A NOTE ON THE WEIGHTED NORM INEQUALITY FOR THE ONE-SIDED MAXIMAL OPERATOR

LAI QINSHENG

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Abstract. Let $M_+^g$ be the one-sided maximal function. In this note we obtain some necessary and sufficient conditions in order that the weighted weak type inequality holds for $M_+^g$. Meanwhile, some necessary or sufficient conditions for the weighted inequality for $M_+^g$ are given.

1. Introduction

Given a positive and locally integrable function $g$ on the real line $R$, the one-sided maximal function $M_+^g$ is defined by

\begin{equation}
M_+^g f(x) = \sup_{h>0} \frac{1}{g(x, x+h)} \int_x^{x+h} |f(y)|g(y) \, dy
\end{equation}

for $f \in L_{\text{loc}}^1(g(x) \, dx)$, where $g(x, x+h) = \int_x^{x+h} g(y) \, dy$.

Symmetrically, the left maximal function $M_-^g$ can be defined in the obvious way.

Recently, weighted inequalities for the operators $M_+^g$ and $M_-^g$ have been extensively studied (see [9], [6], [5], [4]). In this note we will characterize the pairs of weights $(w, v)$ such that $M_+^g$ (or $M_-^g$) is of weak type $(1, q)$ with $1 \leq q$. Secondly, we will give a necessary condition on the pairs $(w, v)$ in order that the weak type $(p, q)$ inequality with $0 < p < q < \infty$ holds for $M_+^g$ (or $M_-^g$) with respect to the measures $wdx$ and $vdx$. This result shows that most weight functions do not verify the weighted weak type $(p, q)$ inequality when $1 < p < q < \infty$. In the case of $1 \leq p$ and $0 < q < p$ we will introduce a $B_+^g(p, q)$ (or $B_-^g(p, q)$) condition and prove that this condition is sufficient for the weak type $(p, q)$ inequality and necessary for the strong type $(p, q)$ inequality for the operator $M_+^g$ (or $M_-^g$). Finally, we will prove analogies of Sawyer’s theorem in [10] and Verbitsky’s theorem in [11]. They characterize the weak type $(p, q)$ inequality when $1 \leq p < \infty$ and $0 < q < p$.

Throughout this paper, $w$ and $v$ always are weight functions by which we mean nonnegative measurable functions taking values in $[0, \infty]$. For a given weight $w$ and measurable set $E$, $w(E) = \int_E w(x) \, dx$ and $\chi_E(x)$ denotes the characteristic function of $E$. Particularly, $|E|$ is the Lebesgue measure of $E$ and $w((a, b))$ is written by $w(a, b)$. For $p \geq 1$, let $p' = \frac{p}{p-1}$ be its conjugate index, and we adopt the usual

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conventions for multiplication in \([0, \infty]\), namely, \(\infty \cdot t = t \cdot \infty = \infty\) for \(0 < t \leq \infty\), \(0 \cdot \infty = 0\), \(\infty^{-1} = 0\) and \(0^{-1} = \infty\).

Now we state our results. We just present our theorems concerning \(M^+_g\), since their counterparts for \(M^-_g\) can be obtained similarly.

**Theorem 1.** Suppose \(1 \leq q < \infty\). Then the weak type \((1, q)\) inequality

\[
(1.2) \quad (\lambda^q w(\{x \in R: M^+_g f(x) > \lambda\})^{1/q} \leq C \int_R |f(x)|^q v(x) \, dx
\]

holds for all \(\lambda > 0\) and measurable \(f\), if and only if \((w, v) \in A^+_g(1, q)\), that is,

\[
(1.3) \quad \sup_{a < b < c} \left( \frac{w(a, b)^{1/q}}{g(a, c)} \text{ess sup}_{x \in (b, c)} \left( \frac{g(x)}{v} \right) \right) = A < \infty.
\]

Moreover, for the best constants \(A\) and \(C\), we have \(A \leq C \leq 4A\).

