THE $\mu$-PIP AND INTEGRABILITY OF A SINGLE FUNCTION

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Abstract. Two examples are given that answer in the negative the following question asked by E. M. Bator: If $f : \Omega \to X^*$ is bounded and weakly measurable and for each $x^{**}$ in $X^{**}$ there is a bounded sequence $(x_n)$ in $X$ such that $x^{**}f = \lim_n fx_n$ a.e., does it follow that $f$ is Pettis integrable?

1. Introduction

Given a finite (complete) measure space $(\Omega, \Sigma, \mu)$, a Banach space $X$ is said to have the $\mu$-Pettis Integrability Property ($\mu$-PIP) if every weakly measurable bounded function $f : \Omega \to X$ is Pettis integrable. In [1] E. Bator shows that a dual space $X^*$ has the $\mu$-PIP with respect to a perfect measure $\mu$ if and only if for every bounded weakly measurable $f : \Omega \to X^*$, $\|w^* - \int f \, d\mu\| = \|D - \int f \, d\mu\|$. In [2] and [4], it is shown how the above statement can be strengthened by dropping the assumption that the measure space be perfect. In fact,

Theorem 1. Let $X^*$ be a dual space. The following are equivalent:

1. $X^*$ has the $\mu$-PIP.
2. [2] For every weakly measurable bounded function $f : \Omega \to X^*$,
   \[ \|w^* - \int f \, d\mu\| = \|D - \int f \, d\mu\|. \]
3. [4] For every weakly measurable bounded function $f : \Omega \to X^*$,
   \[ w^* - \int f \, d\mu = 0 \text{ implies } D - \int f \, d\mu = 0. \]

The following corollary, proven for perfect measures in [1], and in general in [2], follows easily:

Corollary 1. A dual space $X^*$ has the $\mu$-PIP if and only if

\[ (*) \left\{ \begin{array}{l} \text{For every bounded weakly measurable function } f : \Omega \to X^* \\ \text{and each } x^{**} \text{ in } X^{**}, \text{ there exists a bounded sequence } (x_n) \text{ in } X \\ \text{such that } fx_n \to x^{**}f \text{ almost everywhere.} \end{array} \right\} \]

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Consider the following property of a particular weakly measurable and bounded function \( g : \Omega \rightarrow X^* \):

\[
\text{(**) } \begin{cases} 
  \text{For each } x^{**} \text{ in } X^{**} \text{ there exists a bounded sequence } (x_n) \text{ in } X \\
  \text{such that } gx_n \rightarrow x^{**}g \text{ almost everywhere.}
\end{cases}
\]

In [1] Bator asks if property (**) implies Pettis integrability of such a function \( g \). In [2], this question is answered in the negative. The example given is based on one by Talagrand and assumes the existence of two-valued measurable cardinals.

As noted in [5, p. 189], it is consistent with the usual axioms of set theory (ZFC) that there are no two-valued measurable cardinals and it is impossible to prove that their existence is consistent with ZFC. The purpose of this note is to show, assuming only the existence of a first uncountable ordinal, that property (**) does not imply Pettis integrability. The first example shows that certain functions that are weak*-zero, but not weakly-zero, give rise to families of functions that satisfy (**) but fail to be Pettis integrable. It also shows that functions with strong measurability properties (universally weakly measurable functions, see [5]) can satisfy (**) and still fail to be Pettis integrable. The second example shows that even in the best possible situation, where \( X^* \) is a dual of a separable space, Bator’s question has a negative answer.

Let us fix some notation and terminology. The dual of a Banach space \( X \) will be denoted by \( X^* \). Given a complete finite measure space \( (\Omega, \Sigma, \mu) \), a function \( f : \Omega \rightarrow X^* \) is called weakly measurable (resp. weak* measurable) if for all \( x^{**} \) in \( X^{**} \) (resp. all \( x \) in \( X \)) the scalar-valued function \( x^{**}f \) (resp. \( fx \)) is measurable.

