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RANDOM FIXED POINTS OF SET-VALUED OPERATORS

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Abstract. Some random fixed point theorems for set-valued operators are obtained. The measurability of certain marginal maps is also studied. The underlying measurable space is not assumed to be a Suslin family.

1. Introduction

In recent years there have appeared various papers concerning random fixed point theory for single-valued and set-valued random operators; see, for example, [3], [7–12], [14, 15] and references therein. Some authors (see, e.g., [3], [10], [12]) carried their work out in a framework in which the underlying measurable space (Ω, Σ) is a Suslin family (see the definition in [13]) (in particular, when (Ω, Σ) admits a σ-finite complete measure μ). In this case, since all the various kinds of measurability are equivalent (see [13]), a deterministic fixed point theorem may correspond to a random fixed point theorem (cf. [12]). However, the situation is quite different if the underlying measurable space (Ω, Σ) is not assumed to be a Suslin family. Indeed, in such case, it remains an open question (cf. [14]) whether the (deterministic) fixed point property for nonexpansive mappings implies the random fixed point property for (random) nonexpansive mappings.

The purpose of the present paper is to prove several random fixed point theorems for set-valued operators under a framework where the underlying measurable space (Ω, Σ) is not assumed to be a Suslin family. Our approach employs a technique from Itoh [7]. This also enables us to show the measurability of certain marginal maps without assuming the measurable space (Ω, Σ) to be a Suslin family.

2. Preliminaries

By a measurable space we mean a pair (Ω, Σ), where Ω is a nonempty set and Σ is a sigma-algebra of subsets of Ω. Let (X, d) be a metric space. Then we denote by 2^X the family of all nonempty subsets of X, by $CB(X)$ the family of all elements A of 2^X which are closed and bounded, by $K(X)$ the family of all elements A of

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$CB(X)$ which are compact, and by H the Hausdorff metric on $CB(X)$ induced by d, i.e.,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for $A, B \in CB(X)$, where $d(z,E) = \inf\{d(z,u)\colon u \in E\}$ is the distance from a point $z \in X$ to a subset $E \subset X$. A set-valued operator $T\colon \Omega \to X$ is called (Σ-)measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega\colon T(\omega) \cap B \neq \varnothing\}$$

belongs to Σ. Note that in Himmelberg [6], this is called weakly measurable. Since in the present paper only this type of measurability is used, we omit the term 'weakly' for simplicity. Note also that if $T(\omega) \in K(X)$ for all $\omega \in \Omega$, then T is measurable if and only if $T^{-1}(F) \in \Sigma$ for all closed subsets F of X. A measurable (single-valued) function $x\colon \Omega \to X$ is called a measurable selector of a measurable set-valued operator $T\colon \Omega \to 2^X$ if $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. Let M be a nonempty closed subset of X. Then an operator $T\colon \Omega\times M \to 2^X$ is called a random operator if, for each fixed $x \in M$, the operator $T(\cdot, x)\colon \Omega \to 2^X$ is measurable. A measurable operator $x\colon \Omega \to X$ is said to be a random fixed point of a random operator $T\colon \Omega \times M \to 2^X$ if $x(\omega) \in M \cap T(\omega, x(\omega))$ for all $\omega \in \Omega$. When X is a normed linear space and M is a convex subset of X, a set-valued operator $T\colon M \to 2^X$ is said to be convex if its graph

$$G(T) := \{(x,y) \in M \times X\colon y \in T(x)\}$$

is convex. Note that we say that $T\colon M \to 2^X$ is closed- (convex-, closed convex-, etc.)valued if, for each $x \in M, T(x)$ has that particular property. If C is a closed convex subset of a Banach space X, then a set-valued mapping $T\colon C \to CB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \le k\|x-y\|, \qquad x,y \in C. \tag{2.1}$$

If (2.1) holds true when $k = 1$, then T is called nonexpansive. Recall that a set-valued mapping $T\colon C \to CB(X)$ is demiclosed at 0 if, for any sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \in C$ and $y_n \in Tx_n$ for all $n \ge 1$, the conditions $x_n \to x$ weakly and $y_n \to 0$ strongly imply that $0 \in Tx$. Recall also that the Kuratowski measure of noncompactness of a nonempty bounded subset B of X is the number

$$\alpha(B) = \inf\{\varepsilon > 0\colon B \text{ can be covered by a finite number of subsets of } X \text{ of diameter less than } \varepsilon\}.$$

Then a set-valued operator $T\colon C \to 2^X$ is called condensing if, for each bounded subset B of C with $\alpha(B) > 0$, there holds the inequality

$$\alpha(T(B)) < \alpha(B).$$

Here $T(B) = \bigcup_{x\in B} Tx$.

