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THE CLASSIFICATION OF TWO-COMPONENT CUNTZ-KRIEGER ALGEBRAS

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ABSTRACT. Cuntz-Krieger algebras with exactly one nontrivial closed ideal are classified up to stable isomorphism by the Cuntz invariant. The proof relies on Rørdam's classification of simple Cuntz-Krieger algebras up to stable isomorphism and the author's classification of two-component reducible topological Markov chains up to flow equivalence.

1. PRELIMINARIES

Let $A \in \mathrm{M}_n(\{0,1\})$ be nondegenerate (no zero rows or columns) such that no irreducible component is a permutation matrix. The Cuntz-Krieger C^*-algebra (CK-algebra) $\mathcal{O}_A$ associated to A is defined in [CK] as the C^*-algebra generated by partial isometries $s_1, s_2, \ldots, s_n$, satisfying the relations

$$1 = s_1 s_1^* + s_2 s_2^* + \cdots + s_n s_n^*,$$

$$s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^*.$$

The assumption ruling out permutation components is needed to guarantee that $\mathcal{O}_A$ is uniquely defined up to isomorphism [CK],[C1].

Cuntz showed in [C1] that there is a bijective correspondence between the closed ideals of $\mathcal{O}_A$ and the hereditary subsets of the poset Γ_A of irreducible components of A. In particular, $\mathcal{O}_A$ is simple if and only if A is irreducible, and $\mathcal{O}_A$ has exactly one nontrivial closed ideal if and only if A is indecomposable (that is, Γ_A is not a union of two order-disconnected proper subsets) and has exactly two irreducible components. In this note, we will discuss exclusively the latter case. For A decomposable with two irreducible components, the associated algebra $\mathcal{O}_A$ is simply the direct sum of its two simple CK-subalgebras (ideals). In this case, the classification of two-component CK-algebras is trivially reduced to the classification of simple CK-algebras which has been accomplished just recently in [R],[C3].

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Given $A \in \mathrm{M}_n(\{0,1\})$, one can also define a topological Markov chain (i.e., shift of finite type) σ_A associated to A. If Σ is a set of n symbols, then σ_A is the shift homeomorphism of the compact space

$$X_A = \{\, x = (x_i) \in \Sigma^{\mathbb{Z}} \mid A(x_i, x_{i+1}) = 1 \,\}$$

defined by $(\sigma_A x)_i = x_{i+1}$. (See [Bo] for symbolic dynamics background and references.)

Two topological Markov chains (or their adjacency matrices) are said to be flow equivalent (FE) if their suspension flow spaces are homeomorphic under a homeomorphism that maps flow lines onto flow lines and preserves the orientation of the flow lines. Parry and Sullivan [PS] found matrix generators of flow equivalence and showed that $\det(I - A)$ is an invariant of flow equivalence. Later Bowen and Franks [BF] showed that the finitely generated abelian group

$$BF(A) := \frac{\mathbb{Z}^n}{(I - A)\mathbb{Z}^n}$$

is an invariant of flow equivalence, which we will call the Bowen-Franks group of σ_A (or A). Franks showed in [F] that the above two invariants completely characterize flow equivalence of irreducible topological Markov chains whose adjacency matrices are not a permutation. Note that $BF(A)$ determines $\det(I - A)$ up to its sign: $0, +$, or $-$, which we will denote by $sgn\,(A)$.

The stabilization of $\mathcal{O}_A$ is $\bar{\mathcal{O}}_A := \mathcal{O}_A \otimes \mathcal{K}$, where $\mathcal{K}$ is the the algebra of all compact operators on a Hilbert space. $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ are said to be stably isomorphic, if $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$.

Theorem 1.1 (Cuntz-Krieger [CK], [C1]). *$A \sim_{FE} A'$ implies $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$. Moreover,* $\mathrm{Ext}\,(\mathcal{O}_A) \cong BF(A)$, $K_0(\mathcal{O}_A) \cong BF(A^t)$ *and* $K_1(\mathcal{O}_A) \cong \mathrm{Ker}\,(I - A^t)$ *(on* $\mathbb{Z}^n$*).*

For $\mathcal{O}_A$ simple, Mikael Rørdam showed recently in [R] that $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$ if and only if $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_{A'})$. On the other hand, it was shown in [H] that for a nonpermutation irreducible matrix A, any automorphism of $BF(A)$ can be realized by a flow equivalence $A \sim_{FE} A$. Using these new results, Cuntz showed that $\mathcal{O}_A \cong \mathcal{O}_{A'}$ (unital isomorphism) if and only if there is an isomorphism $\alpha_0 : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_{A'})$ such that $\alpha_0[1_A] = [1_{A'}]$ ([R],[C3]). Thus the classification of simple CK-algebras, a problem that had been open for more than twelve years, was solved completely.

