FINITE FACTORIZATION DOMAINS

D.D. ANDERSON AND BERNADETTE MULLINS

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Abstract. An integral domain \( R \) is a finite factorization domain if each nonzero element of \( R \) has only finitely many divisors, up to associates. We show that a Noetherian domain \( R \) is an FFD \( \iff \) for each overring \( R' \) of \( R \) that is a finitely generated \( R \)-module, \( U(R')/U(R) \) is finite. For \( R \) local this is also equivalent to each \( R/[R : R'] \) being finite. We show that a one-dimensional local domain \((R, M)\) is an FFD \( \iff \) either \( R/M \) is finite or \( R \) is a DVR.

In their study of factorization [2], the first author, D.F. Anderson, and M. Zafrullah introduced the notion of a finite factorization domain (FFD). An integral domain \( R \) is an FFD if every nonzero element of \( R \) has only a finite number of nonassociate divisors. The three authors continued their investigation of FFD’s in [3], and F. Halter-Koch studied FFD’s and their monoid analog in [9]. Earlier, A. Grams and H. Warner [8] introduced the related concept of idf-domains. An integral domain \( R \) is an idf-domain (for irreducible-divisor-finite) if each nonzero element of \( R \) has only finitely many nonassociate irreducible divisors.

We adopt the following definitions and notation. For an integral domain \( R \) with quotient field \( K \), \( U(R) \) is the group of units of \( R \) and \( G(R) = K^*/U(R) \), partially ordered by \( aU(R) \leq bU(R) \iff a|b \) in \( R \), is the group of divisibility of \( R \). Clearly \( G(R) \) is order-isomorphic to the group \( \text{Prin}(R) \) of nonzero principal fractional ideals of \( R \) ordered by reverse inclusion. We sometimes call an irreducible element of an integral domain an atom and an integral domain \( R \) is said to be atomic if every nonzero, nonunit element of \( R \) is a finite product of atoms. For an integral domain \( R, R^* = R - \{0\} \) and \( \bar{R} \) is the integral closure of \( R \). For a survey of factorization in integral domains, the reader is referred to [2–3] and for standard definitions and results from commutative ring theory to [6] and [11].

We begin by giving several equivalent conditions for an integral domain to be an FFD.

Theorem 1. For an integral domain \( R \), the following conditions are equivalent:

1. \( R \) is an FFD,
2. every nonzero (principal) ideal of \( R \) is contained in only finitely many principal ideals,
3. for each \( x \in G(R) \) with \( x \geq 0 \), the interval \([0, x]\) is finite,
4. for any infinite collection of distinct principal ideals \( \{(r_\alpha)\} \) of \( R \), \( \bigcap_\alpha (r_\alpha) = 0 \),

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(5) every nonzero element of $R$ has only a finite number of factorizations, up to associates, and
(6) $R$ is an atomic idf-domain.

Proof. Clearly (1)–(5) are equivalent and (1) $\Rightarrow$ (6). (6) $\Rightarrow$ (1) [2, Theorem 5.1].□

We next collect a number of examples and results concerning FFD’s.

**Example 1.** Let $R$ be an integral domain with $R/(a)$ finite for each $0 \neq a \in R$. Then $R$ is an FFD. In particular, if $R$ is a subring of the integral closure of $Z$ in a finite extension of $\mathbb{Q}$, $R$ is an FFD. Rings with the property that each proper homomorphic image is finite were studied by K. Levitz and J. Mott [12]. They observed that an integral domain $R$ has the property that $R/I$ is finite if and only if $R$ is an FFD.

**Example 2.** $R$ is an FFD $\iff R[[X]]$, a set of indeterminates over $R$, is an FFD [2, Proposition 5.3]. By Example 7 below we can also add the equivalence: $R[[X_a] \cup \{X^{-1}_a\}]$ is an FFD. Later, in Example 10, we will show that $R$ an FFD $\not\Rightarrow R[[X]]$ is an FFD.

**Example 3.** A Krull domain is an FFD. (This is remarked in the paragraph above Example 2.)

Let us define an integral domain $R$ to be a strong FFD if each nonzero element of $R$ has only finitely many divisors. And we define $R$ to be a strong idf-domain if each nonzero element of $R$ has only finitely many divisors which are either units or atoms. Several characterizations of strong FFD’s are given in Theorem 5.