Recall now that a set-valued operator $T\colon C \to 2^X$ is said to be upper semicontinuous on C if $\{x \in C\colon Tx \subset V\}$ is open in C whenever $V \subset X$ is open; T is said to be lower semicontinuous if $T^{-1}(V) := \{x \in C\colon Tx \cap V \neq \varnothing\}$ is open in C whenever $V \subset X$ is open; and T is said to be continuous if it is both upper and lower semicontinuous (cf. [1] and [5] for details). We note here that there is another different kind of continuity for set-valued operators: $T\colon X \to CB(X)$ is said to be continuous on X (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0$

whenever $x_n \to x$. It is not hard to see (cf. Deimling [5]) that the two kinds of continuity for a set-valued operator T are equivalent if Tx is compact for every $x \in X$. Throughout this paper, continuity for a set-valued operator always means the former one (i.e., upper-lower semicontinuity).

Now we say that a random set-valued operator $T\colon \Omega \times C \to 2^X$ is continuous (nonexpansive, condensing, etc.) if, for each fixed $\omega \in \Omega$, the operator $T(\omega, \cdot)\colon C \to 2^X$ is continuous (nonexpansive, condensing, etc.).

For later convenience, we list the following three propositions, in which (Ω, Σ) denotes a measurable space and (X, d) is a complete separable metric space.

Proposition 2.1 (cf. [1])**.** *If $T\colon \Omega \to 2^X$ is a measurable closed-valued operator, then T has a measurable selector.*

Proposition 2.2 (Castaing's Representation Theorem)**.** *If $T\colon \Omega \to 2^X$ is a closed-valued map, then the following are equivalent*:

(i) *T is measurable.*
(ii) *For each $x \in X$, the function $d(x, T(\omega))$ is measurable.*
(iii) *There exists a sequence $\{f_n(\omega)\}$ of measurable selectors of T such that*

$$\mathrm{cl}\{f_n(\omega)\} = T(\omega) \quad \textit{for all } \omega \in \Omega.$$

(*Here* $\mathrm{cl}\, A$ *denotes the closure of $A \subset X$ in X.*)

Proposition 2.3 ([7])**.** *Suppose $\{T_n\}$ is a sequence of measurable set-valued operators from Ω to $CB(X)$ and $T\colon \Omega \to CB(X)$ is an operator. If, for each $\omega \in \Omega, H(T_n(\omega), T(\omega)) \to 0$, then T is measurable.*

The following result is hinted in [2] and [8].

Proposition 2.4. *Let C be a closed convex separable subset of a Banach space X and (Ω, Σ) be a measurable space. Suppose $f\colon \Omega \to C$ is a function that is w-measurable, i.e., for each $x^* \in X^*$, the dual space of X, the numerically-valued function $x^* f\colon \Omega \to (-\infty, \infty)$ is measurable, then f is measurable.*

Proof. The conclusion follows from the separability of C and from Theorem 1.2 of Bharacha-Raid's book [2]. □

Also an easy application of Proposition 3 of Itoh [7] leads to the following result.

Proposition 2.5. *Let C and (Ω, Σ) be as in Proposition 2.4. Let $T\colon \Omega \times C \to CB(X)$ be a random continuous operator. Then for any $s > 0$, the operator $G\colon \Omega \to C$ given by*

$$G(\omega) = \{x \in C\colon d(x, T(\omega, x)) \le s\}, \qquad \omega \in \Omega,$$

is measurable.

3. The results

In this section we prove several random fixed point theorems for random multi-valued operators. We also study the measurability of certain marginal maps. We always assume that (Ω, Σ) is a measurable space with Σ a σ-algebra of subsets of Ω and do not assume (Ω, Σ) to be a Suslin family.

