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SOME REMARKS ON THE VARIATION OF CURVE LENGTH AND SURFACE AREA

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Abstract. Consider the curve $C = \{(t, f(t) : 0 \leq t \leq 1\}$, where f is absolutely continuous on $[0,1]$. Then C has finite length, and if U_ϵ is the ϵ-neighborhood of f in the uniform norm, we compare the length of the shortest path in U_ϵ with the length of f. Our main result establishes necessary and sufficient conditions on f such that the difference of these quantities is of order ϵ as $\epsilon \to 0$. We also include a result for surfaces.

1. Introduction

Let f be absolutely continuous on $[0,1]$, and consider the curve $C = \{(t, f(t)) : 0 \leq t \leq 1\}$. Then C has finite length given by

$$L(f) = \int_0^1 \sqrt{1 + (f'(s))^2}\, ds,$$

and since $L(f) = L(g)$ when f differs from g by a constant, we assume throughout that $f(0) = 0$. Let

$$\mathcal{G} = \{h : h \text{ is absolutely continuous on } [0,1], h(0) = 0\},$$

and for $\delta > 0$ set

$$L(f, \delta) = \inf_{\|f-h\| \leq \delta, h \in \mathcal{G}} L(h),$$

where $\|h\| = \sup_{0 \leq s \leq 1} |h(s)|$ denotes the usual sup-norm. In a probability problem encountered recently (see [G] and [KLT] for background and the related problem), we were motivated to consider the asymptotic behavior of $L(f) - L(f, \delta)$ as $\delta \to 0$. In particular, we want to determine for which absolutely continuous f do we have

$$\lim_{\delta \to 0} (L(f) - L(f, \delta))/\delta = c_f \tag{1.1}$$

where $0 < c_f < \infty$, and what is c_f? The answer appears in Theorem 1 below. A verbal description of the shortest path in a general region connecting two given

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points of that region can be found in [L, p. 36]. However, we found it of no use in this problem. Now we need some further notation.

If f is a real-valued measurable function on $[0,1]$, then by a version of f we mean a measurable function g, finite everywhere on $[0,1]$, such that $g = f$ almost everywhere with respect to Lebesgue measure. If f has a version of bounded variation, then it is easy to see that there is one and only one function λ of bounded variation on $[0,1]$ such that $\lambda = f$ a.e., λ is right continuous on $[0,1)$, and left continuous at one. We call this the canonical version of f. Finally, we write $V(g)$ to denote the variation of a function g on $[0,1]$.

The answer to the question above is the following result.

Theorem 1. *Let f be absolutely continuous on $[0,1]$ with $f(0) = 0$, but $\|f\| > 0$. Then*

$$\lim_{\delta\to 0}(L(f) - L(f,\delta))/\delta = c_f \tag{1.2}$$

where $0 < c_f < \infty$ iff $\gamma(s) = f'(s)/(1 + (f'(s))^2)^{1/2}$ has a version of bounded variation on $[0,1]$. Furthermore,

$$c_f = |\lambda(1)| + V(\lambda) \tag{1.3}$$

where λ is the canonical version of $\gamma(s)$ on $[0,1]$.

Remarks. (A) If $f(t) \equiv 0$ on $[0,1]$, then $L(f,\delta) = L(f) = 1$ and $c_f = 0$. Hence the assumption $\|f\| > 0$ is a necessity if we want $c_f > 0$.

(B) If we replace the sup-norm by an L^p-norm, $1 \le p < \infty$, in the definition of $L(f,\delta)$, then it is natural to ask if (1.2) still holds with $c_f \in (0,\infty)$. The following example shows this is not the case. Indeed, our first example shows that even for the simplest choice of $f(t)$, (1.2) fails for any L^p-norm, $1 \le p < \infty$.

(C) Example 1 also suggests that the proper rate of convergence when $1 \le p < \infty$ and $\gamma(s)$ has a version of bounded variation might be $\delta^{p/(p+1)}$. Example 2 shows this is not the case.

