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BANACH ALGEBRAS WITH UNIQUE UNIFORM NORM

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ABSTRACT. Commutative semisimple Banach algebras that admit exactly one uniform norm (not necessarily complete) are investigated. This unique uniform norm property is completely characterized in terms of each of spectral radius, Silov boundary, set of uniqueness, semisimple norms; and its connection with recently investigated concepts like spectral extension property, multiplicative Hahn Banach extension property and permanent radius are revealed. Several classes of Banach algebras having this property as well as those not having this property are discussed.

1. INTRODUCTION

A *uniform norm* on a Banach algebra $(A, \|\cdot\|)$ is a submultiplicative norm (not necessarily complete) $|\cdot|$ satisfying the square property $|x^2| = |x|^2$ $(x \in A)$. By [1, Chapter 2], A is commutative and semisimple iff the spectral radius r $(= r_A)$ is a uniform norm iff A admits a uniform norm. A has *unique uniform norm property* (*UUNP*) if r is the only uniform norm on A. This paper is concerned with Banach algebras with UUNP. Note that equivalent uniform norms are equal and any uniform norm $|\cdot| \leq r$.

A recent trend ([9], [10], [11], [14]) in Banach algebras is to investigate incomplete algebra norms; in the same spirit, it is interesting to characterize intrinsically UUNP and its influence on A. The spectral extension property (SEP) of [10] implies UUNP. In fact, it is claimed in [10] that A has SEP iff "every closed set of uniqueness contains the Silov boundary" (statement (2) of [10, Theorem 1]). However, there is a gap in the proof that this statement implies SEP (personal correspondence with Dr. Meyer). Theorem 2.3 shows that this statement is equivalent to UUNP. Thus the problem whether UUNP implies SEP is open. Also, the apparent similarity between the C^*-property $\|x^*x\| = \|x\|^2$ and the square property $\|x^2\| = \|x\|^2$ of the complete norms on a C^*-algebra and on a uniform Banach algebra (uB-algebra) suggests a structural analogy between these two classes of Banach algebras (in spite of the fact that, contrary to the case of uB-algebras, a C^*-algebra has a unique, not necessarily complete, C^*-norm). Banach *-algebras with unique C^*-norm have already been investigated in [2].

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Throughout $(A, \|\cdot\|)$ is a commutative semisimple Banach algebra. Let $N =$ set of all norms (necessarily submultiplicative) on A; sN = all semisimple norms, $p \in sN$ if the completion A_p of (A, p) is semisimple [14]; $qN = \{p \in N : r \leq p\}$. The set qN is the set of all spectral norms [12] (called Q-norms in [11], [14]); uN = all uniform norms. Then $uN \subseteq sN \subseteq N$, $uN \cap qN = \{r\}$. Let $\Delta(A)$ be the Gelfand space, $x \in A \to \hat{x} \in C_0(\Delta(A))$ be the Gelfand transform and ∂A be the Silov boundary [10].

2. Results

An extension B of A means a Banach algebra B containing A as a subalgebra. A has *spectral extension property* (*SEP*) [10] if for any extension B of A, $r_A(x) = r_B(x)$ $(x \in A)$; equivalently $r(x) \leq |x|$ $(x \in A, |\cdot| \in N)$. This is equivalent to A being a *permanent Q-algebra* [14] i.e., $qN = N$.

Proposition 2.1. *The following are equivalent.*

(1) *A has UUNP* (*resp. SEP*).

(2) *Every norm is sN* (*resp. N*) *is spectral.*

(3) *r is minimal in sN* (*resp. minimum in N*) *in the usual partial order $|\cdot|_1 \leq |\cdot|_2$ in N.*

If r is minimal in N, does A have SEP? A has *unique semisimple norm property* (*USNP*) if any two semisimple norms on A are equivalent. A is a *uniform Banach algebra* (*uB-algebra*) (resp. *uB-equivalent*) if $\|\cdot\|$ is a uniform norm (resp. $\|\cdot\|$ is equivalent to a uniform norm).

Proposition 2.2. *A has USNP iff A is uB-equivalent and has UUNP.*

The non-uB-equivalent Banach algebras $C^k[0,1]$ and the group algebra $L^1(G)$ (G-infinite, abelian) has UUNP but not USNP.