**Example 4.** Any subring $R$ of $k[[X_a]]$, where $\{X_a\}$ is any set of indeterminates over $k$ with $k$ either a finite field or $\mathbb{Z}$, is a strong FFD and hence an FFD. Also, see Theorem 5.

**Example 5.** Let $T$ be an integral domain of the form $K + M$ where $M$ is a nonzero maximal ideal of $T$ and $K$ is a subfield of $T$. Let $D$ be a subring of $K$ and $R = D + M$. Then $R$ is an FFD $\iff T$ is an FFD, $D$ is a field, and $K^*/D^*$ is finite [2, Proposition 5.2]. Thus for fields $F_1 \subseteq F_2$, $R = F_1 + XF_2[X]$ (or $F_1 + XF_2[[X]]$) is an FFD $\iff F_2^2/F_1^2$ is finite which by Brandis’ Theorem [5] is equivalent to $F_1 = F_2$ or $F_2$ is finite.

**Example 6.** An integral domain $R$ is a bounded factorization domain (BFD) if for each nonzero nonunit $x$ of $R$, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$ as a product of irreducible elements of $R$, then $n \leq N(x)$. Clearly an FFD is a BFD. A Noetherian domain is always a BFD [2, Proposition 2.2], but $R = \mathbb{R} + X\mathbb{C}[X]$ is not an FFD. Here $\tilde{R} = \mathbb{C}[X]$ is an FFD and $U(\tilde{R}) \cap R = U(R)$, but see Example 8.

**Example 7** ([2, Example 5.4]). Let $k$ be a field and $T = \{n + i/n! \mid 0 \leq i \leq n! - 1, \ n = 0, 1, 2, \ldots\}$, an additive submonoid of $\mathbb{Q}^+$. Then the monoid domain $R = k[X; T]$ is a one-dimensional FFD. However, $R_S = k[X; Q]$ where $S = \{X^t \mid t \in T\}$ and $\tilde{R} = k[X; Q^+]$ are not atomic and hence not FFD’s. However, note that if $R$ is an FFD and $R \subseteq R_S$ is an inert extension (an extension $A \subseteq B$ is an inert extension if whenever $xy \in A$ for nonzero $x, y \in B$, then $xu, yu^{-1} \in A$ for
some \( u \in U(B) \)), then \( R_S \) is an FFD \([3, \text{ Theorem 2.1}]\). An important case where \( R \subseteq R_S \) is inert is when \( S \) is generated by principal primes or \( S \) is a splitting multiplicatively closed subset of \( R \) (i.e., for each \( x \in R \), \( x = as \) where \( a \in R \), \( s \in S \) and \( aR \cap tR = aR \) for all \( t \in S \)). Conversely, if \( S \) is a splitting multiplicatively closed subset of \( R \) generated by principal primes and \( R_S \) is an FFD, then \( R \) is an FFD \([3, \text{ Theorem 3.1}]\).

**Example 8.** Let \( R \subseteq T \) be a pair of integral domains and let \( K \) be the quotient field of \( R \). If \( U(T) \cap K = U(R) \), then the map \( \varphi_+ : \text{Prin}(R)_+ \to \text{Prin}(T)_+ \) given by \( \varphi_+(xR) = xT \) is injective (the converse is also true), so \( T \) an FFD \( \Rightarrow \) \( R \) is an FFD \([2, \text{ page 17}]\). See Theorem 3 for a generalization.

**Example 9** \([3, \text{ Theorem 5.2}]\). Let \( \{R_\alpha\} \) be a directed family of FFD’s such that for each \( \alpha \leq 0 \), \( R_\alpha \subseteq R_\beta \) is an inert extension. Then \( \lim R_\alpha \) is an FFD.

**Theorem 2.** Let \( R = \bigcap_\alpha R_\alpha \) be a locally finite intersection of FFD’s \( \{R_\alpha\} \). Then \( R \) is an FFD.