If \( E \) is in \( \Sigma \), the Dunford integral of \( f \) over \( E \), denoted by \( D - \int_E f \, d\mu \), is the member of \( X^{**} \) defined by the equation \( (D - \int_E f \, d\mu)(x^{**}) = \int_E x^{**}f \, d\mu \) for all \( x^{**} \in X^{**} \). If \( D - \int_E f \, d\mu \) belongs to \( X^* \) for all \( E \) in \( \Sigma \), then \( f \) is said to be Pettis integrable.

The weak* integral of \( f \) over \( E \), denoted by \( w^* - \int_E f \, d\mu \), is the member of \( X^* \) defined by the equation \( (w^* - \int_E f \, d\mu)(x) = \int_E fx \, d\mu \) for all \( x \in X \).

A function \( f : \Omega \rightarrow X^* \) is said to be weakly equivalent to zero (resp. weak* equivalent to zero) if for all \( x^{**} \) in \( X^{**} \) (resp. all \( x \) in \( X \)), \( x^{**}f = 0 \) \( \mu \)-a.e. (resp. \( fx = 0 \) \( \mu \)-a.e.).

## 2. Examples

When considering property (**), we must pay close attention to the restrictions made on the range-space, that is, the space in which the function is valued. Consider for example a function \( g \) into \( X^* \). When viewed as a function into \( X^{***} \), it satisfies (**), by default, but certainly does not have to be Pettis integrable.

### Example 1.
This example is tailored after a well-known example of Phillips; see [3]. Let \( \omega_1 \) be the first uncountable ordinal, \( \Sigma \) be the \( \sigma \)-algebra of all countable and co-countable subsets of \([0, \omega_1]\), and define \( \mu : \Sigma \rightarrow \{0, 1\} \) by the equation

\[
\mu(A) = \begin{cases} 
  0 & \text{if } A \text{ is countable,} \\
  1 & \text{if } A^c \text{ is countable.}
\end{cases}
\]

Define a function \( f : [0, \omega_1] \rightarrow l_\infty[0, \omega_1] = (l_1[0, \omega_1])^* \) by the equation

\[
[f(s)](t) = \begin{cases} 
  0 & \text{if } t < s, \\
  1 & \text{if } t \geq s.
\end{cases}
\]

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Claim 1. $f$ is weakly measurable.

The dual of $l_\infty[0, \omega_1]$ is the space of all bounded and additive measures on $2^{[0, \omega_1]}$. Fix such a measure $\beta$.

There exists a countable subset $R$ of $[0, \omega_1]$ and a unique decomposition $\beta = \beta_1 + \beta_2$ into bounded additive measures such that for any $A, \beta_1(A) = \beta_1(A \cap R)$ and $\beta_2$ vanishes on countable sets. As

$$\beta_1 f(s) = \int_{[0, \omega_1]} [f(s)](t) d\beta_1(t) = \beta_1(\{t \in R \cap [s, \omega_1]\})$$

and

$$\beta_2 f(s) = \int_{[0, \omega_1]} [f(s)](t) d\beta_2(t) = \beta_2([s, \omega_1]),$$

it follows that $\beta f = \beta_1 f + \beta_2 f = \beta_2([0, \omega_1]) \mu$-almost everywhere.

Claim 2. $f$ is not Pettis integrable.

Indeed, the weak$^*$ integral of $f$ is identically zero, but for any $\beta = \beta_1 + \beta_2$, and any set $E$, $\int_E \beta f(s) d\mu(s) = \beta_2([0, \omega_1])\mu(E)$.

Now, define $\tilde{f} : [0, \omega_1] \to l_\infty[0, \omega_1]$ by the equation

$$\tilde{f}(s) = f(s) + \chi_{[0, \omega_1]}(s).$$

Then $\tilde{f}$ is weakly measurable, not Pettis integrable, but satisfies property (**) in fact, for any $\beta = \beta_1 + \beta_2$ in the dual of $l_\infty[0, \omega_1]$,

$$\beta \tilde{f} = \beta([0, \omega_1]) + \beta_2([0, \omega_1]) \mu \text{-a.e.}$$

Consequently,

$$\beta \tilde{f} = \{ \beta \tilde{f} \} \cdot \gamma \tilde{f},$$

where $\gamma$ is any positive norm-one element of $l_1[0, \omega_1]$.

