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A CHARACTERIZATION OF SYSTEMS OF PARAMETERS

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ABSTRACT. A criterion is given for a set of elements to form a system of parameters on a module over a local ring.

Let R be a Noetherian local ring of dimension n. We recall that a sequence of elements $x_1, \dots, x_n$ in the maximal ideal m of R is called a *system of parameters* if the quotient $R/(x_1, \dots, x_n)$ has dimension zero, and that the dimension n of R is the smallest integer for which a sequence with this property exists. Similarly, if M is an R-module of dimension k, we say that a sequence of elements $x_1, \dots, x_k$ of the maximal ideal of R forms a system of parameters for M if the quotient $M/(x_1, \dots, x_k)M$ has finite length. Assume that M is a finitely generated maximal Cohen-Macaulay module, so that its depth and its Krull dimension are both equal to n. In this note we give a criterion for a sequence of elements in the ideal $(x_1, \dots, x_n)$ to form a system of parameters on M.

Let $y_1, \dots, y_n$ be a sequence of n elements in the ideal generated by $x_1, \dots, x_n$. We may then write each y_i as a sum of multiples of the x_j, so that there is a matrix $A = (a_{ij})$ such that

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

for each i. We abbreviate the column vector with entries $x_1, \dots, x_n$ to $\mathbf{x}$ and use the notation (x) to denote the sequence $x_1, \dots, x_n$ or the ideal they generate; we use similar notation for $\mathbf{y}$ and (y). Thus we may write the above equations in the form $\mathbf{y} = A\mathbf{x}$. Let d be the determinant of A.

Theorem. *Let M be a maximal Cohen-Macaulay module, let $x_1, \dots, x_n$ be a system of parameters on M, and let $\mathbf{y} = A\mathbf{x}$ be as above. Then $y_1, \dots, y_n$ is a system of parameters on M if and only if the map induced by multiplication by d from $M/(x)M$ to $M/(y)M$ is injective.*

Before proving this theorem, we make several comments. First of all, the fact that multiplication by the determinant d sends (x) to (y) follows from Cramer's rule (or some equivalent). Second, one implication, that the map is injective if

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the elements form a system of parameters, is well known and comes up in the construction of the local cohomology module $H^n_m(M)$. Whereas the most common construction of this module is as a limit of $M/(x_1^t, \dots, x_n^t)M$, where the maps are defined by multiplication by $x_1^{t-1} \dots x_n^{t-1}$, it can also be defined as the limit over all systems of parameters for M, and in this definition the maps are defined by the determinant as above. For details, see Kunz [1].

The question as to whether the converse also holds came up in the study of a case of one of the homological conjectures. Conjectures of Peskine and Szpiro imply that it is not possible to have a module of finite projective dimension over a Cohen-Macaulay ring whose Krull dimension is at least one but whose restriction to a component has dimension zero. Our theorem implies that a certain type of counterexample cannot exist. We discuss this connection in more detail below.

A classical criterion for homogeneous elements to form a system of parameters in a polynomial ring is given by the resultant (see Macaulay [2], Chapter 1). This criterion coincides with the one presented here only in the case in which the elements (x) are the variables defining the polynomial ring and (y) are linear polynomials in (x), where both criteria give the elementary result that n linear polynomials in n variables generate the ideal (x) if and only if the determinant of their coefficients is non-zero. In our theorem the ring R can be any local ring; in particular, it does not need to be Cohen-Macaulay. However, since the hypothesis and conclusion hold for a local ring if and only if they hold for its completion, we may assume that R is complete. Thus by the Cohen structure theorem we may assume that there exists a regular local ring S which maps onto R.

Proof of the Theorem. As mentioned above, one implication is well known. If elements (y) form a system of parameters, then it is known that the map induced by the determinant is injective; we sketch a short proof. If (y) form a system of parameters, we can express some powers of the x_i in terms of (y), so that if $\mathbf{x^t}$ denotes the column vector with entries x_i^t, we can write $\mathbf{x^t} = B\mathbf{y}$ for some matrix B. We then have $\mathbf{x^t} = BA\mathbf{x}$, and by taking the product of the determinants, we reduce to the case in which $(y) = (x^t)$. However, in this case we may use the diagonal matrix with entries x_i^{t-1}, so the conclusion is that the map given by multiplication by $x_i^{t-1} \cdots x_n^{t-1}$ from $M/(x)M$ to $M/(x^t)M$ is injective. This result follows from the fact that M is Cohen-Macaulay of dimension n, so that (x) is a regular sequence on M.