Example 1. Let $f(t) = t, 0 \le t \le 1$, and set

$$h_\delta(t) = \begin{cases} t, & 0 \le t \le 1 - x_{\delta,p}, \\ 1 - x_{\delta,p}, & 1 - x_{b,p} \le t \le 1, \end{cases}$$

where $x_{\delta,p} = ((p+1)\delta^p)^{1/(p+1)}$. Then for $1 \le p < \infty$ fixed and $\delta > 0$ sufficiently small

$$\|f - h_\delta\|_p^p = \int_{1-x_{\delta,p}}^1 (t - (1 - x_{\delta,p}))^p dt = x_{\delta,p}^{p+1}/(p+1) = \delta^p.$$

Furthermore,

$$L(h_\delta) = \sqrt{2}(1 - x_{\delta,p}) + x_{\delta,p},$$

and hence using the L^p-norm in the definition of $L(f,\delta)$ we have

$$\begin{aligned} \lim_{\delta\to 0}(L(f) - L(f,\delta))/\delta &\ge \varliminf_{\delta\to 0}(L(f) - L(h_\delta))/\delta \\ &= \varliminf_{\delta\to 0}(\sqrt{2} - 1)x_{\delta,p}/\delta = \infty \end{aligned}$$

since $1 \le p < \infty$.

Example 2. In this example, we only consider the L_1-norm. Its purpose is to show that if $f(t) = \sqrt{1-(1-t)^2}, 0 \le t \le 1$, then

$$\lim_{\delta \to 0} (L(f) - L(f,\delta))/\delta = 1,$$

and hence the obvious conjecture suggested by Example 1 is false. Given $f(t)$ as above, we have that the functions γ and λ of Theorem 1 are $\gamma(s) = \lambda(s) = 1-s, 0 \le s \le 1$. To compute

$$L(f,\delta) = \inf_{\|f-h\|_1 \le \delta, h \in \mathcal{G}} L(h),$$

we easily see that it suffices to take $h \in \mathcal{G}$ such that $0 \le h(s) \le f(s)$, $0 \le s \le 1$, and $h'(s) \ge 0$ a.e. on $[0,1]$. Thus if $g = h - f$, then $g(s) \le 0$ on $[0,1]$ and the constraint $\|f-h\|_1 = \delta$ implies $\int_0^1 g(s)ds = -\delta$. Looking ahead to the proof of Theorem 1 we see from (2.5) and integration by parts that any $h \in \mathcal{G}$ constrained as above satisfies

$$L(h) \ge L(f) + \int_0^1 g(s)ds$$

since $\gamma(s) = 1-s$. Hence $L(h) \ge L(f) - \delta$, so

$$\overline{\lim_{\delta \to 0}} (L(f) - L(f,\delta))/\delta \le 1.$$

Hence it remains to show we can find a function $h_\delta \in \mathcal{G}$ such that $\|f - h_\delta\|_1 = \delta$ and

$$\underline{\lim_{\delta \to 0}} (L(f) - L(f,\delta))/\delta = 1.$$

We use Lagrange multipliers to find the function h_δ (see, for example, [T]), and recall that we may assume $0 \le h_\delta(s) \le f(s), 0 \le s \le 1, h_\delta \in \mathcal{G}$, and $h_\delta' \ge 0$ on [0,1]. To find the possible (local) extremal points for $\int_0^1 \sqrt{1+(y')^2}dt$ on $\mathcal{D} = \{y \in C^1[0,1] : y(0) = 0\}$ under the constraining relation

$$\int_0^1 (\sqrt{1-(1-t)^2} - y)dt = \delta, \tag{1.4}$$

we have for some β and all $v(t) \in C^1[0,1], v(0) = 0$, that

$$\int_0^1 \frac{y'}{\sqrt{1+(y')^2}} v'(t)dt = \beta \int_0^1 v(t)dt. \tag{1.5}$$

Integration by parts on the right side of (1.5) implies

$$\int_0^1 \frac{y'}{\sqrt{1+(y')^2}} v'(t)dt = \beta \int_0^1 (1-t)v'(t)dt. \tag{1.6}$$

Since (1.6) holds for all $v(t) \in C^1[0,1], v(0) = 0$, we have

$$y'/(\sqrt{1+(y')^2}) = \beta(1-t), \quad 0 \le t \le 1. \tag{1.7}$$

Solving (1.7) when $\beta > 0$ ($\beta \leq 0$ does not provide a solution for our problem since then $y' < 0$), we have

$$y = \int_0^t \frac{\beta(1-x)}{\sqrt{1-\beta^2(1-x)^2}}dx = \frac{1}{\beta}\sqrt{1-\beta^2(1-t)^2} - \frac{1}{\beta}\sqrt{1-\beta^2}.$$