A set $F \subseteq \Delta(A)$ is a *set of uniqueness* if $|x|_F = \sup\{|\phi(x)| : \phi \in F\}$ is a norm on A. A has *semisimple SEP* if every semisimple norm on A is spectral. By [10], the *permanent radius* on A is $r_p(x) = \inf\{r_{A_p}(x) : p \in N\}$; analogously the *semisimple permanent radius* on A is $r_s(x) = \{r_{A_p}(x) : p \in sN\} = \inf\{|x| : |\cdot| \in sN\}$.

Theorem 2.3. *The following are equivalent.*

(1) *A has UUNP.*

(2) *∂A is the smallest closed set of uniqueness.*

(3) *If $F \subseteq \Delta(A)$ is closed and not containing ∂A, then there exists $a \in A$ such that $r(a) > 0$ and $\hat{a}|_F = 0$.*

(4) *If $F \subseteq \Delta(A)$ is closed and does not contain ∂A, then there exists $a \in A$ such that $r_s(a) > 0$ and $\hat{a}|_F = 0$.*

(5) *Every complex homomorphism in ∂A has a multiplicative linear extension to any semisimple commutative extension of A.*

(6) *A has semisimple SEP.*

(7) *$r = r_s$ on A.*

(8) *For any semisimple commutative extension B of A and $a \in A$, $\partial(\mathrm{Sp}_A(a)) \subseteq \mathrm{Sp}_B(a) \cup \{0\}$.*

Does UUNP imply SEP? If ∂A is a minimal closed set of uniqueness, does A have UUNP? Also, (i) $r_p \leq r_s \leq r$; (ii) $r = r_p$ iff A has SEP; (iii) $r = r_s$ iff A has UUNP; and (iv) if every norm in N majorizes a semisimple norm, then $r_s = r_p$. Does the converse hold?

We call A *weakly regular* if for every proper closed set $F \subseteq \Delta(A)$, there exists $x \neq 0$ in A such that $\hat{x}|_F = 0$. The algebra in [10, Example 1] is weakly regular but not regular. A has *strong semisimple SEP* (resp. *semisimple multiplicative Hahn Banach property* (*semisimple MHBP*)) if for any semisimple commutative extension B of A, $\mathrm{Sp}_B(x) \cup \{0\} = \mathrm{Sp}_A(x) \cup \{0\}$ $(x \in A)$ (resp. every $\phi \in \Delta(A)$ has a multiplicative linear extension to B). Corresponding to Theorems 2, 3 and Corollary 1 in [10], it is easily shown that (i) A has strong semisimple SEP iff A has UUNP and $\hat{x}(\Delta(A)) \cup \{0\} = \hat{x}(\partial A) \cup \{0\}$ $(x \in A)$; (ii) A has semisimple MHBP iff A has UUNP and $\partial A = \Delta(A)$ iff A is weakly regular; (iii) if A has UUNP (and if the dimension of A is greater than one), then A cannot be an integral domain. Does the converse of (iii) above hold, at least, in uB-algebras? Note that the property of a uB-algebra of being an integral domain is a kind of analyticity property [8, Section 7.4, p. 172 ff].

A has (P)- *property* [10], [11] if for every nonzero closed ideal I of A, there exists $x \in A$ such that $r_p(x) > 0$.

Proposition 2.4. *The following are equivalent.*

(1) *A has SEP.*

(2) *A has UUNP and $r_s = r_p$; i.e. A has UUNP and every norm dominates a semisimple norm.*

(3) *A has UUNP and (P)-property.*

Proposition 2.5. *Let $|\cdot| \in uN$. Then $|\cdot|$ is minimal in sN iff the completion B of $(A, |\cdot|)$ has UUNP and for every nonzero closed ideal I of B with $I = k(h(I))$ (i.e., kernel of hull of I), $I \cap A \neq \{0\}$.*

If sN has a minimal element, does A have UUNP? Given $p \in sN$, $|x|_p = \lim_{n\to\infty} p(x^n)^{1/n}$ $(x \in A)$ defines a uniform norm on A and $|\cdot|_p \leq p$. Thus every semisimple norm dominates a uniform norm. Thus is the essential argument in the proof of Proposition 2.1, the details being omitted.