**Remark 1.** The good weights for the weak type \((p, p)\) inequalities \((p \geq 1)\) were established in [6], [9] and [5]. However, in those articles the condition \(A^+_g(1, 1)\) was represented by

\[
(1.4) \quad M^+_g \left( \frac{w}{g} \right)(x) \leq C \frac{v(x)}{g(x)} \quad \text{for a.e. } x \in R.
\]

Later on, in [4], we established the weak type inequalities for \(M^+_g\) in Orlicz classes. In the particular case of \(1 < p \leq q < \infty\) we obtained the \(A^+_g(p, q)\) condition, i.e.

\[
\sup_{a < b < c} \frac{w(a, b)^{1/q} \sigma(b, c)^{1/p'}}{g(a, c)} = A < \infty,
\]

where \(\sigma = g^{p'}v^{1-p'}\). Our \(A^+_g(1, q)\) condition can be considered as the limiting case of \(A^+_g(p, q)\) as \(p \to 1\). Furthermore, it has an extended version in the case \(0 < q < p = 1\) (cf. Theorem 4). This sort of \(A^+_g(1, q)\) conditions for \(M^+_g\) are new in the literature, although the parallel conditions for usual Hardy-Littlewood maximal operator are well known.

**Theorem 2.** Let \(0 < p \leq q < \infty\) and \((w, v)\) is a pair of weight functions. Suppose the weak type \((p, q)\) inequality

\[
(1.5) \quad (\lambda^q w(\{x \in R: M^+_g f(x) > \lambda\})^{1/q} \leq C \left( \int_R |f(x)|^p v(x) \, dx \right)^{1/p}
\]

holds for all \(\lambda > 0\) and measurable \(f\). Then we have:

(i) For a.e. \(x \in R\), either \(w(x) = 0\) or \(v(x) = \infty\) if \(p < q\).

(ii) There exists a constant \(B\) such that

\[
(1.6) \quad w(x) \leq Bv(x) \quad \text{a.e. } x \in R
\]

if \(p = q\). Moreover, \(B \leq (2C)^p\) for the best constants \(B\) and \(C\) in (1.6) and (1.5) respectively.

**Remark 2.** Theorem 2 indicates that if \(v\) is finite a.e. (e.g. locally integrable) and \(p < q\), then the weight \(w\) must be zero a.e. But, the \(A^+_g(p, q)\) \((1 \leq p \leq q < \infty)\) always contain nontrivial pairs of weight functions, i.e. neither \(w(x) \equiv 0\) nor \(v(x) \equiv \infty\). Here we shall give an example.
Example 1. Since $g$ is locally integrable, there exists $M > 0$ such that

$$|E| = |\{x \in R : g(x) \leq M\}| > 0.$$  

Choose an interval $(a_1, a_2)$ satisfying $|(a_1, a_2) \cap E| > 0$, and fix $a_3 < a_4 < a_1$ arbitrary. Let $w(x) = \chi_{[a_3, a_4]}(x)$, $v(x) = 1$ for $x \in (a_1, a_2) \cap E$ and $v(x) = \infty$ elsewhere. Then $(w, v) \in A^+_{g}(p, q)$ for all $1 \leq p \leq q < \infty$.

Indeed, we only need to verify the $A^+_{g}(p, q)$ conditions for $a < b < c$ with $(a, b) \cap (a_3, a_4) \neq \emptyset$ and $(b, c) \cap (a_1, a_2) \neq \emptyset$. In these cases, it is obvious that $g(b, c) \geq g(a_4, a_1), w(a, b) \leq a_4 - a_3, \sigma(b, c) \leq M'(a_2 - a_1)(p > 1)$ and $\text{ess sup}_{x \in (b, c)} \frac{g(x)}{v(x)} \leq M(p = 1)$. Then the conclusion follows easily.

Definition 1. Suppose $1 \leq p < \infty$ and $0 < q < p$. We say that a pair of weight functions $(w, v) \in B^+_{g}(p, q)$ if

\[
\sum_j \left( \frac{w(a_j, b_j) \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^{1/r} \leq B < \infty \quad \text{(when } p > 1),
\]

\[
\sum_j \left( \frac{w(a_j, b_j) \sigma(b_j, c_j)^{1/q}}{g(a_j, c_j)} \text{ess sup}_{x \in (b_j, c_j)} \left( \frac{g(x)}{v(x)} \right)^{1/r} \right) \leq B < \infty \quad \text{(when } p = 1)
\]

for all sequences $\{(a_j, c_j)\}$ of pairwise disjoint intervals and $b_j \in (a_j, c_j)$. Where $\sigma = \sigma_{g, v, p} = g^p v^{1-p}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. We shall keep these notations throughout this paper.

Theorem 3. Suppose $1 \leq p < \infty$, $0 < q < p$ and $(w, v)$ is a pair of weight functions. If $(w, v) \in B^+_{g}(p, q)$, then the weak type $(p, q)$ inequality (1.5) holds for all $\lambda > 0$ and measurable $f$. Moreover, for the best constants $B$ in Definition 1 and $C$ in (1.5) we have $C \leq 2^{2+1/p'}B$.