Assume now that the map induced by the determinant of A from $M/(x)M$ to $M/(y)M$ is injective; we must show that $y_1, \dots, y_n$ is a system of parameters on M. Denote this map from $M/(x)M$ to $M/(y)M$ by ϕ. The proof uses a composition of two maps, one of which is a map between two Ext modules both of which are of finite length, and the other of which is a map defined by Koszul complexes which can be computed explicitly. The conclusion will be that $M/(y)M$ is a module of finite length, so that $y_1, \dots, y_n$ is a system of parameters as desired.

Let S be a regular local ring which maps onto R, and let s be the dimension of S. We note that all R modules are also S-modules. We next apply the functor $\mathrm{Ext}^s_S(-, S)$ to the map ϕ. Since ϕ is injective and S is regular (so Gorenstein) of dimension s, it follows that ϕ induces a surjective map η from $\mathrm{Ext}^s_S(M/(y)M, S)$ to $\mathrm{Ext}^s_S(M/(x)M, S)$. We wish to compare η with a map induced by a map between the Koszul complexes defined over R by the sequences (x) and (y). To accomplish this, we must lift the Koszul complexes to complexes of free S-modules.

Let $F_\bullet$ be a minimal free resolution of M over S. Since M is Cohen-Macaulay of dimension n, $F_\bullet$ is a complex of length $s-n$. In addition, again using the fact that M is Cohen-Macaulay of dimension n, we have that $\operatorname{Ext}_S^i(M,S)$ is zero for $i \neq s-n$, while $\operatorname{Ext}_S^{s-n}(M,S)$ is an R-module with the same support as M. As a result, the complex $\operatorname{Hom}_S(F_\bullet, S)$ is a free resolution of $\operatorname{Ext}_S^{s-n}(M,S)$ over S. We denote $\operatorname{Ext}_S^{s-n}(M,S)$ by M^*.

Since the map from S to R is surjective, we may lift the sequences (x) and (y) and the matrix A to sequences (x') and (y') and a matrix A' with entries in S such that, using the same notation as above, we have $\mathbf{y}' = A'\mathbf{x}'$. Let d' be the determinant of A'; then the image of d' in R is d. Let $K_\bullet(S,(x'))$ and $K_\bullet(S,(y'))$ denote the Kozsul complexes defined by (x') and (y') respectively. We claim that the map induced by the determinant d' in degree zero from $K_\bullet(S,(x'))$ to $K_\bullet(S,(y'))$ can be lifted to a map $\phi_\bullet$ of complexes whose component in degree n is the identity map. To see this, we consider the dual map from $K_\bullet(S,(y'))$ to $K_\bullet(S,(x'))$ which lifts the map from $S/(y')S$ onto $S/(x')S$; since $(y') = A'(x')$, we may use the exterior powers of A' to lift this map, which will induce multiplication by d' in degree n. The dual of this latter map is then the map $\phi_\bullet$ we are looking for. However, we really need maps lifting the Koszul complexes of (x) and (y) on M, so we must tensor this map with the resolution $F_\bullet$ of M over S. Let $E_\bullet(x')$ denote $K(S,(x')) \otimes_S F_\bullet$, let $E_\bullet(y')$ denote $K(S,(y')) \otimes_S F_\bullet$, and let $\psi_\bullet$ denote $\phi_\bullet \otimes F_\bullet$. The homology of $E_\bullet(x')$ and $E_\bullet(y')$ can be computed by considering the homology of the double complexes of the tensor products $K(S,(x')) \otimes_S F_\bullet$ and $K(S,(y')) \otimes_S F_\bullet$. When we take the homology of these double complexes in the direction of $F_\bullet$, they reduce to the Koszul complexes over M of (x) and (y) respectively. Thus the corresponding spectral sequences of the double complexes degenerate, and the homology of $E_\bullet(x')$ and $E_\bullet(y')$ is the same as the homology of the Koszul complexes of (x) and (y) over M. In particular, since M is a maximal Cohen-Macaulay module and (x) is a system of parameters for M, $E_\bullet(x')$ is a free resolution of $M/(x)M$ over S. We also know that the homology of $E_\bullet(y')$ in degree zero is $M/(y)M$ and that the map induced by $\psi_\bullet$ on homology in degree zero is ϕ.

Now let $G_\bullet$ be a free resolution of $M/(y)M$ over S. We may lift the identity map on $M/(y)M$ to a map from $E_\bullet(y')$ to $G_\bullet$. We thus have maps

$$E_\bullet(x') \to E_\bullet(y') \to G_\bullet$$

such that the first map induces ϕ and the second map induces the identity on homology in degree zero. Thus the composition may be used to compute the surjective map η from $\operatorname{Ext}_S^s(M/(y)M,S)$ to $\operatorname{Ext}_S^s(M/(x)M,S)$ described in the previous paragraph. This map on Ext modules is obtained by applying the functor $\operatorname{Hom}_S(-,S)$ to the above sequence and taking maps on homology in degree s.

