d$ for every $z \in \overline{co}\left(\frac{x_n - x_0}{n}\right)$. Then we obtain

$$
d = d(0, \overline{co}\left(\frac{x_n - x_0}{n}\right)).
$$

Let $w_0$ be another point satisfying (3.4). Then for $j \in J(z_0 - w_0)$, we have

$$
\|z_0 - w_0\|^2 = (z_0 - w_0, j) = \mu_n\left(\frac{x_n - x_0}{n} - \frac{x_n - x_0}{n}, j\right) = 0,
$$

and hence $z_0 = w_0$. This completes the proof.

Corollary 3.4. With the same assumptions as in Theorem 3.3, we have the following:

(i) If $E$ is strictly convex, then the weak $\lim_{n \to \infty} \frac{x_n}{n}$ exists and coincides with $P_K0$ with $\|P_K0\| = d$.

(ii) If $E^*$ has a Fréchet differentiable norm, then the strong $\lim_{n \to \infty} \frac{x_n}{n}$ exists and coincides with $P_K0$.

Proof. (i) Since a reflexive Banach space $E$ is strictly convex, the set

$$
\{z \in \overline{co}\left(\frac{x_n - x_0}{n}\right) : \|z\| = d\}
$$

consists of exactly one point and $d(0, K) = \|P_K0\|$. This point equals $z_0$ in Theorem 3.3. Let $\left\{\frac{x_n}{n}\right\}$ be a subsequence of $\left\{\frac{x_n}{n}\right\}$ such that $\left\{\frac{x_n}{n}\right\}$ converges weakly to $p \in K$. Then since

$$
\|p\| \leq \lim inf \|\frac{x_n}{n}\| = \lim \frac{x_n}{n} = \|P_K0\|,
$$

we have $p = z_0 = P_K0$. This implies that $\left\{\frac{x_n}{n}\right\}$ converges weakly to $P_K0$, which completes the proof.

(ii) It follows from Lemma 2.1 that $\left\{\frac{x_n}{n}\right\}$ converges strongly to $P_K0$.

Remark. (1) Since our study is equivalent to the study of the asymptotic behavior of the sequence $\left\{\frac{T \cdot x}{n}\right\}$ in $E$, where $T$ is a nonexpansive mapping from an arbitrary subset $C$ of $E$ into itself and $x \in C$, Theorem 3.3 is an improvement of Theorem 5 in [11].

(2) Corollary 3.4 is of interest in view of using the mean point. Compare this with Corollary 3.2 in [2].

(3) Corollary 3.4 also contains the previous corresponding results in [5, 6, 7, 8, 9, 10].
References


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