On the other hand, if the strong type $(p, q)$ inequality

\[
\left( \int_R [M^+_g f(x)]^q w(x) \, dx \right)^{1/q} \leq C \left( \int_R |f(x)|^p v(x) \, dx \right)^{1/p}
\]

holds for all measurable $f$, then $(w, v) \in B^+_{g}(p, q)$ with $B \leq C$.

Remark 3. We will give some examples of $B^+_{g}(p, q)$ functions, which shows that the $B^+_{g}(p, q)$ condition is not sufficient for the strong type inequality.

Example 2. Suppose $(u, v) \in A^+_{g}(p, p)$ ($p \geq 1$). Choose $\Omega \subset R$ with $u(\Omega) < \infty$. Let $w = u \chi_{\Omega}$. Then $(w, v) \in B^+_{g}(p, q)$ for all $0 < q < p$.

Example 3. Choose $\Omega \subset R$ bounded. Let $w = g$, and $v = g$ on $\Omega$ and $\infty$ elsewhere. Then $(w, v) \in B^+_{g}(p, q)$ for all $1 \leq q < p$. Particularly, setting $g = 1$ and $\Omega = [0, 1]$, we obtain a simple example which satisfies $B^+_{g}(p, 1)$ (therefore the weak type $(p, 1)$ inequality), but the function $f = \chi_{\Omega}$ causes the strong type $(p, 1)$ inequality to fail.

For $q > 1$, an example follows from the corresponding example given by Muckenhoupt in the case $p = q = 2$ (see [7], p. 218).

Example 4. Set $g = 1$. Let $w(x) = x \log(-x)$ on $[-\frac{1}{2}, 0)$ and $0$ elsewhere, and $v(x) = x^2 [\log(-x)]^4$ on $[-\frac{1}{2}, 0)$ and $\infty$ elsewhere. Then $(w, v) \in B^+_{g}(3, 2)$, but the function $f(x) = -\chi_{[-\frac{1}{2}, 0]}(x)/(x \log^2(-x))$ violates the strong type $(3, 2)$ inequality.
**Definition 2.** Suppose \( 1 \leq p < \infty \) and \( 0 < q < p \). For every \((a, b) \subset \mathbb{R}\), let

\[
\rho(a, b) = \sup_{c > b} \frac{w(a, b)^{1/p}}{\sigma(b, c)^{1/p'}} \quad (p > 1),
\]

\[
\rho(a, b) = \sup_{c > b} \left( \frac{w(a, b)}{g(a, c)} \right)^{1/q} \left( \frac{g(x)}{v(x)} \right)^{1/p'} \quad (p = 1).
\]

Define \( \Phi(x) = \Phi_{p, q, w, v}(x) = \sup \{ \rho(a, b) x(a, b) \} \).

**Theorem 4.** Suppose \( 1 \leq p < \infty \) and \( 0 < q < p \). The following statements are equivalent.

(i) The weak type inequality (1.5) holds.

(ii) There exists a constant \( B > 0 \) such that for every sequence \( \{ (a_j, b_j) \} \) of pairwise disjoint \((a_j, b_j)\) and \( c_j > b_j \) arbitrary, the inequality

\[
w \left( \bigcup_j (a_j, b_j) \right)^{1/q} \leq B \left\{ \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)^{1/p'}} \right)^p \right\}^{1/p} \quad (p > 1)
\]

or

\[
w \left( \bigcup_j (a_j, b_j) \right)^{1/q} \leq B \sum_j \frac{g(a_j, c_j)}{\text{ess sup}_{x \in (b_j, c_j)} \left( \frac{g(x)}{v(x)} \right)} \quad (p = 1)
\]

holds.

(iii) The function \( \Phi(x) \) defined in Definition 2 is in \( L^{r, \infty}(w) \), i.e.

\[
\| \Phi \|_{r, \infty, w} = \sup_{\lambda > 0} \lambda^r w(\{ \Phi(x) > \lambda \})^{1/r} < \infty.
\]

Moreover, for the best constants \( B \) and \( C \) we have

\[
2^{1-1/r} \| \Phi \|_{r, \infty, w} \leq B \leq C \leq (\| \Phi \|_{r, \infty, w} + 4^p)^{1/r}.
\]

In spite of Theorem 4, Theorem 3 is still worthwhile, because the \( B^*_+(p, q) \) condition is more convenient in applications. For instance, it is not easy to test Examples 3 and 4 by condition (ii) or (iii).