2. Stable isomorphisms of two-component CK-algebras

Cuntz [C1] began the study of nonsimple CK-algebras and flow equivalence of reducible topological Markov chains. He especially discussed the case in which $\mathcal{O}_A$ has exactly one nontrivial closed ideal. In that case, the indecomposable defining $\{0,1\}$-matrix A, modulo conjugation by a permutation matrix, has the block upper-triangular form

$$A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}, \tag{2-1}$$

where the square submatrices A_1 and A_2 are essentially irreducible (that is, their maximum nondegenerate principal submatrices are irreducible) and neither irreducible component is a permutation. Some important properties of endomorphisms of simple $\mathcal{O}_A$ were proved by studying the corresponding two-component CK-algebras [C4].

Suppose $A_1 \in M_n(\mathbb{Z}_+)$ and $A_2 \in M_m(\mathbb{Z}_+)$. Then X can be identified with an element of $\mathbb{Z}^n \times \mathbb{Z}^m$ through

$$X = \sum_{\substack{1\le i\le n \\ 1\le j\le m}} X(ij)(e_i \otimes f_j),$$

where $e_1, \dots, e_n$ and $f_1, \dots, f_m$ are the standard bases of $\mathbb{Z}^n$ and $\mathbb{Z}^m$. Let $q_1 : \mathbb{Z}^n \to \mathbb{Z}^n/(I - A_1)\mathbb{Z}^n$ and $q_2 : \mathbb{Z}^m \to \mathbb{Z}^m/(I - A_2^t)\mathbb{Z}^m$ be the quotient maps, then X defines naturally an equivalence class $[X] := (q_1 \otimes q_2)(X) \in BF(A_1) \otimes BF(A_2^t)$.

Consider

$$A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}, \qquad A' = \begin{pmatrix} A_1' & X' \\ 0 & A_2' \end{pmatrix}. \tag{2-2}$$

Definition 2.1. Let $A,\ A' \in \mathrm{M}_k(\mathbb{Z}_+)$ be indecomposable with two irreducible components and having the forms as in (2-2), where A_1, A_2, A_1' and A_2' are essentially irreducible (but can be essential permutations). The Cuntz invariant $\mathcal{C}(A)$ of A is the pair $([X],\ BF(A_1) \otimes BF(A_2^t))$, modulo product-type isomorphisms. Precisely, we say $\mathcal{C}(A) = \mathcal{C}(A')$ if there exist isomorphisms $\delta :\ BF(A_1) \to BF(A_1')$ and $\eta :\ BF(A_2^t) \to BF(A_2'^t)$ such that $(\delta \otimes \eta)([X]) \equiv [X']$ in $BF(A_1') \otimes BF(A_2'^t)$ (see [C1],[H]).

Theorem 2.2 (Cuntz [C1])**.** *Let A and A' be indecomposable $\{0,1\}$-matrices and have the forms of (2-2) with essentially irreducible nonpermutation diagonal blocks. Then $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$ implies $\mathcal{C}(A) = \mathcal{C}(A')$. In particular, $A \sim_{FE} A'$ implies $\mathcal{C}(A) = \mathcal{C}(A')$.*

Remark 2.3. The invariant $\mathcal{C}(A)$ was derived in [C1] via C^*-algebra extensions (KK-theory), in the case where the diagonal blocks A_1 and A_2 of A are not essential permutation matrices. Given A and A', for simplicity we suppose all the diagonal blocks are irreducible, there are short exact sequences $\rho : \bar{\mathcal{O}}_{A_2} \rightarrowtail \mathcal{O}_A \twoheadrightarrow \mathcal{O}_{A_1}$ and $\rho' : \bar{\mathcal{O}}_{A_2'} \rightarrowtail \mathcal{O}_{A'} \twoheadrightarrow \mathcal{O}_{A_1'}$. Cuntz showed that if $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$, then there is a product-type isomorphism $\beta : \mathrm{Ext}(\mathcal{O}_{A_1}, \mathcal{O}_{A_2}) \to \mathrm{Ext}(\mathcal{O}_{A_1'}, \mathcal{O}_{A_2'})$ such that $\beta[\rho] = [\rho']$. So the pair $([\rho], \mathrm{Ext}(\mathcal{O}_{A_1}, \mathcal{O}_{A_2}))$ is a stable isomorphism invariant. Cuntz also showed that $[\rho] \in \mathrm{Ext}(\mathcal{O}_{A_1}) \otimes K_0(\mathcal{O}_{A_2}) \subset \mathrm{Ext}(\mathcal{O}_{A_1}, \mathcal{O}_{A_2})$ and the pair $([X] \cong [\rho], BF(A_1) \otimes BF(A_2^t) \cong \mathrm{Ext}(\mathcal{O}_{A_1}) \otimes K_0(\mathcal{O}_{A_2}))$ is a stable isomorphism invariant for $\mathcal{O}_A$.