Hence from the constraining relation (1.4), β is the solution of the equation

$$\frac{\pi}{4} - \frac{1}{2\beta}\left(\frac{1}{\beta}\arcsin\beta - \sqrt{1-\beta^2}\right) = \delta,$$

and it is easy to see that $\beta \to 1$ as $\delta \to 0$. Thus

$$\begin{aligned}
&\lim_{\delta\to 0}(L(f) - L(h_\delta))/\delta \\
&\quad = \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_0^1 \left(\sqrt{1+(f')^2} - \sqrt{1+(y')^2}\right)dt\right) \\
&\quad = \lim_{\beta\to 1}\left(\frac{\pi}{2} - \frac{1}{\beta}\arcsin\beta\right) \Big/ \left(\frac{\pi}{4} - \frac{1}{2\beta}\left(\frac{1}{\beta}\arcsin\beta - \sqrt{1-\beta^2}\right)\right) \\
&\quad = 1.
\end{aligned}$$

The first proof we obtained for Theorem 1 was very constructive, but this approach failed when we tried to prove a similar result for surfaces. Our second approach is contained in the proof of Theorem 1 below, and applies to surfaces as well. Unfortunately, the result we can prove for surfaces is not as complete as Theorem 1 in that the precision of (1.3) is lacking. The scaling idea in our second approach emerged in some discussions with Tom Ilmanen, and we thank him for his interest in these results.

2. Proof of the theorem

Throughout $f \in \mathcal{G}$ and $\gamma(s) = f'(s)/(1+(f'(s))^2)^{1/2}$. If $F(x) = (1+x^2)^{1/2}$, then $F'(x) = x/(1+x^2)^{1/2}$, $F''(x) = (1+x^2)^{-3/2}$, and for $f, g \in \mathcal{G}$, Taylor's formula implies for almost all $s \in [0,1]$ that

$$\begin{aligned}
(1+(h'(s))^2)^{1/2} &= (1+(f'(s))^2)^{1/2} \\
&\quad + \frac{f'(s)}{(1+(f'(s))^2)^{1/2}}(h'(s) - f'(s)) + \frac{1}{2}\frac{(h'(s)-f'(s))^2}{(1+(\tau(s))^2)^{3/2}}
\end{aligned} \tag{2.1}$$

where $\tau(s)$ is between $f'(s)$ and $h'(s)$. The proof will now proceed via a sequence of three lemmas. We write $C_0^1[0,1]$ to denote the continuously differentiable functions on $[0,1]$ which are zero at zero.

Lemma 1. *Let $f \in \mathcal{G}$ with $\|f\| > 0$ and assume*

$$T(g) = \int_0^1 \gamma(s)g'(s)ds \qquad (g \in \mathcal{G}) \tag{2.2}$$

where $\gamma(s) = f'(s)/(1+(f'(s))^2)^{1/2}$. *If*

$$\Lambda = \sup_{\substack{g\in\mathcal{G}\\ \|g\|\le 1}} T(g) < \infty, \tag{2.3}$$

then

$$\lim_{\delta\to 0}(L(f) - L(f,\delta))/\delta = c_f \tag{2.4}$$

where $0 < c_f < \infty$. *Furthermore,* $c_f = \Lambda$.

Proof. Since f is absolutely continuous with $f(0) = 0$, (2.1) implies with $g = h - f \in \mathcal{G}$ that

$$L(h) \ge L(f) + T(g). \tag{2.5}$$

Now $L(f,\delta) = \inf_{\substack{\|f-h\|\le\delta\\ h\in\mathcal{G}}} L(h)$, and hence (2.3), (2.5), symmetry, and homogeneity imply

$$L(f,\delta) \ge L(f) - \Lambda\delta. \tag{2.6}$$

Thus,

$$\overline{\lim_{\delta\to 0}}(L(f) - L(f,\delta))/\delta \le \Lambda\,. \tag{2.7}$$