Theorem 3.1. *Suppose that C is a nonempty weakly compact convex separable subset of a Banach space $X, T\colon \Omega \times C \to CB(X)$ is a random continuous operator such that for each $\omega \in \Omega, I - T(\omega, \cdot)$ is demiclosed at 0, and the deterministic operator $T(\omega, \cdot)\colon C \to CB(X)$ has a fixed point. Then T has a random fixed point.*

Proof. For every $\omega \in \Omega$, set

$$F(\omega) = \{x \in C\colon x \in T(\omega, x)\}.$$

Also, for each integer $n \geq 1$, set

$$F_n(\omega) = \left\{x \in C\colon d(x, T(\omega, x)) \leq \frac{1}{n}\right\}.$$

Clearly, $F(\omega) \subset F_n(\omega)$, $F(\omega)$ is nonempty by assumption, and $F_n(\omega)$ is closed as T is continuous. Moreover, by Proposition 2.5, each F_n is measurable. It then follows from Proposition 2.1 that each F_n admits a measurable selector x_n. We thus have

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n}. \tag{3.1}$$

Define a map $G\colon \Omega \to CB(C)$ by

$$G(\omega) = \text{w-cl}\{x_n(\omega)\}, \qquad \omega \in \Omega.$$

(Here w-cl denotes the closure under the weak topology of X.) Since C is separable, the weak topology on C is metrizable. Let d_w be a metric on C which induces the weak topology on C. Then $G\colon \Omega \to (C, d_w)$ is measurable and closed-valued. It follows again from Proposition 2.1 that G has a w-measurable selector x which is measurable by Proposition 2.4. Now we claim that this x is a random fixed point of T. Since x is measurable, it remains to show that for each $\omega \in \Omega, x(\omega) \in T(\omega, x(\omega))$. In fact, by definition of G, for a fixed $\omega \in \Omega$, we have a subsequence $\{x_{n_j}(\omega)\}$ of $\{x_n(\omega)\}$ converging weakly to $x(\omega)$. It follows from (3.1) that for each $i \geq 1$ there is a $y_i(\omega) \in T(\omega, x_{n_i}(\omega))$ such that

$$\|x_{n_i}(\omega) - y_i(\omega)\| \leq \frac{2}{n_i}.$$

Now we have $x_{n_i}(\omega) - y_i(\omega) \in (I - T(\omega, \cdot))x_{n_i}(\omega), x_{n_i}(\omega) \to x(\omega)$ weakly and $x_{n_i}(\omega) - y_i(\omega) \to 0$ strongly. From the demiclosedness of $I - T(\omega, \cdot)$ at 0, it then follows that $0 \in (I - T(\omega, \cdot))x(\omega)$; that is, $x(\omega) \in T(\omega, x(\omega))$. This completes the proof. □

Corollary 3.1. *Suppose C is a weakly compact convex separable subset of a Banach space X and $T\colon \Omega \times C \to C$ is a nonexpansive random operator such that for each $\omega \in \Omega, I - T(\omega, \cdot)$ is demiclosed at 0. Suppose also that for each $\omega \in \Omega$, the (deterministic) nonexpansive mapping $T(\omega, \cdot)\colon C \to C$ has a fixed point. Then T has a random fixed point.*

Remark 3.1. Corollary 3.1 removes the assumption that the space X is strictly convex in Theorem 1(ii) of Xu [14].

Corollary 3.2 ([8]). *Suppose X is a Banach space satisfying Opial's condition, C is a weakly compact convex separable subset of X, and $T\colon \Omega \times C \to K(C)$ is a nonexpansive random operator. Then T has a random fixed point.*

Proof. In fact, Opial's condition of X implies that $I - T$ is demiclosed at 0. □

In Theorem 3.1 above, if we assume T is convex, then the demiclosedness of $I - T$ at 0 can be dropped.