In [H], we reproved the Cuntz invariant of flow equivalence using matrix techniques from symbolic dynamics and showed the following.

Theorem 2.4 ([H])**.** *Let A and A' be as in Definition 2.1. Then $A \sim_{FE} A'$ if and only if*

(1) $A_i \sim_{FE} A_i',\ i = 1, 2$.

(2) $\mathcal{C}(A) = \mathcal{C}(A')$.

If the irreducible components of A and A' are not permutation matrices, then (1) *can be replaced by* $(1')$: $sgn\,(A_i) = sgn\,(A'_i)$, $i = 1, 2$.

Now we are ready to show that $\mathcal{C}(A)$ alone is a complete stable isomorphism invariant for $\mathcal{O}_A$, as conjectured in [H].

To every matrix $A \in \mathrm{M}_n(\mathbb{Z}_+)$, we associate two matrices $A_{\_} \in \mathrm{M}_{n+2}(\mathbb{Z}_+)$ and $A_{\sim} \in \mathrm{M}_{n+3}(\mathbb{Z}_+)$ given by

$$A_{\_} = \begin{pmatrix} & & & & | & 0 & 0 \\ & & & & | & \vdots & \vdots \\ & & A & & | & 0 & 0 \\ & & & & | & 1 & 0 \\ - & - & - & - & - & - & - \\ 0 & \cdots & 0 & 1 & | & 1 & 1 \\ 0 & \cdots & 0 & 0 & | & 1 & 1 \end{pmatrix} \quad \text{and} \quad A_{\sim} = \begin{pmatrix} & & & | & 0 & 0 & 1 \\ & A & & | & \vdots & \vdots & \vdots \\ & & & | & 0 & 0 & 1 \\ - & - & - & - & - & - & - \\ 1 & \cdots & 1 & | & 0 & 0 & 1 \\ 0 & \cdots & 0 & | & 0 & 1 & 1 \\ 0 & \cdots & 0 & | & 1 & 1 & 1 \end{pmatrix},$$

respectively ([C2],[R]). It is easy to check that $\det(I - A_{\_}) = \det(I - A_{\sim}) = -\det(I - A)$.

Lemma 2.5. *Let $e_1, \ldots, e_n, e_{n+1}, e_{n+2}, e_{n+3}$ be the standard basis for $\mathbb{Z}^{n+3}$ (column space). Define*

$$\delta_{\_} : BF(A) \to BF(A_{\_}) \quad by \quad [\sum_{i=1}^{n} \xi_i e_i] \mapsto [\sum_{i=1}^{n} \xi_i e_i + 0 \cdot e_{n+1} + 0 \cdot e_{n+2}]$$

and

$$\delta_{\sim} : BF(A) \to BF(A_{\sim}) \quad by \quad [\sum_{i=1}^{n} \xi_i e_i] \mapsto [\sum_{i=1}^{n} \xi_i e_i + 0 \cdot e_{n+1} + 0 \cdot e_{n+2} + 0 \cdot e_{n+3}],$$

where $\xi_i \in \mathbb{Z}$. Then both $\delta_{\_}$ and $\delta_{\sim}$ are group isomorphisms. Similarly, we can define group isomorphisms

$$\eta_{\_} : BF(A^t) \to BF(A^t_{\_}) \quad and \quad \eta_{\sim} : BF(A^t) \to BF(A^t_{\sim}).$$

Proof. Linear algebra. □

Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$; then

$$A_{\_} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Denote by $\mathcal{O}_2$ and $\mathcal{O}_{2\_}$ the CK-algebras defined by the two matrices above. Cuntz showed in [C2] that $\mathcal{O}_2 \cong \mathcal{O}_{2\_}$ implies $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A\_}$ for any $\{0,1\}$-matrix A. This result of Cuntz, together with Franks' classification of irreducible topological Markov chains up to flow equivalence in [F], as well as Theorem 1.1, reduces the classification of simple CK-algebras up to stable isomorphism to the question: Is $\mathcal{O}_2 \cong \mathcal{O}_{2\_}$? It has been open for many years until recently, Rørdam [R] provides a positive answer.