On the other hand, by integrating (2.1) we see

$$L(h) \le L(f) + T(g) + e(g) \tag{2.8}$$

where $0 \le e(g) \le \int_0^1 (g'(s))^2 ds$. Thus fix $\epsilon > 0$, and observe that (2.3) implies the linear functional T has a unique continuous extension, call it $\tilde{T}$, to all of $C_0[0,1]$, and we also have

$$\Lambda = \sup_{\substack{g\in C_0[0,1]\\ \|g\|\le 1}} \tilde{T}(g). \tag{2.9}$$

Since $C_0^1[0,1]$ is sup-norm dense in $C_0[0,1]$, and hence also in $\mathcal{G}$, and T is symmetric, we have $g_0 \in C_0^1[0,1]$ with $\|g_0\| \le 1$ such that

$$T(g_0) < -\Lambda + \epsilon. \tag{2.10}$$

Let $g_\delta = \delta g_0$ for $\delta > 0$. Then $g_\delta \in C_0^1[0,1]$, $\|g_\delta\| \le \delta$, and by homogeneity $T(g_\delta) < \delta(-\Lambda + \epsilon)$. Furthermore, since $g_0 \in C_0^1[0,1]$, we have $\int_0^1 (g_0'(s))^2 ds = b < \infty$, hence

$$e(g_\delta) \le \delta^2 b. \tag{2.11}$$

Thus (2.8) implies if $h_\delta = g_\delta + f$ that

$$L(h_\delta) \le L(f) + \delta(-\Lambda + \epsilon) + \delta^2 b. \tag{2.12}$$

Hence by definition of $L(f,\delta)$ we have

$$L(f,\delta) \le L(f) + \delta(-\Lambda + \epsilon) + \delta^2 b, \tag{2.13}$$

and therefore

$$\Lambda - \epsilon \le \underline{\lim_{\delta\to 0}}(L(f) - L(f,\delta))/\delta\,. \tag{2.14}$$

Combining (2.7) and (2.14), we have (2.4) with $c_f = \Lambda < \infty$. That $\Lambda > 0$ follows easily since $f \in \mathcal{G}$ with $\|f\| > 0$. Hence Lemma 1 is proven.

Lemma 2. *If* $\Lambda = \sup_{\substack{g\in\mathcal{G}\\ \|g\|\le 1}} T(g)$ *where* $T(g)$ *is as in (2.2), then* $\Lambda < \infty$ *iff* γ *has a version of bounded variation on* $[0,1]$. *Furthermore, if* $\Lambda < \infty$, *then* $\Lambda = |\lambda(1)| + V(\lambda)$ *where* λ *is the canonical version of* γ.

Proof. Assume γ has a version of bounded variation on $[0,1]$ and let λ be the canonical version of γ. Since $\mathcal{G} \supseteq C_0^1[0,1]$, we then have for $g \in C_0^1[0,1]$ that

$$\begin{aligned} T(g) = \int_0^1 \gamma(s)g'(s)ds &= \int_0^1 \lambda(s)g'(s)ds \\ &= \int_0^1 \lambda(s)dg(s) \\ &= \lambda(1)g(1) - \int_0^1 g(s)d\lambda(s). \end{aligned} \tag{2.15}$$

Since λ is of bounded variation, (2.15) implies T has a unique extension to $C_0[0,1]$ such that

$$\sup_{\|g\|\le 1} T(g) < \infty,$$

and hence $\Lambda < \infty$. Further, since λ is right continuous on $[0,1)$ and left continuous at 1 (remember λ is canonical), we have

$$T(g) = \int_0^1 g d\mu, \qquad g \in C_0[0,1], \tag{2.16}$$

where $\mu = \lambda(1)\delta_1 - \lambda$, and δ_1 and λ are mutually singular. Hence $V(\mu) = |\lambda(1)| + V(\lambda)$, and by the Riesz Representation Theorem (and duality), we have

$$\Lambda = \sup_{\substack{g\in\mathcal{G}\\ \|g\|\le 1}} T(g) = \sup_{\|g\|\le 1} T(g) = |\lambda(1)| + V(\lambda). \tag{2.17}$$