Proposition 2.6. *Let B be a dense ideal of $(A, \|\cdot\|)$ such that B is a Banach algebra with some norm $|\cdot|$. If A has UUNP, then B has UUNP.*

Proofs.

Proof of Proposition 2.2. Let A have USNP. Then A is uB-equivalent, since r (being a uniform norm) and the $\|\cdot\|$ are semisimple. Let $|\cdot|$ be in uN. Then $|\cdot|$ is equivalent to r, hence are equal. Thus A has UUNP. Conversely, suppose A is uB-equivalent and has UUNP. Since $\|\cdot\| \in sN$, it is enough to show that any $|\cdot| \in SN$ is equivalent to $\|\cdot\|$. Let $|\cdot|$ be in sN. By Proposition 2.1, $r \leq |\cdot|$ on A. Since the inclusion map of A into B (the completion of $(A, |\cdot|)$) is a homomorphism from A into a semisimple commutative Banach algebra, there exists $k > 0$ such that $|\cdot| \leq k\|\cdot\|$ on A. Thus $r \leq |\cdot| \leq k\|\cdot\|$. Since A is uB-equivalent and r is the only uniform norm, $|\cdot|$ is equivalent to $\|\cdot\|$. □

Proof of Theorem 2.3. (1) implies (2). Let $F \subseteq \Delta(A)$ be a closed set of uniqueness. Then $|x|_F = \sup\{|\phi(x)| : \phi \in F\}$ is a uniform norm on A. Since A has UUNP, $r = |\cdot|_F$. By definition of ∂A, $\partial A \subseteq F$. This proves (2), and (2) implies (3) is clear.

(3) implies (4). Let $F \subseteq \Delta(A)$ be closed, not containing ∂A. Then, there exists $a \in A$ such that $r(a) > 0$ and $\hat{a}|_F = 0$. Clearly, $r_s(a) \leq r(a)$. Suppose $r_s(a) < r(a)$.

By definition of r_s, $r_s(a) \leq |a| < r(a)$, for some $|\cdot| \in uN$. Define $K = \{\phi \in \Delta(A) : \phi$ is continuous with $|\cdot|\}$. Then K is closed and $|x|_K = \sup\{|\phi(x)| : \phi \in K\} = |x|$ $(x \in A)$. Hence, K contains ∂A, and so $r \leq |\cdot|_K = |\cdot|$. In particular, $r(a) \leq |a|$, which is a contradiction.

(4) implies (5) is analogous to (3) implies (4) of [10, Theorem 1].

(5) implies (6). Let $|\cdot| \in sN$. Then the completion B of $(A, |\cdot|)$ is a semisimple commutative extension of A. By assumption, for any $x \in A$, $r(x) = \sup\{|\phi(x)| : \phi \in \partial A\} \leq \sup\{|\phi(x)| : \phi \in \Delta(B)\} \leq |x|$. Hence (6) is proved, and (6) implies (7) is clear from the definition of r_s.

(7) implies (8). Let $(B, |\cdot|)$ be a semisimple commutative extension of A and $x \in A$. Since $r_B \in uN$ and $r_s = r$, one has $r = r_B$. This implies that $|\cdot|$ is spectral on A. By [9, p. 67], $\mathrm{Sp}_A(x) \cup \{0\} = \mathrm{Sp}_{A^\sim}(x) \cup \{0\}$ where $A^\sim$ is the completion of $(A, |\cdot|)$. By [7, Theorem 6.2.1, p. 61], $\partial(\mathrm{Sp}_{A^\sim}(x)) \subseteq \mathrm{Sp}_B(x) \cup \{0\}$ $(x \in A^\sim)$. Hence (8) is proved.

(8) implies (1). Let $|\cdot| \in uN$. Then, $|\cdot| \leq r$ and the completion B of $(A, |\cdot|)$ is a semisimple extension of A. Hence $\partial(\mathrm{Sp}_A(x)) \subseteq \mathrm{Sp}_B(x) \cup \{0\}$ $(x \in A)$, so that $r_A \leq r_B = |\cdot|$ on A. Thus A has UUNP. □

Proof of Proposition 2.4. (1) implies (2) follows from $r_p \leq r_s \leq r$ and A having SEP implies $r_p = r$.