Proof. Let \( 0 \neq d \in R \) be a nonunit. Let \( \alpha_1, \ldots, \alpha_n \) be the indices for which \( d \) is not a unit in \( R_\alpha \). If \( e \in R \) with \( dR \subseteq eR, \) then \( dR_\alpha \subseteq eR_\alpha \) for each \( \alpha \). Hence \( e \) is a unit in each \( R_\alpha \) except possibly for \( \alpha_1, \ldots, \alpha_n \). Since for each \( R_{\alpha_1}, \ldots, R_{\alpha_n}, dR_{\alpha_1} \) is contained in only finitely many principal ideals (in \( R_{\alpha_1} \)), the same holds for \( dR \). So \( R \) is an FFD. Alternatively, note that \( G(R) \) is order-isomorphic to a subgroup of \( \bigoplus G(R_\alpha) \) (with the cardinal sum order) and apply Theorem 1.

**Remark 1.** While a locally finite intersection of domains each satisfying ACCP also satisfies ACCP, a locally finite intersection of idf-domains need not be an idf-domain \([8]\).

The next two theorems which generalize \([9, \text{ Theorem 7}]\) are straightforward modifications of its proof.

**Theorem 3.** Let \( R \subseteq S \) be a pair of integral domains where \( R \) has quotient field \( K \). If \( S \) is an FFD with \((U(S) \cap K^+)/U(R) \) finite, then \( R \) is an FFD.

Proof. Observe that \((U(S) \cap K^+)/U(R) = \ker \hat{\varphi} \) where \( \hat{\varphi} : G(R) \to G(S) \) is given by \( \hat{\varphi}(rU(R)) = rU(S) \). Now \( \ker \hat{\varphi} \) is finite \( \Rightarrow \) each \( \varphi^{-1}(sU(S)) \) is finite \( \Rightarrow \) each \( \varphi^{-1}(s) \) is finite where \( \varphi : \text{Prin}(R) \to \text{Prin}(S) \) is given by \( \varphi(rR) = rS \). Now since \( S \) is an FFD, \( \{xS \mid x \in S, x \in S\} \) is finite. Thus \( \varphi^{-1}(\{xS \mid xS \supseteq aS, x \in S\}) \) (where \( \varphi = \varphi_{\text{Prin}(R)} \)) is finite. But

\[
\{bR \mid bR \supseteq aR, b \in R\} \subseteq \varphi^{-1}(\{xS \mid x \in S, x \in S\})
\]

Thus \( R \) is an FFD.

**Theorem 4.** Suppose that \( R \subseteq S \) is a pair of integral domains with \([R : R] \neq 0\). Then \( R \) an FFD \( \Rightarrow \) \( U(S)/U(R) \) is finite and \( S \) is an FFD.

Proof. Let \( 0 \neq a \in [R : S] \) be a nonunit. For \( u \in U(S), a^2 = (ua)(u^{-1}a) \) and \( ua, u^{-1}a \in [R : S] \subseteq R \). Hence for each \( u \in U(S), a^2R \subseteq uaR \). Since \( R \) is an FFD, the set \( \{uaR \mid u \in U(S)\} \) is finite. So there exist \( u_1, \ldots, u_n \in U(S) \) so that for any \( u \in U(S), uaR = u_i aR \) for some \( i \). Thus there exists \( \lambda \in U(R) \) with \( ua = \lambda u_i a \) and hence \( u = \lambda u_i \). Thus \( U(S)/U(R) = \{u_1U(R), \ldots, u_nU(R)\} \) is finite.
Let $0 \neq s \in S$. Suppose that $ss \subseteq s_0S$, so $s = s_0s'$, where $s, s_0, s' \in S$. Let $0 \neq d \in [R : S]$, so $(ds_0)(ds'_0) = d^2s$. Hence $ds_0R \supseteq d^2sR$. Since $R$ is an FFD, \{ds_0R\} is finite. Hence \{s_0R\} is finite and thus \{s_0S\} is finite. So $S$ is also an FFD. □

**Corollary 1.** Let $R$ be an FFD and let $S$ be the complete integral closure of $R$. Then $U(S)/U(R)$ is torsion. In particular, $U(R)/U(R)$ is torsion.

Proof. Let $u \in U(S)$. Then $u \in U(R[u^{-1}])$ and $[R : R[u^{-1}]] \neq 0$. By Theorem 4, $U(R[u^{-1}])/U(R)$ is finite. So $uU(R)$ has finite order in $U(S)/U(R)$. □

**Remark 2.** Let $F_1 \subseteq F_2$ be a pair of fields where $F_1$ is an infinite algebraic extension of $Z$. Then $R = F_1[X]$ is a one-dimensional local domain with $R = F_1[X]$ and $G(R) \cong Z\oplus F_1^*$. Here $U(R)$ is a Krull domain and hence an FFD.