The following is the main result of this note.

Theorem 2.6. *Let A and A' be indecomposable $\{0,1\}$-matrices with two irreducible components as in Theorem 2.2. Then $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$ if and only if $\mathcal{C}(A) = \mathcal{C}(A')$, that is, they have the same Cuntz invariant as defined in Definition 2.1.*

Proof. By Theorem 2.2, it suffices to prove the sufficiency.

Suppose $\mathcal{C}(A) = \mathcal{C}(A')$ (see Definition 2.1 and (2-2)).

If $sgn\,(A_i) = sgn\,(A'_i)$, $i = 1, 2$, then by Theorems 1.1 and 2.4, we have $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$.

Otherwise, suppose first that $sgn\,(A_1) = -sgn\,(A'_1)$, and $A_2 = A'_2$. We shall derive a "diagonal block changing" method for CK-algebras, analogous to the Blocking Lemma for flow equivalence in [H].

Let

$$A_* = \begin{pmatrix} A_{1\,*} & | & X_* \\ - & - & - \\ 0 & | & A_2 \end{pmatrix}, \quad \text{where} \quad A_{1\,*} := (A_1)_\sim,\ X_* := \begin{pmatrix} & X & \\ - & - & - \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$A_{**} = \begin{pmatrix} A_{1\,**} & | & X_{**} \\ - & - & - \\ 0 & | & A_2 \end{pmatrix}, \quad \text{where} \quad A_{1\,**} := (A_{1\,*})_-, \quad X_{**} := \begin{pmatrix} & X_* & \\ - & - & - \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}.$$

By Lemma 2.5, it is easily seen that

$$\mathcal{C}(A) = \mathcal{C}(A_*) = \mathcal{C}(A_{**}).$$

Note that $\det(I - A_1) = \det(I - A_{1\,**})$ and $sgn\,(A'_1) = sgn\,(A_{1\,*})$. By Theorems 1.1 and 2.4, we have $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A_{**}}$, and $\bar{\mathcal{O}}_{A'} \cong \bar{\mathcal{O}}_{A_*}$. Thus in order to show that $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$, it suffices to show that $\mathcal{O}_{A_*} \cong \mathcal{O}_{A_{**}}$. Here we need a trick of Cuntz [C2] to show that $\mathcal{O}_2 \cong \mathcal{O}_{2\_}$ implies $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A\_}$, which was elaborated in the Appendix of [R]. It turns out that the argument there for simple CK-algebras can be adapted for two-component CK-algebras as follows.

Since $\mathcal{O}_2 \cong \mathcal{O}_{2\_}$ (Rørdam [R]), as was shown in the Appendix, there is a C^*-algebra

$$\mathcal{E} = C^*(s_1, s_2, e) = C^*(t_1, t_2, t_3, t_4, e) \tag{2-3}$$

with unit 1, where e is a one-dimension projection, s_i's and t_j's are partial isometries satisfying

$$1 = s_1 s_1^* + s_2 s_2^*, \quad s_1^* s_1 = s_2^* s_2 = 1;$$

$$\begin{aligned} 1 &= t_1 t_1^* + t_2 t_2^* + t_3 t_3^* + t_4 t_4^*, \\ t_1^* t_1 &= t_1 t_1^* + t_2 t_2^*, \\ t_2^* t_2 &= t_1 t_1^* + t_2 t_2^* + t_3 t_3^*, \\ t_3^* t_3 &= t_2 t_2^* + t_3 t_3^* + t_4 t_4^*, \\ t_4^* t_4 &= t_3 t_3^* + t_4 t_4^*; \end{aligned} \tag{2-4}$$

and $es_1 = s_1 e = e$, $et_1 = t_1 e = e$.

Suppose the sizes of A_1 and A_2 are n and m, respectively. Let $\mathfrak{A}$ be the algebra $\mathbb{C} \oplus \cdots \oplus \mathbb{C} \oplus \mathcal{E}$, with $n+m$ copies of $\mathbb{C}$, and denote by $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ the one-dimension projections onto each of these $n+m$ copies. Represent $\mathfrak{A}$ on a Hilbert space H such that every nonzero projection in $\mathfrak{A}$ becomes infinite on H.