(2) implies (3). From $r_s = r_p$ and Theorem 2.3, $r = r_p$. Hence, semisimplicity of A implies A has (P)-property.

(3) implies (1). Let $|\cdot| \in N$. By [11, Theorem 2, p. 80], there exists a minimal norm $|\cdot|_0$ in N such that $|\cdot|_0 \leq |\cdot|$. By [9, Theorem 2, p. 49], $|\cdot|_0 \in uN$. Since A has UUNP, $r = |\cdot|_0 \leq |\cdot|$. Hence A has SEP. □

Proof of Proposition 2.5. Let $|\cdot| \in uN$ be minimal in sN. Let $|\cdot|_1$ be a uniform norm on B. Then $|\cdot|_1 \leq |\cdot|$ on B. By minimality of $|\cdot|$, $|\cdot|_1 = |\cdot|$ on A, hence on B. Thus B has UUNP. Let I be a nonzero closed ideal of B such that $k(h(I)) = I$. Then B/I is semisimple, by [7, Theorem 7.3.1, p. 174]. Define a uniform seminorm on B as $|x|_1 = r_{B/I}(x + I)$. Then $\ker(|\cdot|_1) = \{x \in B : |x|_1 = 0\} = I$. Suppose $I \cap A = \{0\}$. Then $|\cdot|_1 \in uN$ and $|\cdot|_1 \leq |\cdot|$ on A. Since $|\cdot|$ is minimal in sN, $|\cdot|_1 = |\cdot|$ on A, so on B. This is a contradiction. For the converse, let $|\cdot|_1 \in sN$ such that $|\cdot|_1 \leq |\cdot|$. Then we may assume that $|\cdot|_1 \in uN$. Since $|\cdot|_1$ is $|\cdot|$-continuous, it can be extended as a uniform seminorm on B, denoted by $|\cdot|_1$ itself. This has to be a norm on B, otherwise $I = \ker(|\cdot|_1)$ is a nonzero closed ideal of B, $k(h(I)) = I$, $I \cap A = \{0\}$. Thus $|\cdot|, |\cdot|_1$ are uniform norms on B. Since B has UUNP, they are equal on B, so on A. □

Proof of Proposition 2.6. Let $|\cdot|_1$ be a uniform norm on B. Since B is an ideal in A, $r_B = r_A$ on B, hence $|\cdot|_1 \leq \|\cdot\|$ on B. Let the extension of $|\cdot|_1$ as a uniform seminorm on A be denoted by $|\cdot|_1$ itself. Let $x \in A$, $|x|_1 = 0$. Then there exists $\{x_n\}$ in B such that $x_n \to x$, hence $xx_n \to x^2$ in $\|\cdot\|$ and $|\cdot|_1$. Since B is an ideal and $|\cdot|_1$ is a norm on B, $xx_n = 0$, hence $x^2 = 0$. Since A is semisimple, $x = 0$. Thus $|\cdot|_1 \in uN$. Since A has UUNP, $|\cdot|_1 = r$ on A, so $|\cdot|_1 = r_A = r_B$ on B. □

3. Examples

(3.1). If A is regular (in particular, if A has an orthogonal basis [6]), then A has SEP [10], hence UUNP. This includes (i) Banach sequence algebras c_0, c, l^p $(1 \leq p \leq \infty)$, (ii) the convolution algebras $L^p(T)$ $(1 \leq p < \infty)$ on the unit circle T, (iii) the Hardy spaces $H^p(U)$ $(1 < p < \infty)$ with Hadamard product [6]. By Proposition 2.6, every Segal algebra [4] on a locally compact abelian group G has UUNP. The measure algebra $M(G)$ has UUNP iff G is discrete. Indeed, let F be the closure of $\hat{G}$ in $\Delta(M(G))$. Then $|\mu|_F = \sup\{|\phi(u)| : \phi \in F\}$ is a uniform norm on $M(G)$. If $M(G)$ has UUNP, $\partial M(G) \subseteq F$. This happens only if G is discrete [13, p. 329].

