PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 2, February 1996

ON ANTICONFORMAL AUTOMORPHISMS OF RIEMANN SURFACES WITH NONEMBEDDABLE SQUARE

ANTONIO F. COSTA

(Communicated by Albert Baernstein II)

ABSTRACT. In this paper we present an example of an anticonformal automorphism whose square has prime order and is not embeddable. We prove that every embeddable automorphism of odd order of a compact Riemann surface is the square of an orientation-reversing self-homeomorphism. Finally we study whether a conformal involution is the square of an orientation-reversing automorphism.

A smooth surface embedded in $\mathbf{R}^3$ inherits a conformal structure from $\mathbf{R}^3$. A Riemann surface is embeddable if it is conformally equivalent to a smooth surface which is embedded in $\mathbf{R}^3$. It has been shown that every Riemann surface is embeddable (see [G] and [R1]). If X is a Riemann surface and f is a conformal automorphism of X, we say that f is embeddable when there is a conformal embedding $d : X \to \mathbf{R}^3$ such that dfd^{-1} is the restriction of a rotation. R. A. Rüedy [R2] has given necessary and sufficient conditions for an automorphism to be embeddable. In [Z1], using [R2], it is claimed that an automorphism of prime order of a compact Riemann surface is embeddable if it is the square of an anticonformal automorphism. In this paper we present an example of an anticonformal automorphism whose square has prime order and is not embeddable. Therefore Theorem 1 of [Z1] is not valid. Theorem 1 has been used in [Z2] and [Z3], leading to some mistakes which were later corrected in [BC] and [Y]. We also give an alternative proof of Theorem 2 of [Z1] asserting that every embeddable automorphism of odd order of a compact Riemann surface is the square of an orientation-reversing self-homeomorphism. Finally we study whether a conformal involution is the square of an orientation-reversing automorphism.

Let f be a conformal automorphism of prime order of a Riemann surface X and $\mathrm{Fix}(f)$ be the fixed point set of f. Given a point $p \in \mathrm{Fix}(f)$, there exists a chart (U, ϕ) on X such that $\phi(p) = 0$ and $\phi f \phi^{-1}(z) = z \exp i\alpha$. Now $\alpha = \alpha(f, p)$ is unique up to a multiple of 2π, and it is independent of the choice of the chart. We normalize α by requiring $-\pi < \alpha \leq \pi$.

The following remark will be very useful.

Remark 1. If the genus n of X satisfies $n > 1$, then there is a surface fuchsian group Λ uniformizing X and a fuchsian group Δ containing Λ such that X/f can

Received by the editors December 30, 1993 and, in revised form, April 29, 1994 and September 16, 1994.

1991 *Mathematics Subject Classification.* Primary 30F99.

The author was partially supported by DGICYT PB 92-0716 and EU project CHRX-CT93-408.

be uniformized as orbifold by Δ. If f has order r, then there is a monodromy epimorphism $\omega : \Delta \to \mathbf{Z}_r = \Delta/\Lambda$.

Assume that r is prime; then the group Δ has signature $(g'; [r, r, \ldots^{s} \ldots, r])$. Let $x_1, \ldots, x_s$ be the elliptic elements of a canonical generator system of Δ. The fixed point of the elliptic transformation x_i projects by $D \to X = D/\Lambda$ on a fixed point of f that we shall call p_i, $i = 1, \ldots, s$. Then $\alpha(f, p_i) = \alpha(f, p_j)$ (respectively $\alpha(f, p_i) = -\alpha(f, p_j)$) if and only if $\omega(x_i) = \omega(x_j)$ (resp. $\omega(x_i) = -\omega(x_j)$). A similar remark applies in the cases $n = 0$ and $n = 1$, replacing the fuchsian groups by spherical or euclidean crystallographic groups.