Theorem 3.2. *Suppose C is a weakly compact convex separable subset of a Banach space X and $T\colon \Omega \times C \to CB(X)$ is a random, continuous and convex set-valued operator. Suppose also for each $\omega \in \Omega$, the map $T(\omega, \cdot)\colon C \to CB(X)$ has a fixed point. Then T has a random fixed point.*

Proof. As in the proof of Theorem 3.1, we set for every $\omega \in \Omega$ and integer $n \geq 1$,

$$F(\omega) = \{x \in C\colon x \in T(\omega, x)\} \quad \text{and} \quad F_n(\omega) = \left\{x \in C\colon d(x, T(\omega, x)) \leq \frac{1}{n}\right\}.$$

Then $F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega)$ and each F_n is measurable by Proposition 2.5. Moreover, since T is continuous and convex, $F(\omega)$ and $F_n(\omega)$ are closed and convex and hence weakly closed. As in the proof of Theorem 3.1, let d_w be a metric on C that produces the weak topology on C, and let H_w be the corresponding Hausdorff metric. We now show that

$$\lim_{n\to\infty} H_w(F_n(\omega), F(\omega)) = 0, \qquad \omega \in \Omega. \tag{3.2}$$

In fact, if we denote by $a(\omega)$ the limit appearing in (3.2) which exists since $\{F_n(\omega)\}$ decreases to $F(\omega)$, then it is easily seen that

$$H_w(F_n(\omega), F(\omega)) = \sup_{y \in F_n(\omega)} d_w(y, F(\omega)) \geq a(\omega).$$

If $a(\omega) > 0$, then for each integer $n \geq 1$, there is some $y_n \in F_n(\omega)$ such that

$$d_w(y_n, F(\omega)) > \frac{1}{2} a(\omega). \tag{3.3}$$

Since (C, d_w) is compact, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $d_w(y_{n_j}, y) \to 0$ for some $y \in C$, i.e., $\{y_{n_j}\}$ converges weakly to y. It then follows from (3.3) that

$$d_w(y, F(\omega)) \geq \frac{1}{2} a(\omega) > 0. \tag{3.4}$$

On the other hand, since $\{F_n(\omega)\}$ is a decreasing sequence of closed subsets in (C, d_w), we get

$$y \in \bigcap_{n=1}^{\infty} F_n(\omega) = F(\omega).$$

This contradicts (3.4). Hence (3.2) is proven. Now by Propositions 2.1 and 2.3, F is w-measurable and has a w-measurable selector x which is in turn measurable by Proposition 2.4. This x is clearly a random fixed point of T. □

Theorem 3.3. *Let C be a closed convex separable subset of a Banach space X and $T\colon \Omega \times C \to CB(X)$ be a continuous condensing random operator such that for each $\omega \in \Omega, T(\omega, C)$ is bounded. Suppose also that for each $\omega \in \Omega$, the deterministic map $T(\omega, \cdot)\colon C \to CB(X)$ has a fixed point. Then T has a random fixed point.*

Proof. As before, set for each $\omega \in \Omega$ and integer $n \geq 1$,

$$F(\omega) = \{x \in C\colon x \in T(\omega, x)\} \quad \text{and} \quad F_n(\omega) = \left\{x \in C\colon d(x, T(\omega, x)) \leq \frac{1}{n}\right\}.$$

Then $F(\omega)$ is nonempty and closed since $T(\omega,\cdot)$ is upper semicontinuous. Also, each F_n is measurable by Proposition 2.5. We now show that for each $\omega \in \Omega$,

$$\lim_{n\to\infty} H(F_n(\omega), F(\omega)) = 0. \tag{3.5}$$

Since $\{F_n(\omega)\}$ is decreasing and $\bigcap_{n=1}^{\infty} F_n(\omega) = F(\omega)$, the limit in the left side of (3.5) exists. Denote the limit by $b(\omega)$. Then we have

$$H(F_n(\omega), F(\omega)) = \sup_{y\in F_n(\omega)} d(y, F(\omega)) \geq b(\omega).$$