To complete the computation, we compute the cohomology in degree s of the complex $\operatorname{Hom}(E(x'),S)$, and similarly for (y'). We have

$$\begin{aligned}\operatorname{Hom}(E(x'),S) &= \operatorname{Hom}(K(S,(x')) \otimes F_\bullet, S)\\ &\cong \operatorname{Hom}(K(S,(x')),S) \otimes \operatorname{Hom}(F_\bullet,S).\end{aligned}$$

Since $\operatorname{Hom}(F_\bullet,S)$ is a free resolution of M^* of length $s-n$, and since the complex $\operatorname{Hom}(K(S,(x')),S)$ is again isomorphic to a Koszul complex on (x'), taking cohomology in the direction of $\operatorname{Hom}(F_\bullet,S)$ reduces this complex to the Koszul complex

$K^{\bullet}(M^*, (x))$ with degree shifted by $s-n$. The cohomology of $K^{\bullet}(M^*, (x))$ in degree n is $M^*/(x)M^*$, so we finally conclude that the cohomology of $\mathrm{Hom}(E(x'), S)$ in degree s is $M^*/(x)M^*$. We may use the same argument to show that the cohomology in degree s of $\mathrm{Hom}(E(y'), S)$ is $M^*/(y)M^*$. Since the map induced by $\psi_{\bullet}$ in degree n is the identity, the map induced on cohomology in degree s by the map from $\mathrm{Hom}(E(y'), S)$ to $\mathrm{Hom}(E(x'), S)$ is the natural projection from $M^*/(y)M^*$ onto $M^*/(x)M^*$.

Putting these pieces together, we thus have a sequence

$$\mathrm{Ext}^s_S(M/(y)M, S) \to M^*/(y)M^* \to M^*/(x)M^* = \mathrm{Ext}^s_S(M/(x)M, S),$$

where the composition is surjective and the outer modules have finite length. The second map is an isomorphism modulo the maximal ideal of R, so if the composition is surjective, the first one must also be surjective by Nakayama's Lemma. Thus, since $\mathrm{Ext}^n_S(M/(y), S)$ has finite length, $M^*/(y)M^*$ must also have finite length, and (y) must be a system of parameters on M^*. Since M and M^* have the same support, (y) is also a system of parameters on M.

Applying this theorem to the ring R itself we obtain:

Corollary. *Let R be a Cohen-Macaulay local ring, (x) a system of parameters and $\mathbf{y} = A\mathbf{x}$ as above. Then (y) is a system of parameters if and only if the map given by multiplication by the determinant of A from $R/(x)$ to $R/(y)$ is injective.*

We remarked above that the original motivation for considering this question came from considering a special case of a conjecture on modules of finite projective dimension. The "Strong Intersection Conjecture" of Peskine and Szpiro states that if M is a finitely generated module of finite projective dimension over a local ring R, and if N is a finitely generated module such that $M \otimes_R N$ has finite length, we have

$$\dim(N) \leq \mathrm{grade}_R(M)$$

(see Peskine-Szpiro [3], Chapitre II.0, conjecture (e)). We recall that the *grade* of a module is the maximum length of a regular sequence contained in its ahnnihilator. In particular, this conjecture would imply that if R is a Cohen-Macaulay ring and P is a minimal prime ideal, then, since $\dim(R/P) = \dim(R)$, if $M \otimes_R (R/P)$ has finite length, we must have $\mathrm{grade}(M) = \dim(R)$, so that M must also have finite length.

We give a brief outline of the example we were considering. Let Q be the ring $k[x, y, a, b, c, d]/(ax+by, cx+dy)$, where k is a field. The ring Q has two minimal prime ideals, which are generated by x, y and by $ad-bc$ respectively. Suppose that M is a finitely generated Q-module of finite projective dimension such that $M \otimes Q/(x, y)$ is a module of finite length. Suppose also that M can be lifted to a $k[x, y, a, b, c, d]$-module F which is free and finitely generated over $k[x, y]$; this hypothesis is reasonable since $M \otimes Q/(x, y)$ has finite length. It can then be verified that, since M has finite projective dimension, we must have

$$M = F/(ax+by, cx+dy)F.$$

Now the exact sequence of maps of free modules

$$\to S^2 \xrightarrow{(x\ \ y)} S \xrightarrow{ad-bc} S \xrightarrow{\binom{x}{y}} S^2 \to$$

can be extended to an exact complex of finitely generated free modules infinitely in both directions; thus when tensored with M it must still be exact since M has finite projective dimension. Since $M = F/(ax+by, cx+dy)F$, exactness states that the map induced by the determinant $ad-bc$ from $M/(x,y)M = F/(x,y)F$ to $M = F/(ax+by, cx+dy)F$ is injective. Thus the theorem implies that $ax+by, cx+dy$ is a system of parameters for F, so that M has finite length and the conjecture is true in this case.

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