**Remark 4.** The equivalence between (i) and (ii) holds even for arbitrary \( 0 < q < \infty \), and this is a Sawyer’s type characterization for the one-sided maximal function (cf. [10]). The condition (iii) is a Verbitsky’s type condition (cf. [11]). However, the case of \( p = 1 \) is new and the proof is new.

### 2. Proofs of the theorems

**Proof of Theorem 1.** Necessity. The argument is based on the following elementary fact about the weak type \((p, q)\) inequality (1.5).

For every \( a < b < c \) and measurable set \( S \subset (b, c) \), it follows from the weak type inequality (1.5) easily that

\[
(2.1) \quad \frac{g(S)}{g(a, c)} w(a, b)^{1/q} \leq C v(S)^{1/p},
\]

on setting \( f(x) = \chi_S(x) \).
Then consequence is this: if the weak type \((1,q)\) \((2.3)\)

**Remark 5.** This completes the proof of Theorem 1.

It follows from the inequality (2.1) that

\[
\frac{w(a,b)^{1/q}}{g(a,c)} \leq C \frac{\nu(S_\lambda)}{g(S_\lambda)} < C\lambda.
\]

Letting \(\lambda \to \alpha\), we get

\[
\frac{w(a,b)^{1/q}}{g(a,c)} \leq C\alpha. \tag{2.2}
\]

**Sufficiency.** We may assume that \(f \geq 0\) is bounded and has compact support. Let \(\lambda > 0\) and \(\Omega_\lambda = \{x : \Omega_\lambda^+ f(x) > \lambda\}\). It is well known that (see [6]) \(\Omega_\lambda = \bigcup (a_j, b_j)\) where the intervals \((a_j, b_j)\) are bounded pairwise disjoint and

\[
\lambda \leq \frac{1}{g(x,b_j)} \int_{a_j}^{b_j} f(x)g(x) \, dx \quad \text{for every } x \in [a_j, b_j]. \tag{2.3}
\]

Following the idea used in [5], we set \(x_j = a_j\) and \(x_k \in (a_j, b_j)\) such that

\[
\int_{x_j}^{b_j} f(x)g(x) \, dx = \frac{1}{2^n} \int_{a_j}^{b_j} f(x)g(x) \, dx \quad (k = 1, 2, \ldots).
\]

Then \(\{x_j\}\) is increasing, \(\lim_{k \to \infty} x_j = b_j\) and

\[
\lambda \leq \frac{4}{g(x_j, x_{j+1})} \int_{x_j}^{x_{j+1}} f(x)g(x) \, dx. \tag{2.4}
\]

Then we have

\[
\lambda^q w(\Omega_\lambda) = \sum_j \sum_{k=0}^{\infty} \lambda^q w(x_j^k, x_j^{k+1}) \\
\leq 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^k, x_j^{k+1})}{g(x_j^k, x_j^{k+2})^q} \left( \int_{x_j^{k+1}}^{x_j^{k+2}} f g \right)^q \quad \text{(by (2.4))} \\
\leq 4^q \sum_j \sum_{k=0}^{\infty} \frac{w(x_j^k, x_j^{k+1})}{g(x_j^k, x_j^{k+2})^q} \left( \text{ess sup}_{x \in (x_j^{k+1}, x_j^{k+2})} \left( \frac{q}{v} \right) \right)^q \left( \int_{x_j^{k+1}}^{x_j^{k+2}} f v \right)^q \\
\leq (4A)^q \sum_j \sum_{k=0}^{\infty} \left( \int_{x_j^{k+1}}^{x_j^{k+2}} f v \right)^q \quad \text{(by } A_y^+(1,q) \text{ condition (1.3))} \\
\leq \left( 4A \int f(x)v(x) \, dx \right)^q.
\]

This completes the proof of Theorem 1.

**Remark 5.** The necessity of Theorem 1 is still true when \(0 < q < 1\). An immediate consequence is this: if the weak type \((1,q)\) inequality holds for \(M_y^+\) with respect to \((w,v)\) and \(\text{ess sup}_{(a,b)} (\frac{q}{v}) = \infty\), then \(w(x) = 0\) a.e. \(x \in (-\infty,a)\).
Proof of Theorem 2. The result is derived from the following propositions.