**Corollary 2.** $R[[X]]$ is an FFD $\Rightarrow R$ is completely integrally closed. Hence for $R$ Noetherian, $R[[X]]$ is an FFD $\Rightarrow R$ is integrally closed.

Proof. Let $\alpha \in K$, the quotient field of $R$, be almost integral over $R$, so $[R : R[\alpha]] \neq 0$. Hence $[R[[X]] : R[\alpha][X]] \neq 0$. Thus $R[[X]]$ an FFD $\Rightarrow U(R[\alpha][X])/U(R[[X]])$ is finite. Suppose that $\alpha \notin R$. Then $\{(1+\alpha X^m)U(R[\alpha][X])\}^{\infty}_{m=1}$ is an infinite subset of $U(R[\alpha][X])/U(R[[X]])$. For if $(1+\alpha X^m)U(R[\alpha][X]) = (1+\alpha X^m)U(R[[X]])$ for $0 < m < n$, then $(1+\alpha X^m)(1+\alpha X^{-m})^{-1} \in U(R[\alpha][X]) \subseteq R[[X]]$. But

$$(1+\alpha X^m)(1+\alpha X^{-m})^{-1} = 1 - \alpha X^m + \cdots,$$

a contradiction.

Suppose that $R$ is Noetherian. If $R$ is integrally closed, then $R$ is a Krull domain. Hence $R[[X]]$ is also a Krull domain and hence an FFD. □

**Example 10.** Let $F_1 \subseteq F_2$ be a pair of finite fields. Then $R = F_1 + YF_2[Y]$ is a nonintegrally closed one-dimensional local domain with finite residue field $F_1$ and hence is an FFD. Since $R$ is not integrally closed, $R[[X]]$ is not an FFD.

We next use Theorem 4 to characterize strong FFD’s.

**Theorem 5.** For an integral domain $R$ the following conditions are equivalent.

1. $R$ is a strong FFD.
2. $R$ is an atomic strong idf-domain.
3. $R$ is an FFD and $U(R)$ is finite.
4. For any set of indeterminates $\{X_\alpha\}$ over $R$, every subring of $R[[X_\alpha]]$ is a strong FFD.
5. Every subring of $R[X]$ is an FFD.

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) By Theorem 1, $R$ is an FFD. Since 1 has only finitely many unit factors, $U(R)$ is finite. (3) $\Rightarrow$ (1) Clear. (3) $\Rightarrow$ (4) $R[[X_\alpha]]$ is a FFD (Example 2) and $U(R[[X_\alpha]]) = U(R)$ is finite. Let $S$ be a subring of $R[[X_\alpha]]$. Since $0 \neq f \in S$ has only finitely many factors in $R[[X_\alpha]]$ by (1) $\Rightarrow$ (3), $f$ certainly has only finitely many factors in $S$. So $S$ is a strong FFD. (4) $\Rightarrow$ (5) Clear. (5) $\Rightarrow$ (3) $R$ is a subring of $R[X]$ and hence an FFD. Let $R_0$ be the prime subring of $R$ and let $S = R_0 + XR[X]$. By hypothesis, $S$ is an FFD. Now $X \in [S : R[X]]$, so by Theorem 4, $U(R[X])/U(S)$ is finite. But $U(R[X]) = U(R)$
and $U(S) = U(R_0)$, so $U(R)/U(R_0)$ is finite. Since $U(R_0)$ is finite, $U(R)$ is itself finite. □

Remark 3. (1) Let $R = \bigcup_{n=1}^{\infty} GF(p^n)$, $p$ a prime. Then $R$ is an infinite field with every proper subring a finite field and hence a strong FFD. But $R$ is not a strong FFD. Thus in Theorem 5 we cannot add the condition: every subring of $R$ is an FFD. This example also shows that a direct limit of strong FFD’s, while an FFD, need not be a strong FFD.