Remark 1. The above example shows that for any finite measure space $(\Omega, \Sigma, \mu)$, any bounded function $f : \Omega \to X^*$ which is weakly a constant, that is, for every $x^*, x^* f = c_{x^*} \ (= \text{constant}) \ a.e.$, and weak$^*$ equivalent to zero but not weakly equivalent to zero, gives rise to a family of functions satisfying (**).

Indeed, if $f$ is such a function, the family $\{ f + x^* : x^* \in X^* \setminus \{0\} \}$ satisfies (**), but none of the functions are Pettis integrable.

Remark 2. A function $f : \Omega \to X$ defined on a compact Hausdorff space $\Omega$ is said to be universally weakly measurable if for every Radon measure $\mu$ on $\Omega$, the scalar functions $x^* f, x^* \in X^*$, are $\mu$-measurable.

Phillips [3] has constructed a bounded function $f : [0, 1] \to l_\infty[0, 1]$ such that $x^* f$ is Borel measurable for all $x^*$ in $l_\infty[0, 1]^*$, and hence, $f$ is universally weakly measurable. With respect to the Lebesgue measure on $[0, 1]$, $f$ is weak$^*$, but not weakly, equivalent to zero. Furthermore, $f$ is weakly constant in the sense of the above remark, and again by the same remark, property (**) fails to imply Pettis integrability even in the case where $f$ satisfies the stronger assumption of being universally weakly measurable.

Example 2. There exists a function $f$ with values in $l_\infty(\mathbb{N})$ that satisfies (**), but fails to be Pettis integrable.
Let $\Omega = (\{0, 1\}^N, \Sigma, \mu)$ be as in [5, Theorem 13-2-1] and let $f : \{0, 1\}^N \to l_\infty(\mathbb{N})$ be the function that assigns to each point $a \in \{0, 1\}^N$ its characteristic function $\chi_a$.

Write $l_\infty(\mathbb{N})^* = l_1(\mathbb{N}) \oplus c_0^\perp$. In [5, Theorem 13-3-3] it is shown that for any $\gamma$ in $c_0^\perp$,

$$\gamma f = k_\gamma (= \text{constant}) \quad \mu\text{-a.e.}$$

Hence, for $x^* = (\alpha_i) \oplus \gamma$ in $l_1(\mathbb{N}) \oplus c_0^\perp$,

$$x^* f = (\alpha_i) f + \gamma = (\alpha_i) f + k_\gamma \quad \mu\text{-a.e.}$$

Let $\{e_i : i \in \mathbb{N}\}$ be the standard basis for $l_\infty(\mathbb{N})$ and define $S : l_\infty(\mathbb{N}) \to l_\infty(\mathbb{N})$ by the equation

$$S(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots).$$

Define a function $\tilde{f} : \{0, 1\}^N \to l_\infty(\mathbb{N})$ by the equation

$$\tilde{f}(a) = e_1 + S(f(a)).$$

If $x^* = (\alpha_i) \oplus \gamma \in l_\infty(\mathbb{N})^*$ and we write $(\beta_i) \oplus \delta$ to denote $S^*(x^*)$, then

$$x^* \tilde{f} = x^*(e_1) + x^* S(f) = \alpha_1 + S^*(x^*) f = \alpha_1 + \{(\beta_i) \oplus \delta\} f$$

$$= (\alpha_1 + k_\delta) + (\beta_i) f = (\alpha_1 + k_\delta, \beta_1, \beta_2, \beta_3, \ldots) \tilde{f}.$$

Hence, $\tilde{f}$ satisfies property $(**)$, but is not Pettis integrable since $f$ is not.

REFERENCES


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