Proposition 1. Suppose weak type \((p,q)\) inequality (1.5) holds for given \(0 < p, q < \infty\). If \(w\) is not locally integrable, then there exists \(a \in [-\infty, \infty]\) such that \(w\) is locally integrable on \((\infty,a)\) and \(v(x) = \infty\) a.e. \(x \in (a, \infty)\). On the other hand, if \(v(x) = 0\) on \(S\) with \(|S| > 0\), then there exists \(b \in [-\infty, \infty]\) such that \(w(x) = 0\) a.e. \(x \in (-\infty,b)\) and \(v(x) > 0\) a.e. \(x \in (b, \infty)\).

Proposition 2. Under the same conditions as those in Theorem 2, suppose \(v(x) > 0\) a.e. and \(w(x)\) is locally integrable on an interval \((a,b)\). Then we have

(i) For a.e. \(x \in (a,b)\), either \(w(x) = 0\) or \(v(x) = \infty\) if \(p < q\).
(ii) \(w(x) \leq (2C)^p v(x)\) a.e. \(x \in (a,b)\) if \(p = q\). (Where \(C\) is the best constant in (1.6).)

Proof of Proposition 1. Let

\[
E = \{ x \in R: \text{there exists} \delta > 0 \text{ such that } w(x - \delta, x + \delta) < \infty \}
\]

and put \(a = \inf\{R \setminus E\}\). It is obvious that \(w\) is locally integrable on \((\infty,a)\).

Meanwhile, for \(a < c < d\) and measurable \(S \subset (c,d)\) arbitrary, there exists \(e \in R \setminus E\) with \(a \leq e < c\). It follows from the definition of \(R \setminus E\) that \(w(e - \delta, c) = \infty\) for any \(\delta > 0\). Hence the inequality (2.1) shows that \(v(S) = \infty\), which implies \(v(x) = \infty\) a.e. \(x \in (a, \infty)\), since \(S \subset (c,d)\) and \((c,d) \subset (a, \infty)\) are arbitrary.

In order to prove the rest, we set \(\bar{E} = \{ x \in R: v(x) = 0 \}\). Let

\[
b = \text{ess sup}(I(x)\chi_{\bar{E}}(x)),
\]

where \(I(x) = x\). For every \(x < b\), we have \(|\bar{E} \cap (x,b)| > 0\). Then the inequality (2.1) implies \(w(x_1, x) = 0\) for all \(x_1 < x\). The required conclusion follows. \(\square\)

Proof of Proposition 2. In order to produce a contradiction to (i) we assume that there exists a constant \(M > 0\) such that

\[
|E| = \left| \left\{ x \in (a,b): w(x) > \frac{1}{M}, v(x) < M \right\} \right| > 0.
\]

Write \(v_E(x) = v(x)\chi_{\bar{E}}(x)\), then \(v_E\) is locally integrable. Choose \(x_0 \in E\) being a Lebesgue point of \(g(x), w(x), v_E(x)\) and a point of density of \(E\).

Since \(x_0\) is a point of density of \(E\), for every positive integer \(n\), there exists an \(h_n > 0\) such that

\[
1 - \frac{1}{n} < \frac{|E \cap (x_0 - h_n, x_0 + h_n)|}{2h_n} \leq 1
\]

and \(h_n \to 0\) as \(n \to \infty\). For these \(h_n\) we have

\[
1 - \frac{1}{n} < \frac{|E \cap (x_0 - h_n, x_0)|}{2h_n} + \frac{|E \cap (x_0, x_0 + h_n)|}{2h_n}
\]

\[
\leq \frac{1}{2} + \frac{|E \cap (x_0, x_0 + h_n)|}{2h_n},
\]

therefore

\[
\frac{1}{2} - \frac{1}{n} < \frac{|E \cap (x_0, x_0 + h_n)|}{2h_n}.
\]

(2.5)
Let \( S = E \cap (x_0, x_0 + h) \). It follows from (2.1) that
\[
\frac{g(E \cap (x_0, x_0 + h_n))}{g(x_0 - h_n, x_0 + h_n)} w(x_0 - h_n, x_0) \leq C v_E(x_0, x_0 + h_n)^{1/p}.
\]

By use of (2.5) we have
\[
\left(1 - \frac{1}{n}\right) \frac{2h_n}{g(x_0 - h_n, x_0 + h_n)} \frac{g(E \cap (x_0, x_0 + h_n))}{|E \cap (x_0, x_0 + h_n)|} \left(\frac{w(x_0 - h_n, x_0)}{h_n}\right)^{1/q} \leq C h_n^{1/p - 1/q} \left(\frac{v_E(x_0, x_0 + h_n)}{h_n}\right)^{1/p}.
\]

Inequality (2.5) shows that the sequence \( \{E \cap (x_0, x_0 + h_n)\} \) shrinks to \( x_0 \) nicely (see [8], p. 140). On taking \( n \to \infty \), it follows from the Lebesgue differential theorem, more precisely its slightly generalized version (see [8], p. 141), that
\[
\frac{1}{2} w(x_0) \leq 0.
\]
This contradicts the fact that \( w(x_0) > \frac{1}{M} \).