(2) Let $R$ be a subring of the integral closure of $\mathbb{Z}$ in a finite field extension. By Example 1, every subring of $R$ is an FFD. However, since $U(R)$ may be infinite (e.g., $R = \mathbb{Z}[\sqrt{2}]$), $R$ need not be a strong FFD.

We next characterize Noetherian FFD’s.

**Theorem 6.** For a Noetherian integral domain $R$, the following conditions are equivalent.

1. $R$ is an FFD.
2. If $S$ is an overring of $R$ with $S$ a finitely generated $R$-module, then $U(S)/U(R)$ is finite.
3. There is an FFD overring $R'$ of $R$ which is integral over $R$ such that if $S$ is an overring of $R$ with $R \subseteq S \subseteq R'$ where $S$ is a finitely generated $R$-module, then $U(S)/U(R)$ is finite.

**Proof.** (1) ⇒ (2) Theorem 4. (2) ⇒ (3) Take $R' = R$, the integral closure of $R$. Now $R$ is a Krull domain and hence an FFD. (3) ⇒ (1) Suppose that $R$ is not an FFD. Then there exist nonzero $d_1, d_2, \ldots \in R$ with $dR \subseteq d_nR$ such that the $d_nR$ are distinct principal ideals of $R$. Now $d_1R' \subseteq d_nR'$ and $R'$ is an FFD, so the set $\{d_nR'\}$ is finite. Re-indexing, if necessary, we can assume that $d_1R' = d_nR'$ for each $n \geq 1$. Now $\{d_n\}$ is a finitely generated ideal of $R$, say $\{d_n\} = (d_1, \ldots, d_m)$. So each $d_n = d_1^{\alpha_1} \cdots d_m^{\alpha_m}$. Now $d_1R' = d_nR'$ gives that each $d_n^{\alpha_n}$ is a unit in $R'$. Since $S = R\left[\frac{d_1}{d_1}, \ldots, \frac{d_m}{d_1}\right]$ is a finitely generated $R$-module, $U(S)/U(R)$ is finite. But $\frac{d_n^{\alpha_n}}{d_1^{\alpha_1}} \in U(S)$ (since $\frac{d_n^{\alpha_n}}{d_1^{\alpha_1}} \in U(R')$) and if $\frac{d_n^{\alpha_n}}{d_1^{\alpha_1}}U(R) = \frac{d_n^{\alpha_n}}{d_1^{\alpha_1}}U(R)$, then $d_nU(R) = d_nU(R)$ and hence $d_nR = d_nR$, a contradiction. □

**Corollary 3** ([9, Theorem 7]). Let $R$ be a Noetherian domain with $\bar{R}$ a finitely generated $R$-module. Then $R$ is an FFD $\iff U(R)/U(R)$ is finite.

Theorem 6 is actually stronger than Corollary 3 in the sense that a Noetherian FFD $R$ need not have $\bar{R}$ a finitely generated $R$-module. The existence of one-dimensional local domains $(R, M)$ with $R/M$ finite and $\bar{R}$ not a finitely generated $R$-module follows from [10, Corollary 1.27]. W. Heinzer also communicated to us that [12, Example 2.9] can be modified as follows to yield an appropriate example.

**Example 11.** (A one-dimensional local FFD $(R, M)$ with $\bar{R}$ not a finitely generated $R$-module.) Let $k$ be a finite field of characteristic $p$ and let $Y \in X[[X]]$ with $X, Y$ algebraically independent over $k$. Let $R = V[Y]$ where $V = k[[X]] \cap k(X, Y^p) \subseteq W = k[[X]] \cap k(X, Y)$. Here $V$ and $W$ are DVR’s with quotient fields $k(X, Y^p)$ and $k(X, Y)$, respectively, and $k(X, Y^p)$ is purely inseparable over $k(X, Y)$ of degree $p$. So $W$ is the integral closure of $V$ in $k(X, Y)$, $R$ has quotient field $k(X, Y)$, and $\bar{R} = W$. By the Krull-Akizuki Theorem, we see that $R$ is a one-dimensional local domain, say with maximal ideal $M$, and $R/M = k$. Hence
(\(R, M\)) is an FFD. However, \(W\) is not a finitely generated \(R\)-module. For if \(W\) were a finitely generated \(R\)-module, then \(W\) would be a finitely generated \(V\)-module. But then since \(W = V + XW\), we get \(W = V\) by Nakayama’s Lemma, a contradiction. Note that here \(G(R) \cong G(W) \oplus (U(W)/U(R))\) where \(G(W) \cong \mathbb{Z}\) and \(U(W)/U(R)\) is a countably infinite elementary \(p\)-primary abelian group. Thus \(U(W)/U(R)\) is a DVR.