Let $0 < r < 1$, $U_r = \{z \in \mathbb{C} : |z| < r\}, A_r = \{f \in C(\overline{U}_1) : f$ is analytic in $U_r\}$, $\|f\| = \sup\{|f(z)| : z \in \overline{U}_1\}$ $(f \in A_r)$. This is [10, Example 2], which has SEP, hence UUNP (unlike the disc algebra); however, it is not weakly regular, not antisymmetric, not analytic algebra and $\Delta(A_r) = \overline{U}_1$ has analytic structure. Likewise, one can compare the uB-algebra $H^\infty(U_1)$ with $A_r^\infty = \{f \in C(U_1) : f$ is bounded on U_1 and analytic on $U_r\}$.

(3.2). For $0 < r < 1$, let $B_r = \{f \in C(\overline{U}_1) : f$ is analytic on $U_1 \backslash \overline{U}_r\}$ with norm $\|f\| = \sup\{|f(z)| : |z| \leq 1\}$. Then B_r does not have UUNP.

Let $w = (w_n)_{-\infty}^{+\infty}$ such that $1 \leq w_{m+n} \leq w_n w_m$. Let $A(w)$ be the convolution algebra of sequences $a = \{a_n\}_{-\infty}^{+\infty}$ with $\|a\| = \sum_{-\infty}^{+\infty} |a_n| w_n < \infty$. By [5, p. 120], $\Delta(A) = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ for some $r, 0 < r \leq R$. If $r < R$, then $A(w)$ cannot have UUNP.

A nonhermitian *-semisimple commutative Banach *-algebra A cannot have UUNP. Indeed, the Gelfand Naimark pseudonorm m is a uniform norm on A. However, r and m cannot be equal, otherwise A is hermitian [3, §35, Theorem 3, p. 188].

Remark. The affirmative answer of the problem whether UUNP imply SEP would imply the following: (i) Weak regularity is equivalent to MHBP. (ii) $r = r_s$ implies $r = r_p$. (iii) In a uB-algebra $(A, \|\cdot\|)$, UUNP is equivalent to the property (K) : $\|\cdot\| \leq |\cdot|$ for all $|\cdot| \in N$. (That $C(X)$ has the property (K) is a well-known theorem of Kaplansky. As exhibited by the disc algebra, this does not hold in uB-algebra.) In view of the above (as well as Proposition 2.4), we believe that the problem may have a negative answer.

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References

1. B. Aupetit, *Propertetes spectrales des algebres de Banach*, Lecture Notes in Math., vol. 735, Springer-Verlag, Berlin, Heidelberg, and New York, 1979. MR **81i:**46055
2. B. A. Barnes, *The properties *-regularity and uniqueness of C^*-norm in a general *-algebra*, Trans. Amer. Math. Soc. **279** (1983), 841–859. MR **85f:**46100
3. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin, Heidelberg, and New York, 1973. MR **54:**11013
4. J. T. Burnham, *Segal algebras and dense ideal in Banach algebras*, Functional Analysis and its Applications (H. G. Garnir, K. R. Unni, and J. H. Williamson eds.), Lecture Notes in Math., vol. 399, Springer-Verlag Berlin and New York, 1974. MR **54:**54
5. I. Gelfand, D. Raikov, and G. Shilov, *Commutative normed rings*, Chelsea, New York, 1964. MR **34:**4940

6. T. Husain and S. Watson, *Topological algebras with orthogonal Schauder bases*, Pacific J. Math. **91** (1980), 339–347. MR **82h:**46064
7. R. Larsen, *Banach algebras*, Marcel Dekker, New York, 1973. MR **58:**7010
8. G. M. Leibowitz, *Lectures on complex function algebras*, Scott Foresman and Company, Glenview, IL, 1970. MR **55:**1072
9. M. J. Meyer, *Submultiplicative norms on Banach algebras*, Ph.D. Thesis, University of Oregon, 1989.
10. ______, *Spectral extension property and extension of multiplicative linear functionals*, Proc. Amer. Math. Soc. **112** (1991), 855–861. MR **91j:**46059
11. ______, *Minimal incomplete norms in Banach algebras*, Studia Math. **102** (1992), 77–85. MR **93c:**46085
12. T. W. Palmer, *Spectral algebras*, Rocky Mountain J. Math. **22** (1992), 293–328. MR **93d:**46079
13. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, NJ, 1960. MR **22:**5903
14. B. J. Tomiuk and B. Yood, *Incomplete normed algebra norms on Banach algebras*, Studia Math. **95** (1989), 119–132. MR **91e:**46063

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