Conversely, if $\Lambda < \infty$, then as in Lemma 1 the linear functional T has a unique continuous extension $\tilde{T}$ to all of $C_0[0,1]$. Hence by the Riesz Representation Theorem there exists a function λ of bounded variation on $[0,1]$ such that for $g \in C_0[0,1]$

$$\tilde{T}(g) = \int_0^1 g(s)d\lambda(s) = \int_0^1 (\lambda(1) - \lambda(s))dg(s), \tag{2.18}$$

where the last equality is by integration by parts. Hence for $g \in C_0^1[0,1]$ we have

$$\int_0^1 (\lambda(1) - \lambda(s))g'(s)ds = \int_0^1 \gamma(s)g'(s)ds,$$

and this implies $\gamma(s) = \lambda(1) - \lambda(s)$ a.s. Thus Lemma 2 is proven.

To complete the proof of Theorem 1 it now suffices to show that

$$\lim_{\delta\to 0}(L(f) - L(f,\delta))/\delta < \infty$$

implies $\Lambda < \infty$. This is accomplished by showing the following:

Lemma 3. *If Λ is given by (2.3) and $\Lambda = \infty$, then*

$$\lim_{\delta \to 0} (L(f) - L(f, \delta))/\delta = \infty, \tag{2.19}$$

and γ does not have a version of bounded variation.

Proof. First we show that if $\Lambda = \infty$, then

$$\sup_{\substack{g \in C_0^1[0,1] \\ \|g\| \le 1}} T(g) = \infty. \tag{2.20}$$

For this we take $M > 0$ arbitrarily large and $\epsilon > 0$ small. Since $\Lambda = \infty$, there exists $g_0 \in \mathcal{G}$, $\|g_0\| = 1$, such that $T(g_0) > (M+1)/(1+\epsilon)$. Now $g_0' \in L^1[0,1]$, and hence there exists $\tilde{g} : [0,1] \to \mathbb{R}$ continuous such that

$$\int_0^1 |\tilde{g}(u) - g_0'(u)| du < \epsilon.$$

Letting $h(s) = \int_0^s \tilde{g}(u) du, 0 \le s \le 1$, we have $h \in C_0^1[0,1]$ and since $\|\gamma\|_\infty \le 1$, we have for $0 < \epsilon < \frac{1}{2}$ that

$$\begin{aligned} T(h) = \int_0^1 \gamma(s) h'(s) ds &= \int_0^1 \gamma(s) \tilde{g}(s) ds \\ &\ge \int_0^1 \gamma(s) g_0'(s) ds - \epsilon \\ &> M/(1+\epsilon). \end{aligned}$$

Since M is arbitrarily large, this gives (2.20) since

$$\|h\| = \sup_{0 \le s \le 1} |\int_0^s \tilde{g}(u) du| \le \|g_0\| + \epsilon \le 1 + \epsilon.$$

To finish the proof we again take $M > 0$ arbitrarily large, and since (2.20) holds, select $g_0 \in C_0^1[0,1]$ such that

$$T(g_0) \le -M$$

and $\|g_0\| \le 1$. Then for $g_\delta = \delta g_0$ and $h_\delta = g_\delta + f$ we have from (2.1) that

$$L(f, \delta) \le L(h_\delta) \le L(f) + \delta(-M) + \delta^2 b$$

where $b = \int_0^1 (g_0'(s))^2 ds < \infty$. Since M is arbitrary, this implies (2.19). The proof of Lemma 3 is now completed by applying Lemma 2 to see $\Lambda = \infty$ implies γ does not have a version of bounded variation.

To complete the proof of Theorem 1 we observe that Lemmas 1 and 3 combine to give (1.2) with $0 < c_f < \infty$ iff $\Lambda < \infty$, and they identify c_f with Λ. Finally, Lemma 2 links $\Lambda < \infty$ to γ having a version of bounded variation as required. Hence the theorem is proved.