Now we prove (ii). For contradiction we suppose
\[
|S| = |\{x \in (a, b) : w(x) > (2C)^p v(x)\}| > 0.
\]
Write \( v_s(x) = v(x) \chi_s(x) \). Choose \( x_0 \in S \) in the same way as that in the proof of (i). Then the same argument as above shows
\[
\frac{1}{2} w(x_0)^{1/p} \leq C v(x_0)^{1/p}.
\]
This contradicts the fact that \( (2C)^p v(x_0) < w(x_0) \).

**Remark 6.** It follows from Proposition 1 that we may assume that \( w(x) \) is locally integrable on \( R \) and \( v(x) > 0 \) a.e. \( x \in R \) in advance, when we study the weak type or strong type inequality for the one-sided maximal operator.

**Proof of Theorem 3.** In order to prove the first part, without loss of generality we may assume that \( f \geq 0 \) is bounded and has compact support. For \( \lambda > 0 \), let \( \Omega_\lambda, (a_j, b_j) \) and \( (x_j^k, x_j^{k+1}) \) be the same as those in the proof of the sufficiency of Theorem 1. Then we can write
\[
\lambda^q w(\Omega_\lambda) = \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^k, x_j^{k+1})
\]
(2.6)
\[
= \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^{2k}, x_j^{2k+1}) + \sum_j \sum_{k=0}^\infty \lambda^q w(x_j^{2k+1}, x_j^{2k+2})
\]
\[
= I + II, \quad \text{say}.
\]
Suppose $p > 1$. We shall only estimate term I in (2.6), and the rest is similar.

(2.7)

\[
I \leq 4^p \sum_{j} \sum_{k=0}^{\infty} \frac{w(x_j^{2k}, x_j^{2k+1})}{g(x_j^{2k}, x_j^{2k+2})^q} \left( \int_{x_j^{2k+1}}^{x_j^{2k+2}} fg \right)^q \tag{by (2.4)}
\]

\[
\leq 4^p \sum_{j} \sum_{k=0}^{\infty} \frac{w(x_j^{2k}, x_j^{2k+1})}{g(x_j^{2k}, x_j^{2k+2})^q} \left( \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^{p} \right)^{q/p} \sigma(x_j^{2k+1}, x_j^{2k+2})^{q/p'} \tag{by Hölder’s inequality}
\]

\[
\leq 4^p \left[ \sum_{j} \sum_{k=0}^{\infty} \left( \frac{w(x_j^{2k}, x_j^{2k+1})^{1/q} \sigma(x_j^{2k+1}, x_j^{2k+2})^{1/p'}}{g(x_j^{2k}, x_j^{2k+2})^{q/p}} \right)^{r/q} \times \left( \sum_{j} \sum_{k=0}^{\infty} \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^{p} \right)^{p/q} \right]^{q/r} \tag{by Hölder’s inequality again}
\]

\[
\leq (4B)^q \left( \sum_{j} \sum_{k=0}^{\infty} \int_{x_j^{2k+1}}^{x_j^{2k+2}} f^{p} \right)^{q/p} .
\]

Thus we get

\[
(\lambda^q w(\Omega_\lambda))^{1/q} \leq 2^{1/p} 4B \left( \int_R f^{p} \right)^{1/p} .
\]

With a few obvious changes (cf. the proof of the sufficiency of Theorem 1) the foregoing procedure is still available for the case of $p = 1$. The first part of Theorem 3 is proved.