Example. Let Γ be an NEC (non-euclidean crystallographic) group with signature (see [M])

$$(1; -, [5, 5], \{-\}),$$

which means that Γ acts as a discrete group of conformal and anticonformal transformations of the complex disc D and D/Γ is a nonorientable 2-orbifold of genus 1 with two conic points having isotropy group of order five. The group Γ has a canonical presentation of the form

$$(d, x_1, x_2; x_1^5 = x_2^5 = 1, d^2 x_1 x_2 = 1).$$

Let $\theta : \Gamma \to \mathbf{Z}_{10}$ be the epimorphism given by $\theta(d) = 1, \theta(x_1) = 2$, and $\theta(x_2) = 6$. The Riemann surface $X = D/\operatorname{Ker}\theta$ has an anticonformal automorphism g of order 10 such that g acts as $\theta^{-1}(1)$ on $D/\operatorname{Ker}\theta$. The conformal automorphism $f = g^2$ has four fixed points, and the orbifold X/f is uniformized by $\theta^{-1}(\langle 2\rangle)$. Hence the monodromy epimorphism for the branched covering $X \to X/f$ is given by the restriction of θ to $\theta^{-1}(\langle 2\rangle)$. A canonical presentation of $\theta^{-1}(\langle 2\rangle)$ is

$$\begin{aligned}(x_1, x_2, (x_2^{-1})^d, (x_1^{-1})^d; x_1 x_2 (x_2^{-1})^d (x_1^{-1})^d = 1,\\ x_1^5 = x_2^5 = ((x_2^{-1})^d)^5 = ((x_2^{-1})^d)^5 = 1).\end{aligned}$$

Since $\theta(x_1) \neq \theta(x_2)$ and $\theta(x_1) \neq -\theta(x_2)$, if p_1 and p_2 are the projections by $D \to X = D/\operatorname{Ker}\theta$ of the fixed points of x_1 and x_2 respectively, then $|\alpha(f, p_1)| \neq |\alpha(f, p_2)|$. Thus by [R2] the automorphism f is not embeddable.

The following theorem is a consequence from [R2].

Theorem 1. *Let X be a compact Riemann surface with an anticonformal automorphism g of order $2r$ with r prime. Then $f = g^2$ is embeddable if and only if $|\alpha(f, p)| = |\alpha(f, q)|$ for any two fixed points of f, p, and q.*

Proof. Let g be an anticonformal automorphism g of order $2r$ with r prime. Since r is prime, then $\operatorname{Fix}(f) = \operatorname{Fix}(\langle f\rangle)$. If p is a fixed point of f, then $g(p)$ is also fixed by f, so f has an even number of fixed points. Assume that p is a fixed point of f. If (U, ϕ) is a chart on X such that $\phi(p) = 0$ and $\phi f \phi^{-1}(z) = z \exp i\alpha$ and c is the conjugation in $\mathbf{C}$, then $(g(U), \psi = c\phi g^{-1})$ is a chart on X such that $\psi(g(p)) = 0$ and $\psi f \psi^{-1}(z) = z \exp i(-\alpha)$. Hence $\alpha(f, p) = -\alpha(f, g(p))$ or $\alpha(f, p) = \alpha(f, g(p)) = \pi$. Thus the only condition of the Theorem of [R2] that f must satisfy in order to be embeddable is $|\alpha(f, p)| = |\alpha(f, q)|$ for any two fixed points p and q.

The following theorem was proved by R. Zarrow in [Z1]. Here we give an alternative proof. □

Theorem 2. *Let X be a Riemann surface and f be an embeddable conformal automorphism of odd order of X. Then f is the square of an orientation-reversing self-homeomorphism of X.*

Proof. Suppose that f is an embeddable conformal automorphism of odd order r. By the Theorem of [R2] the points of $\mathrm{Fix}(f)$ can be distributed in pairs (p_i, p_i') such that $\alpha(f, p_i) = -\alpha(f, p_i')$. Assume that X has genus $n > 1$ (the cases $n = 0$ or $n = 1$ are treated similarly, replacing the hyperbolic crystallographic groups by euclidean or spherical crystallographic groups). Let Δ be the group that uniformizes the orbifold X/f and $x_1, \ldots, x_s, x_1', \ldots, x_s'$ be the elliptic elements of a canonical presentation of Δ. Let $\omega : \Delta \to \mathbf{Z}_r$ be the monodromy epimorphism of the covering $X \to X/f$ which satisfies $\omega(x_i) = -\omega(x_i')$ for $i = 1, \ldots, s$ by hypothesis (see Remark 1). Let n^* be the genus of Δ. Let Γ be an NEC (euclidean or spherical crystallographic) group with signature:

$$(n'; +; [r, \ldots^{s} \ldots, r]; \{(-) \ldots^{k} \ldots (-)\}),$$

where $(n', k) = (n^*/2, 1)$ if n^* is even and $(n', k) = ((n^* - 1)/2, 2)$ if n^* is odd.