Then find partial isometries $x_1, \ldots, x_{n+1}$ on H satisfying

$$x_i x_i^* = f_i, \quad x_i^* x_i = s_2 s_2^* + \sum_{j=1}^{n} A_1(i,j) f_j + \sum_{j=1}^{m} X(i,j) g_j,$$

for $i = 1, \ldots n$ and

$$x_{n+1} x_{n+1}^* = e, \quad x_{n+1}^* x_{n+1} = s_2 s_2^* + f_1 + \cdots + f_n.$$

Let $x_{n+2} = s_1(1-e)$ and $x_{n+3} = s_2$. Then find partial isometries $\hat{x}_1, \ldots, \hat{x}_m$ on H satisfying

$$\hat{x}_i \hat{x}_i^* = g_i, \quad \hat{x}_i^* \hat{x}_i = \sum_{j=1}^{m} A_2(i,j) g_j,$$

for $i = 1, \ldots, m$. Let $x_{n+3+i} = \hat{x}_i$, $i = 1, \ldots, m$. Then it is easy to check as in [R] that the C^*-algebra $\mathfrak{B}_1$ generated by $x_1, \ldots, x_{n+m+3}$ contains $\mathfrak{A}$ and is isomorphic to $\mathcal{O}_{A_*}$.

Next, find partial isometries $u_1, \ldots, u_{n+1}$ in $\mathfrak{A}$ so that $u_i u_i^* = x_i^* x_i$, and

$$u_i^* u_i = t_2 t_2^* + \sum_{j=1}^{n} A_1(i,j) f_j + \sum_{j=1}^{m} X(i,j) g_j \quad (i = 1, \ldots, n),$$

$$u_{n+1}^* u_{n+1} = t_2 t_2^* + f_1 + \cdots + f_n.$$

Such u_i's exist because $s_2 s_2^*$ is equivalent to $t_2 t_2^*$ in $\mathcal{E}$ due to (2-3) and (2-4). Set $y_i = x_i u_i$ (hence $x_i = y_i u_i^*$) for $i = 1, \ldots, n+1$, $y_{n+2} = t_1(1-e)$, $y_{n+3} = t_2$, $y_{n+4} = t_3$, $y_{n+5} = t_4$ and $y_{n+5+i} = \hat{x}_i$, $i = 1, \ldots, m$. Again, we can check that the C^*-algebra $\mathfrak{B}_2$ generated by $y_1, \ldots, y_{n+m+5}$ contains $\mathfrak{A}$ and hence is equal to $\mathfrak{B}_1$. But $\mathfrak{B}_2 \cong \mathcal{O}_{A_{**}}$, therefore $\mathcal{O}_{A_*} \cong \mathcal{O}_{A_{**}}$.

Now consider $A_1 = A_1'$ but $sgn\,(A_2) = -sgn\,(A_2')$. We can change A_2 into A_2' by a stable isomorphism of $\mathcal{O}_A$, with a possible change in the off-diagonal block. The method used to do this is similar to that above, where X_* and X_{**} will be defined correspondingly as matrices obtained from X by adding some zero columns to its right. Therefore, it remains to show that $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$ when they have the same Cuntz invariant and flow equivalent diagonal blocks of their defining matrices. But this follows from Theorems 1.1 and 2.4. □

Remark 2.7. In the proof of Theorem 2.6, we actually have encountered a CK-subalgebra defined by a degenerate $\{0,1\}$-matrix, since A_1 and A_2 are generally essentially irreducible. Being merely essentially irreducible, A_1 and A_2 may not be uniquely defined to make A a 2×2 block upper-triangular matrix. In [C2], A_1 is chosen to be irreducible, while A_2 generally have zero columns and is called the saturation of its irreducible core matrix. However, given a matrix A with its maximal nondegenerate principal submatrix A_{core}, one can see that $\mathcal{O}_A \cong \mathcal{O}_{A_{core}}$ and $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_{A_{core}})$ canonically. Also as was shown in [H], the Cuntz invariant is independent of possible choices of A_1 and A_2 for a 2×2 block upper-triangular matrix A. In fact, in order to compute Cuntz invariant, we can change all essentially irreducible diagonal block matrices into their irreducible cores by flow equivalence (but then the resulting matrix might be over $\mathbb{Z}_+$ instead of over $\{0,1\}$, see [H]). We have seen that Cuntz invariant is a perfect stable isomorphism invariant for two-component CK-algebras. However, it is still not clear how one could classify these algebras up to unital isomorphism.