Now we prove the rest of Theorem 3. Let $p > 1$. Given \( \{(a_j, c_j)\} \) pairwise disjoint and \( b_j \in (a_j, c_j) \), we may assume 0 < \( \sigma(b_j, c_j) < \infty \) for all \( j \). Indeed, if \( \sigma(b_j, c_j) = \infty \) for some \( j \), which means \( \int_R (g(x_j^{2k}, x_j^{2k+2})^{p/v} \right) \left( \frac{w(x_j^{2k}, x_j^{2k+1})}{g(x_j^{2k}, x_j^{2k+2})} \right)^{1/q} \leq 2^{1/p} 4B \left( \int_R f^{p} \right)^{1/p} .
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(\lambda^q w(\Omega_\lambda))^{1/q} \leq 2^{1/p} 4B \left( \int_R f^{p} \right)^{1/p} .
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\]

\[
(\lambda^q w(\Omega_\lambda))^{1/q} \leq 2^{1/p} 4B \left( \int_R f^{p} \right)^{1/p} .
\]
We have
\[
C^p \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r = C^p \int_R f_N(x)^p v(x) \, dx
\]
\[
\geq \left( \int_R (M_{+}^p f_N(x))^q w(x) \, dx \right)^{p/q}
\]
\[
\geq \left( \sum_{j=1}^N \left[ \frac{f_{b_j}^{C_j} f_N g}{g(a_j, c_j)} \right]^q w(a_j, b_j) \right)^{p/q} \quad \text{(by (2.8)).}
\]
\[
= \left\{ \sum \frac{w(a_j, b_j)}{g(a_j, c_j)^q} \left[ \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^{r/p} \sigma(b_j, c_j) \right] \right\}^{p/q}
\]
\[
= \left\{ \sum \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r \right\}^{p/q}.
\]
This is
\[
\left[ \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q} \sigma(b_j, c_j)^{1/p'}}{g(a_j, c_j)} \right)^r \right]^{1/r} \leq C.
\]
Letting \( N \to \infty \), we get \((w, v) \in B^+_p(p, q)\) with \( B \leq C \).

Suppose \( p = 1 \). Keeping Remark 5 in mind, we assume
\[
0 < \beta_j = \text{ess sup}_{(b_j, c_j)} \left( \frac{g}{v} \right) < \infty \quad \text{for every } j.
\]

Given \( \lambda > 1 \), set \( E_j = \{ x \in (b_j, c_j) : \frac{w(x)}{v(x)} > \frac{\beta_j}{\lambda} \} \). Then \( v(E_j) < \infty \) and \( |E_j| > 0 \) which implies \( v(E_j) > 0 \) (cf. Remark 6 we may assume \( v(x) > 0 \) a.e.). Put
\[
f_j(x) = \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j) \beta_j} \right)^r \chi_{E_j}(x) \frac{\chi_{E_j}(x)}{v(E_j)}
\]
and \( f_N(x) = \sum_{j=1}^N f_j(x) \). Then
\[
\int_R f_N(x) v(x) \, dx = \sum_{j=1}^N \left( \frac{w(a_j, b_j)^{1/q} \beta_j}{g(a_j, c_j)} \right)^r
\]
and
\[
\left[ \frac{f_{b_j}^{C_j} f_N g}{g(a_j, c_j)} \right]^q w(a_j, b_j) = \frac{w(a_j, b_j)}{g(a_j, c_j)^q} \left[ \left( \frac{w(a_j, b_j)^{1/q} \beta_j}{g(a_j, c_j)} \right)^r \frac{g(E_j)}{v(E_j)} \right]^q
\]
\[
\geq \left( \frac{1}{\lambda} \right)^q \left[ \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j) \beta_j} \right)^r \frac{g(E_j)}{v(E_j)} \right]^q \left( \frac{\beta_j}{\lambda} v(E_j) \right)
\]
\[
= \left( \frac{1}{\lambda} \right)^q \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j) \beta_j} \right)^r \left( \frac{1}{r} = \frac{1}{q} - 1 \right).
\]
The argument similar to that used above shows
\[
\left[ \sum_{j=1}^{N} \left( \frac{w(a_j, b_j)^{1/q}}{g(a_j, c_j)^{1/r}} \right)^r \right]^{1/r} \leq \lambda C.
\]
Then we obtain \((w, v) \in B^+_g(1, q)\) with \(B \leq C\). This completes the proof of Theorem 3.

**Proof of Theorem 4.** We shall prove the theorem only in the case \(p > 1\), since the necessary modifications for the case \(p = 1\) are indicated in the proof of Theorems 1 and 3.

(i) \(\Rightarrow\) (ii) Similarly to the proof of Theorem 3, the condition \(\sigma(a, b) = \infty\) together with the weak type inequality (1.5) implies that \(w(x) = 0\) for a.e. \(x \in (-\infty, b)\). Therefore we may assume that \(0 < \sigma(b_j, c_j) < \infty\) for all \(j\). Let
\[
f = \left\{ \sum_{j} \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)} \right)^p \frac{\sigma(x)}{v(x)} \chi(b_j, c_j) \right\}^{1/p}.
\]
It follows from (2.8) that \(M^+_f f(x) \geq 1\) on \(\cup(a_j, b_j)\) and the weak type inequality (1.5) yields
\[
w\left( \bigcup (a_j, b_j) \right)^{p/q} \leq C \int_R f^p v = C \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)^{1/p'}} \right)^p.
\]
This is (ii) with \(B \leq C\).