Let $(a_1, b_1, \ldots, a_{n'}, b_{n'}, x_1^*, \ldots, x_s^*,\ c_1, \ldots, c_k, e_1, \ldots, e_k;\ (x_1^*)^r = \cdots = (x_s^*)^r = 1, c_i^2 = 1,\ e_i c_i e_i^{-1} = c_i,\ i = 1, \ldots, k, [a_1, b_1] \ldots [a_{n'}, b_{n'}] x_1^* \cdots x_s^* e_1 \cdots e_k = 1)$ be a canonical presentation of Γ. Let $j : \mathbf{Z}_r \to \mathbf{Z}_{2r}$ be the inclusion given by the multiplication by 2. Let $\theta : \Gamma \to \mathbf{Z}_{2r}$ be the epimorphism defined by $\theta(a_1) = \theta(b_1) = \cdots = \theta(a_{n'}) = \theta(b_{n'}) = 0, \theta(c_i) = r, i = 1, \ldots, k, \theta(e_1) = j(-\sum_{i=1}^{s} \omega(x_i))$, $\theta(e_i) = 0,\ i = 2, \ldots, k$, and $\theta(x_i^*) = j(\omega(x_i))$, $i = 1, \ldots, s$. Then the surface $X' = D/\operatorname{Ker} \theta$ has an anticonformal automorphism g' such that g'^2 and f have the same order and the same valencies on the branched points of the natural projections $X' \to X'/g'^2$ and $X \to X/f$. By the Equivalence Theorem in Section 11 of [N], g'^2 and f are topologically equivalent, i.e. there is a homeomorphism $h : X \to X'$ such that $f = h^{-1} g'^2 h$. Thus $g = h^{-1} g' h$ is the orientation-reversing homeomorphism that we are looking for.

By [R2] every conformal involution of a Riemann surface is embeddable. The following result shows that not every conformal involution is the square of an orientation-reversing self-homeomorphism. □

Theorem 3. *Let X be a Riemann surface of genus n and f be a conformal (embeddable) involution of X with $2s$ fixed points. Then f is the square of an orientation-reversing self-homeomorphism if and only if $n + s \equiv 1 \bmod 4$.*

Proof. Assume that f is the square of an orientation-reversing self-homeomorphism g of X. There is a Riemann surface X' and an anticonformal automorphism g' such that there is a homeomorphism $h : X \to X'$ satisfying $g' = hgh^{-1}$ (see for instance [GM]). Assume that X has genus $n > 1$, and let Γ be the NEC group that uniformizes the orbifold X'/g' (see [S]). Since the order of g' is 4, the orbifold X'/g' has no boundary (see [BC] or [Y]). Then the group Γ has signature

$$(n^*, -, [2, \ldots^{s} \ldots, 2], \{-\}).$$

Assume that

$$(d_1, \ldots, d_{n^*}, x_1^*, \ldots, x_s^*; (x_1^*)^2 = \cdots = (x_s^*)^2 = 1, d_1^2, \ldots, d_{n^*}^2 x_1^* \cdots x_s^* = 1)$$

is a canonical presentation of Γ, and let $\theta : \Gamma \to \mathbf{Z}_4$ be the monodromy epimorphism of the covering $X' \to X'/g'$. Since X/f is orientable, then $\theta(d_j) = 1$ or 3 and $\theta(d_j^2) = 2, j = 1, \ldots, n^*$. As $\theta(x_i^*) = 2, i = 1, \ldots, s$, and $\theta(d_1^2, \ldots, d_{n^*}^2 x_1^* \cdots x_s^*) = 0$, then $n^* + s$ must be even. Thus by the Riemann-Hurwitz formula we have that $n + s \equiv 1 \bmod 4$. (When the genus g of X satisfies $n \leq 1$, the proof is the same, taking for Γ the corresponding euclidean or spherical crystallographic group.)