The Cuntz invariant is rather computable. In some special cases, the invariant has an even simpler equivalent version. Here is an example, where all the CK-algebras associated to diagonal block matrices of A are stably isomorphic to the Cuntz algebra $\mathcal{O}_n$ $(n \geq 2)$.

Corollary 2.8. *Let A and A' be as in Theorem 2.6. Suppose that $K_0(\mathcal{O}_{A_i})$ is a finite cyclic group, $i = 1, 2$. Then $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$ if and only if $K_0(\mathcal{O}_{A_i}) \cong K_0(\mathcal{O}_{A_i'})$, $i = 1, 2$, and $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_{A'})$.*

In particular, if A_1 (A_1') is an $n \times n$ $(n' \times n')$ matrix with all entries being 1 and A_2 (A_2') is an $m \times m$ $(m' \times m')$ matrix also with all entries being 1, where $n, n', m, m' \geq 2$, and X and X' are not a zero matrix, then $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$ if and only if $n = n'$, $m = m'$ and $gcd\{n-1, m-1, l\} = gcd\{n'-1, m'-1, l'\}$, where $l = \sum_{i,j} X(i,j)$ and $l' = \sum_{i,j} X'(i,j)$.

Proof. It suffices to prove the sufficiency. For the first part, as in the proof Theorem 2.6, we can assume that $sgn(A_i) = sgn(A_i') = -$, $i = 1, 2$. Notice that $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_{A'})$ if and only if $BF(A) \cong BF(A')$. Thus we have $A_i \sim_{FE} A_i'$, $i = 1, 2$. It then follows from Proposition 3.7 of [H] that $A \sim_{FE} A'$, and by Theorem 1.1 we have $\bar{\mathcal{O}}_A \cong \bar{\mathcal{O}}_{A'}$.

For the second part of the corollary, it suffices to show that $A \sim_{FE} A'$. Using the inverse operation of state splitting for topological Markov chains (see [F], where a state splitting is given explicitly by a splitting of the i-th row (column) and a replicating of the i-th column (row) of the adjacency matrix; also cf. [Bo]), we can see that A is strongly shift equivalent to $\left(\begin{smallmatrix} n & l \\ 0 & m \end{smallmatrix}\right)$, and A' is strongly shift equivalent to $\left(\begin{smallmatrix} n' & l' \\ 0 & m' \end{smallmatrix}\right)$. Note that strong shift equivalence implies flow equivalence and that the Cuntz invariants are preserved during these moves of flow equivalence. The rest then again follows from Proposition 3.7 of [H]. □

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References

[Bo] M. Boyle, *Symbolic dynamics and matrices*, Combinatorial and Graph-Theoretic Problems in Linear Algebra, IMA Volumes in Math and Appl. Vol. 50 (R. Brualdi, S. Friedland, and V. Klee, eds.), Springer-Verlag, New York 1993, pp. 1–38. MR **94g:**58062

[BF] R. Bowen and J. Franks, *Homology for zero-dimensional basic sets*, Ann. of Math. (2) **106** (1977), 73–92. MR **56:**16692

[C1] J. Cuntz, *A class of C*-algebras and topological Markov chains* II: *reducible chains and the Ext-functor for C*-algebras*, Inventiones Math. **63** (1981), 25–40. MR **82f:**46073b

[C2] ———, *The classification problem for the C*-algebras $\mathcal{O}_A$*, Geometric methods in operator algebras (H. Araki and E. G. Effros, eds.), Longman, New York, 1986, pp. 145–151. MR **88a:**46081

[C3] ______, Personal communication.

[C4] ______, *On the homotopy groups of endomorphisms of a C^*-algebra* (*with applications to topological Markov chains*), Proc. of O.A.G.R. Conference in Neptun, Romania, Pitman, 1984, pp. 124–137. MR **86a:**46093

[CK] J. Cuntz and W. Krieger, *A class of C^*-Algebras and topological Markov chains*, Inventiones Math. **56** (1980), 251–268. MR **82f:**46073a

[F] J. Franks, *Flow equivalence of subshifts of finite type*, Ergodic Theory Dynamical Systems **4** (1984), 53–66. MR **86j:**58078

[H] D. Huang, *Flow equivalence of reducible shifts of finite type*, Ergodic Theory Dynamical Systems **14** (1994), 695–720. CMP 95:04

[PS] W. Parry and D. Sullivan, *A topological invariant for flows on one dimensional spaces*, Topology **14** (1975), 297–299.

[R] M. Rørdam, *Classification of Cuntz-Krieger algebras*, K-theory **9** (1995), 31–58.

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