We have observed that a one-dimensional local domain \((R, M)\) with \(R/M\) finite is an FFD. This raises the question of when is a one-dimensional local domain \((R, M)\) with \(R/M\) infinite an FFD? It follows from our next theorem that \(R\) must be a DVR.

**Theorem 7.** Let \((R, M)\) be a quasilocal domain and let \(R'\) be an overring of \(R\) that is a finitely generated \(R\)-module. Then \(U(R')/U(R)\) is finite if and only if \(R/[R : R']\) is finite. Hence if \(R/M\) is finite, \(U(R')/U(R)\) is finite if and only if \(R \cong R'\).

**Proof.** (\(\Rightarrow\)) Suppose that \(R/[R : R']\) is finite. If \([R : R'] = R, R = R'\) and certainly \(U(R')/U(R)\) is finite. So suppose that \([R : R'] \neq R\). By [1, Lemma 2], \(U(R')/U(R) \cong U(R'/[R : R'])/U(R/[R : R'])\) which is finite since \(R'/[R : R']\) is finite being a finitely generated \(R/[R : R']\)-module.

(\(\Leftarrow\)) Suppose that \(U(R')/U(R)\) is finite. First suppose that \(R/M\) is finite. The proof of [1, Theorem 1] shows that \([R : R']\) is finite. Next suppose that \(R/M\) is infinite. Let \(Q_1, \ldots, Q_n\) be the maximal ideals of \(R'\). As in the proof of [1, Theorem 1] (with \(R'\) playing the role of \(D\)), \((Q_1 \cap \cdots \cap Q_n)/M\) has finite length as an \(R\)-module. The equation \(\bar{q} = (1 + m)\bar{t}_i\) ([1, Theorem 1, line 13 of proof]) shows that the socle \(\text{Soc}((Q_1 \cap \cdots \cap Q_n)/M)\) is finite. (For \((1 + m)\bar{t}_i = \bar{t}_i\) since \(m\bar{t}_i = 0\) for \(i\) in the socle.) Since \(R/M\) is infinite, \(\text{Soc}((Q_1 \cap \cdots \cap Q_n)/M) = M/M\), so \(Q_1 \cap \cdots \cap Q_n = M\). So \(M \subseteq [R : R']\). Suppose that \([R : R'] \neq R\). Then by [1, Lemma 2], \(U(R'/Q_1 \cap \cdots \cap Q_n)/U(R/M)\) is finite. With a change of notation, put \(R/M = K\) and \(R'/Q_1 = K_i\). So \(K_1 \times \cdots \times K_n\) is a vector space over \(K\) and there exist \(t_i \in K_1^i \times \cdots \times K_n\), \(1 \leq i \leq l\), so that every element of \(K_1^i \times \cdots \times K_n\) has the form \(u t_i\) for some \(u \in K^*\). Let \(\{s_j\}\) be the set of \(2^n\) elements of \(K_1 \times \cdots \times K_n\) where each coordinate is 0 or 1. Then \(K_1 \times \cdots \times K_n = \bigcup_i K t_i s_j\) a finite union of one-dimensional subspaces. Since \(K\) is infinite, \(R/M = K = K_1 \times \cdots \times K_n = R'/Q_1 \cap \cdots \cap Q_n = R'/M\). So \(R \cong R'\). □

**Corollary 4.** Let \((R, M)\) be a quasilocal FFD with \(R/M\) infinite. Then \(R\) is integrally closed. Thus a local domain \((R, M)\) with \(R/M\) infinite is an FFD \(\Leftrightarrow R\) is integrally closed.