Thus for each $n \geq 1$, one can choose $y_n \in F_n(\omega)$ such that

$$d(y_n, F(\omega)) > b(\omega) - \frac{1}{n}. \tag{3.6}$$

Set

$$D = \{y_n\}.$$

Since each y_n lies in $F_n(\omega)$, that is,

$$d(y_n, T(\omega, y_n)) \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty, \tag{3.7}$$

it follows that

$$\alpha(T(\omega, D)) \leq \alpha(D).$$

Therefore, $\alpha(D) = 0$ for $T(\omega,\cdot)$ is condensing. This implies that $\{y_n\}$ admits a subsequence $\{y_{n'}\}$ converging to some $y \in C$. By the upper semicontinuity of T, (3.7) yields $d(y, T(\omega, y)) = 0$, i.e., $y \in F(\omega)$, which together with (3.6) implies that $b(\omega) \leq d(y, F(\omega)) = 0$, and (3.5) is thus verified. Now by Proposition 2.3, $F(\omega)$ is measurable and each measurable selector x of F is clearly a random fixed point of T. □

Corollary 3.3. (A partial random version of Browder's Fixed Point Theorem) *Suppose C is a nonempty compact convex subset of a Banach space X and $T\colon \Omega \times C \to KC(X)$ is a random continuous operator. (Here $KC(X)$ is the family of all nonempty compact convex subsets of X.) Suppose in addition either of the following two boundary conditions is satisfied*:

(i) *For each $x \in \partial C$, the boundary of C, and each $\omega \in \Omega$, there exist $y \in T(\omega, x), u \in C$, and $\lambda > 0$ such that*

$$y = x + \lambda(u - x).$$

(ii) *For each $x \in \partial C$ and $\omega \in \Omega$, there exist $y \in T(\omega, x), u \in C$, and $\lambda < 0$ such that*

$$y = x + \lambda(u - x).$$

Then T has a random fixed point.

Proof. As a matter of fact, for each fixed $\omega \in \Omega$, under either of the two boundary conditions (i) and (ii), the map $T(\omega,\cdot)\colon C \to KC(X)$ has a fixed point by Browder's Fixed Point Theorem [4]. □

Remark 3.2. It is unclear whether the conclusions of Theorem 3.3 and Corollary 3.3 are still valid if continuity of T is weakened to upper semicontinuity. (Note that this is true for deterministic mappings T.)

We finally use the above technique to discuss the measurability of a marginal function and a marginal map. The reader is referred to [1] if the underlying measurable space is a σ-finite measure space.

Theorem 3.4. *Suppose C is a weakly closed nonempty separable subset of a Banach space X, $F\colon \Omega \to 2^X$ is measurable with weakly compact values, $f\colon \Omega\times C \to \mathbb{R}^1$ (the field of real numbers) is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r\colon \Omega \to \mathbb{R}^1$ defined by*

$$r(\omega) := \inf_{x\in F(\omega)} f(\omega, x)$$

and the marginal map $R\colon \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x)\colon f(\omega, x) = r(\omega)\}$$

are measurable.

Proof. By Castaing's Representation Theorem (Proposition 2.2), we have a sequence $\{x_n\}$ of measurable selectors of $F(\omega)$ such that

$$\mathrm{cl}\{x_n(\omega)\} = F(\omega), \qquad \omega \in \Omega.$$

It follows from the continuity of f that

$$r(\omega) = \inf_{n\geq 1} f(\omega, x_n(\omega)). \tag{3.8}$$

Since each $f(\cdot, x_n(\cdot))$ is measurable, it follows from (3.8) that the marginal function r is measurable. Next we show the measurability of the marginal map R. To this end, set for each integer $n \geq 1$,

$$R_n(\omega) = \left\{x \in F(\omega)\colon f(\omega, x) \leq r(\omega) + \frac{1}{n}\right\}.$$

Then each R_n is measurable. Moreover, since f is w-l.s.c. and F is weakly compact valued, we see that R_n is also weakly compact valued. Noting that $\{R_n(\omega)\}$ decreases to $R(\omega)$, by a similar argument to that in the proof of Theorem 3.2, we get

$$\lim_{n\to\infty} H_w(R_n(\omega), R(\omega)) = 0, \qquad \omega \in \Omega.$$

(Here, as previously, H_w denotes the Hausdorff metric induced by the weak topology on C.) It follows from Propositions 2.3 and 2.4 that R is measurable. □

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