Suppose that f is a conformal involution of X. Assume that X has genus $n > 1$ (the cases $n = 0$ or $n = 1$ are treated similarly, replacing the hyperbolic crystallographic groups by euclidean or spherical crystallographic groups). Let Δ be the group that uniformizes the orbifold X/f and $x_1, \dots, x_s, x'_1, \dots, x'_s$ be the elliptic elements of a canonical presentation of Δ. Assume that Δ has genus m in its signature. Let $\omega : \Delta \to \mathbf{Z}_2$ be the monodromy epimorphism of the covering $L : X \to X/f$. Let Γ be a NEC (euclidean or spherical crystallographic) group with signature

$$(m+1; -; [2, \overset{s}{\dots}, 2]; \{-\})$$

and

$$(d_1, \dots, d_{m+1}, x_1^*, \dots, x_s^*; (x_1^*)^2 = \dots = (x_s^*)^2 = 1,$$
$$d_1^2, \dots, d_{m+1}^2 x_1^* \cdots x_s^* = 1)$$

be a canonical presentation of Γ. By the Riemann-Hurwitz formula the condition $n + s \equiv 1 \mod 4$ is equivalent to $m + s \equiv 1 \mod 2$. Then we can define $\theta : \Gamma \to \mathbf{Z}_4$ by $\theta(d_1) = \dots = \theta(d_{m+1}) = 1$ and $\theta(x_i^*) = 2$, $i = 1, \dots, s$. Hence the surface $X' = D/\operatorname{Ker}\theta$ has an anticonformal automorphism g' such that g'^2 is topologically equivalent to f (see the Equivalence Theorem in Section 11 of [N]), i.e. there is a homeomorphism $h : X \to X'$ such that $f = h^{-1} g'^2 h$. Thus $g = h^{-1} g' h$ is the orientation-reversing self-homeomorphism that we need. □

Remark 2. There exist Riemann surfaces admitting a conformal automorphism which is the square of an orientation-reversing self-homeomorphism but not the square of an anticonformal automorphism. The construction of such examples is easy using maximal NEC groups (see [BCF]).

Acknowledgment

The author would like to thank the referee for important corrections and suggestions.

References

[BC] E. Bujalance and A. F. Costa, *Orientation reversing automorphisms of Riemann surfaces*, Illinois J. Math. **38** (1994), 616–623. CMP 94:14

[BCF] E. Bujalance, A. F. Costa, and A. Fernández, *Uniformization of Klein surfaces by maximal NEC groups*, Rev. Roumaine Math. Pures Appl. **40** (1995), 39–54.

[G] A. M. Garsia, *An embedding of closed Riemann surfaces in Euclidean space*, Comment. Math. Helv. **35** (1961), 93–110. MR **23:**A2890

[GM] N. Greenleaf and C. L. May, *Bordered Klein surfaces with maximal symmetry*, Trans. Amer. Math. Soc. **274** (1982), 265–283. MR **84f:**14022

[M] A. M. Macbeath, *The classification of non-euclidean crystallographic groups*, Canad. J. Math. **19** (1967), 1192–1205. MR **36:**3890

[N] J. Nielsen, *Die Struktur periodischer Transformationen von Flächen*, Danske Vid Selsk. Mat-Fys. Medd. **1** (1937), 1–77.

[R1] R.A. Rüedy, *Embeddings of open Riemann surfaces*, Comment. Math. Helv. **46** (1971), 214–225. MR **46:**3781

[R2] ———, *Symmetric embeddings of Riemann surfaces*, Princeton Univ. Press, Princeton, NJ, 1971, pp. 406–418. MR **52:**3512

[S] D. Singerman, *Symmetries and pseudosymmetries of hyperbolic surfaces*, Glasgow Math. J. **21** (1980), 39–49. MR **81c:**30080

[Y] K. Yocoyama, *Complete classification of periodic maps on compact surfaces*, Tokyo J. Math. **15** (1992), 247–279.

[Z1] R. Zarrow, *Anticonformal automorphisms of compact Riemann surfaces*, Proc. Amer. Math. Soc. **54** (1976), 162–164. MR **53:**8412

[Z2] ———, *Orientation reversing maps of surfaces*, Illinois J. Math. **23** (1979), 82–92. MR **80k:**57003b

[Z3] ———, *A counting function for orientation reversing maps*, Illinois J. Math. **24** (1980), 469–476. MR **81f:**58014

Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED, 28040 Madrid, Spain

E-mail address: antonio.costa@uned.es