**Proof.** Combine Theorems 4 and 7. □

**Corollary 5.** Let \((R, M)\) be a local domain. Then the following conditions are equivalent.

1. \(R\) is an FFD.
2. If \(R'\) is an overring of \(R\) which is a finitely generated \(R\)-module, then \(U(R')/U(R)\) is finite.
3. If \(R'\) is an overring of \(R\) which is a finitely generated \(R\) module, then \(R/[R : R']\) is finite.
4. Either \(R\) is integrally closed or \(R/M\) is finite and for each proper overring \(R'\) of \(R\) with \([R : R'] \neq 0, [R : R']\) is \(M\)-primary.
(5) For each simple integral overring \( R[\alpha] \) of \( R \), \( U(R[\alpha])/U(R) \) is finite.

(6) Either \( R \) is integrally closed or \( R/M \) is finite and for each simple proper 
integral overring \( R[\alpha] \) of \( R \), \( [R : R[\alpha]] \) is \( M \)-primary.

(7) For each simple integral overring \( R[\alpha] \) of \( R \), \( R/[R : R[\alpha]] \) is finite.

Proof. (1) \( \Leftrightarrow \) (2) Theorem 6. (2) \( \Leftrightarrow \) (3) and (5) \( \Leftrightarrow \) (7) Theorem 7. (3) \( \Leftrightarrow \) (4)

Note that for \( R/M \) finite, \( R/[R : R'] \) is finite \( \Leftrightarrow \) \( R = R' \) or \( [R : R'] \) is \( M \)-primary.
The same proof shows that (6) \( \Leftrightarrow \) (7). (4) \( \Rightarrow \) (6) Clear. (6) \( \Rightarrow \) (4). Suppose that \( R \) is not integrally closed, so \( R/M \) is finite. Let \( R' = R[\alpha_1, \ldots, \alpha_n] \) be a finitely generated \( R \)-module. By hypothesis, for each \( \alpha_i \), there is an \( M \)-primary ideal \( M_i \) with \( M_iR[\alpha_i] \subseteq R \). Then \( M_1 \cdots M_n \) is \( M \)-primary and \( M_1 \cdots M_nR[\alpha_1, \ldots, \alpha_n] \subseteq R \).

**Corollary 6.** A one-dimensional semilocal domain \( R \) is an FFD \( \Leftrightarrow \) for each 
maximal ideal \( M \) of \( R \) with \( R/M \) infinite, \( R_M \) is a DVR.

Proof. Let \( M_1, \ldots, M_n \) be the maximal ideals of \( R \). Since \( G(R) \) is order-isomorphic to \( G(R_{M_1}) \oplus \cdots \oplus G(R_{M_n}) \) (in the cardinal sum order) [4, Theorem 3.2], \( R \) is an 
FFD \( \Leftrightarrow \) each \( R_{M_i} \) is an FFD. Now \( R_{M_i} \) has residue field \( R/M_i \). So \( R_{M_i} \) is an FFD 
\( \Leftrightarrow \) either \( R/M_i \) is finite (Example 1) or \( R_{M_i} \) is a DVR.

**Remark 4.** Suppose in Example 7, we take \( k \) to be an infinite field. Then \( R = k[X; T] \) is a one-dimensional FFD with each residue field infinite, but \( R \) is not integrally closed and \( R \) is not an FFD.

From our previous results, it seems reasonable to conjecture that a Noetherian domain is an FFD if and only if \( R = \bigcap \{R_P \mid \text{ht } P = 1 \} \) where the intersection is locally finite and each \( R_P \) is a one-dimensional FFD. While the implication \( (\Rightarrow) \) does follow from Theorem 2, we show that \( (\Leftarrow) \) need not be true.

**Example 12.** Let \( (R, M) \) be a one-dimensional local domain with \( R/M \) finite 
that is not a DVR. Then \( R[X] \) is an FFD. Since \( R[X] \) is Cohen-Macaulay, \( R[X] = \bigcap \{R[X]_P \mid \text{ht } P = 1 \} \), and the intersection is of course locally finite. Now if \( P \) is a 
height-one prime of \( R[X] \) with \( P \cap R = 0 \), then \( R[X]_P \) is a DVR. But for \( P = M[X] \), 
\( R[X]_{M[X]} = R(X) \) is a one-dimensional local domain with infinite residue field that 
is not a DVR. Hence \( R(X) \) is not an FFD.

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Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242

Current address, B. Mullins: Department of Mathematics, Youngstown State University, Youngstown, Ohio 44555

E-mail address: dan-anderson@uiowa.edu

E-mail address: bmullins@math.ysu.edu