(ii) \(\Rightarrow\) (iii) For \(\lambda > 0\), write \(E_\lambda = \{ \Phi(x) > \lambda \}\). For each \(x \in E_\lambda\) there exist \(a_x < b_x < c_x\) such that \(x \in (a_x, b_x)\) and \(\frac{w(a_x, b_x)^{1/p} \sigma(b_x, c_x)^{1/p'}}{g(a_x, c_x)} > \lambda\). Then for arbitrary compact set \(K \subset E_\lambda\), from \(\{ (a_x, b_x) \}_{x \in K}\) we can choose a finite subfamily of open intervals \(\{ I_i \}\), which still covers \(K\). By use of the covering lemma in [3] (Lemma 4.4, p. 25), one can select a finite subsequence \(\{ (a_j, b_j) \}\) from \(\{ I_i \}\) so that \(\{ (a_j, b_j) \}\) are pairwise disjoint and \(w(\bigcup I_i) \leq 2 \sum_j w(a_j, b_j)\). For the set \(\{ (a_j, b_j) \}\), we have
\[
w\left( \bigcup_j (a_j, b_j) \right)^{p/q} \leq B^p \sum_j \left( \frac{g(a_j, c_j)}{\sigma(b_j, c_j)^{1/p'}} \right)^p (\text{by (ii)})
\]
\[
\leq B^p \lambda^{-p} \sum_j w(a_j, b_j).
\]
Thus we get
\[
w(K) \leq w\left( \bigcup_i I_i \right) \leq 2 \sum_j w(a_j, b_j) \leq 2B^r \lambda^{-r},
\]
which implies \(\|\Phi\|_{r, \infty, w} \leq 2^{1/r} B\).

(iii) \(\Rightarrow\) (i) Let \(f, \Omega_\lambda\) and \(\{ (x^k, x^{k+1}) \}\) be the same as those in the proof of Theorem 1. In addition, we may assume \(\int_R f^p v = 1\). Write
\[
w(\Omega_\lambda) = w(\Omega_\lambda \cap \{ \Phi(x) > \lambda^{q/r} \}) + w(\Omega_\lambda \cap \{ \Phi(x) \leq \lambda^{q/r} \}) = I + II, \quad \text{say.}
\]
For item I in (2.9) we have
\begin{equation}
I \leq w(\{\Phi(x) \leq \lambda^{\theta/r}\}) \leq \|\Phi\|_{r,\infty,w}^{r}\lambda^{-\theta}.
\end{equation}
To estimate II in (2.9), we consider \((x^+_j, x^{k+1}_j)\) having non-empty intersection with \(\{\Phi(x) \leq \lambda^{\theta/r}\}\). It follows from (2.4) and Hölder’s inequality that
\begin{equation}
\begin{aligned}
w(x^+_j, x^{k+1}_j)^{1/p} &\leq 4 \frac{w(x^+_j, x^{k+1}_j)^{1/p} \sigma(x^{k+1}_j, x^{k+2}_j)^{1/p'}}{g(x^+_j, x^{k+2}_j)} \\
&\leq 4\lambda^{\theta/r} \left(\int_{x^{k+1}_j}^{x^{k+2}_j} f^p v \right)^{1/p},
\end{aligned}
\end{equation}
since \((x^+_j, x^{k+1}_j) \cap \{\Phi(x) \leq \lambda^{\theta/r}\} \neq \emptyset\). Raising (2.11) to the power of \(p\) and then summing it over \(j\) and \(k\), we have
\begin{equation}
II \leq 4p\lambda^{-\theta}.
\end{equation}
Combining (2.10) and (2.12), we obtain (1.5) with \(C \leq (\|\Phi\|_{r,\infty,w}^{r} + 4p)^{1/q}\). The proof of Theorem 4 is complete. 

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DEPARTMENT OF PURE MATHEMATICS, THE UNIVERSITY OF LEEDS, LEEDS, LS2 9JT, ENGLAND
Current address: Department of Mathematics & Statistics, University of Edinburgh, King’s Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom
E-mail address: pmt5lq@sun.leeds.ac.uk
E-mail address: qlai@maths.ed.ac.uk

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