3. A result for surfaces

Using the ideas in the proof of Theorem 1 we can establish a result for surfaces. First we describe the necessary notation. Let U be an open subset of $\mathbb{R}^2$. Then $f \in L^1_{\text{loc}}(U)$ if $f \in L^1(V, dxdy)$ for all open sets V such that the closure of V is a compact subset of U. If $f \in L^1_{\text{loc}}(U)$, then the function $g \in L^1_{\text{loc}}(U)$ is said to be the weak partial derivative of f with respect to x in U if

$$\int\int_U f(x,y)\frac{\partial}{\partial x}\phi(x,y)dxdy = -\int\int_U g(x,y)\phi(x,y)dxdy \tag{3.1}$$

for all ϕ continuously differentiable with compact support in U, i.e. $\phi \in C^1_c(U)$. Of course, a similar definition holds for the weak partial derivative of f with respect to y on U, and we denote both the weak partial derivatives and the ordinary partial derivatives of f by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Let $Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, and set $|Df|^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$. If $f : U \to \mathbb{R}$ we say f is in the Sobolev space $W^{1,1}(U)$ if $f \in L^1(U)$ and the weak partial derivatives both exist and are in $L^1(U)$. For $f \in W^{1,1}(U)$ we define the $W^{1,1}(U)$ norm by

$$\|f\|_{W^{1,1}(U)} = \int\int_U (|f| + |Df|)dxdy. \tag{3.2}$$

Let $Q = [0,1] \times [0,1]$, $Q^0 = (0,1) \times (0,1)$, and set

$$\mathcal{H} = C(Q) \cap W^{1,1}(Q^0) \tag{3.3}$$

where $C(Q)$ denotes the continuous functions on Q. If $f \in \mathcal{H}$, then

$$S(f) = \int\int_Q \sqrt{1 + |Df|^2}\, dxdy < \infty\,, \tag{3.4}$$

and $S(f)$ represents the surface area of the graph

$$G_f = \{(x, y, f(x,y)) : (x,y) \in Q\}\,.$$

If $f \in C(Q)$, $\delta > 0$, let

$$S(f,\delta) = \inf_{\substack{h \in \mathcal{H} \\ \|f-h\| \le \delta}} S(h) \tag{3.5}$$

where $\|\cdot\|$ is the sup-norm on $C(Q)$. Throughout this section we also use $\text{Lip}(Q)$ to denote the Lipschitz continuous functions on Q. Our result for surfaces is the next proposition.

Proposition 1. *Let $f \in \mathcal{H}$. If $F(u,v) = (1 + u^2 + v^2)^{1/2}$ and*

$$T_f(h) = \int\int_Q \text{grad}\, F\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot Dh dxdy \quad (h \in \mathcal{H}) \tag{3.6}$$

is such that

$$\Lambda_f = \sup_{\substack{h \in \mathcal{H} \\ \|h\| \le 1}} T_f(h) < \infty, \tag{3.7}$$

then

$$\lim_{\delta\to 0}(S(f)-S(f,\delta))/\delta=\Lambda_f\,, \tag{3.8}$$

and f non-constant implies $\Lambda_f>0$. Furthermore, if $f\in\mathcal{H}$ and

$$\sup_{\substack{h\in Lip(Q)\\ \|h\|\le 1}} T_f(h)=\infty, \tag{3.9}$$

then

$$\lim_{\delta\to 0}(S(f)-S(f,\delta))/\delta=\infty. \tag{3.10}$$

Remark. Proposition 1 clearly lacks the precision of (1.3), but if we replace $\mathcal{H}$ in (3.5) by a linear subspace $\mathcal{M}$ of $\mathcal{H}$ containing $\mathrm{Lip}(Q)$, and satisfying

$$\Lambda_f=\sup_{\substack{h\in\mathcal{M}\\ \|h\|\le 1}} T_f(h)=\sup_{\substack{h\in Lip(Q)\\ \|h\|\le 1}} T_f(h), \tag{3.11}$$

then

$$\lim_{\delta\to 0}(S(f)-S(f,\delta))/\delta=\Lambda_f, \tag{3.12}$$

regardless of whether Λ_f is finite or infinite. This improvement follows since the scaling argument applies even if $\Lambda_f=\infty$ when (3.11) holds. Of course, $\mathcal{M}=\mathrm{Lip}(Q)$ obviously satisfies (3.11), but by applying the approximation ideas in [EG, p. 123] it is easy to see that $\mathcal{M}=\{f:\exists \text{ open set } \ V_f\supseteq Q \ \text{ and } \ f\in W^{1,1}(V_f)\cap C(V_f)\}$ also does.

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