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SHARP MAXIMAL INEQUALITIES FOR STOCHASTIC INTEGRALS IN WHICH THE INTEGRATOR IS A SUBMARTINGALE

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ABSTRACT. We obtain sharp maximal inequalities for strong subordinates of real-valued submartingales. Analogous inequalities also hold for stochastic integrals in which the integrator is a submartingale. The impossibility of general moment inequalities is also demonstrated.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_0$ contains all P-null sets. Suppose X is an adapted right-continuous real-valued submartingale with left limits and H is a predictable process with values in the closed unit ball of $\mathbb{R}^\nu$, where ν is a positive integer. Define an adapted right-continuous process Y with left limits by

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s \, dX_s.$$

We will compare the size of Y with that of X by finding constants β such that for all $\lambda > 0$,

$$\lambda P(Y^* \geq \lambda) \leq \beta \|X\|_1 \tag{1.1}$$

where $\|X\|_1 = \sup_{t\geq 0} \|X_t\|_1$ and $Y^* = \sup_{t\geq 0} |Y_t|$. In this paper we will denote the Euclidean norm of $y \in \mathbb{R}^\nu$ by $|y|$ and the inner product of $y, k \in \mathbb{R}^\nu$ by $y \cdot k$.

If we restrict X to the class of martingales, it is known that the best constant satisfying (1.1) is $\beta = 2$ [2, 3]. By the best constant we mean that for any $\beta < 2$ there exist a martingale X, a predictable process H, and a $\lambda > 0$ such that $\lambda P(Y^* \geq \lambda) > \beta \|X\|_1$. It is also known [5] that if we restrict X to the class of nonnegative submartingales, then the best constant satisfying (1.1) is $\beta = 3$.

In this paper we will show that for the class of real-valued submartingales, the best constant in (1.1) is $\beta = 6$. To do this we shall first prove the analogous inequality and more for discrete-time submartingales. In the last section of this

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paper we shall show that there are no moment inequalities of the form $\|Y\|_p \leq \beta \|X\|_p$ where $1 < p < \infty$ and β is finite and depends only on p. In fact, we shall show that for any $p \in [1, \infty)$, there is no finite β such that $\|Y\|_1 \leq \beta \|X\|_p$. For the case $p = \infty$, see [7] where it is shown that if $\|X\|_\infty = 1$, then there is a constant γ such that for $\lambda > 4$, $P(Y^* \geq \lambda) \leq \gamma \exp(-\lambda/4)$, so, for any $r \in [1, \infty)$, $\|Y\|_r$ is bounded by some constant depending only on r.

2. A maximal inequality for submartingales

Let $f_0, f_1, \dots$ be a real-valued submartingale relative to a filtration $(\mathcal{F}_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ with difference sequence $d_0, d_1, \dots$, and $g_0, g_1, \dots$ an $\mathbb{R}^\nu$-valued process adapted to $(\mathcal{F}_n)_{n\geq 0}$ with difference sequence $e_0, e_1, \dots$, where ν is a positive integer. We say that g is strongly subordinate to f if g is both differentially subordinate and conditionally differentially subordinate to f, i.e. for all $n \geq 0$, $|e_n| \leq |d_n|$ and $|\mathbf{E}(e_{n+1}|\mathcal{F}_n)| \leq |\mathbf{E}(d_{n+1}|\mathcal{F}_n)|$. Note that if for $k \geq 0$, $e_k = h_k d_k$ where $h_k : \Omega \to [-1, 1]$ is $\mathcal{F}_{k-1}$-measurable, then g is strongly subordinate to f. In particular, if g is a ± 1-transform of f, i.e. $e_k = \epsilon_k d_k$ where $\epsilon_k \in \{-1, 1\}$, then g is strongly subordinate to f.

Theorem 2.1. *If $f = (f_n)_{n\geq 0}$ is a submartingale relative to a filtration $(\mathcal{F}_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ is strongly subordinate to f, then for all $\lambda > 0$,*

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n\geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \sup_{n\geq 0} \mathbf{E} f_n^+ - 2\mathbf{E} f_0 \tag{2.1}$$

where $g^* = \sup_{n\geq 0} |g_n|$.

Remarks. If f is a martingale, then $\mathbf{E} f_n^+$ and $\mathbf{E} f_n^-$ are nondecreasing sequences. It then follows from $\mathbf{E} f_0 = \mathbf{E} f_n^+ - \mathbf{E} f_n^-$ that $\|f\|_1 = 2 \sup_{n\geq 0} \mathbf{E} f_n^+ - \mathbf{E} f_0$, where $\|f\|_1 = \sup_{n\geq 0} \|f_n\|_1$. Thus in the martingale case, (2.1) implies that

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n\geq 0}(|f_n| + |g_n|) \geq \lambda) \leq 2 \|f\|_1$$

which is Theorem 4.1 of [4]. If f is a nonnegative supermartingale, (2.1) implies

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n\geq 0}(|f_n| + |g_n|) \geq \lambda) \leq 2\mathbf{E} f_0$$

which is Theorem 8.1 of [5]. Both results are shown to be sharp in the articles quoted. If f is a nonnegative submartingale with $f_0 = 0$, the resulting inequality is not sharp in the case $f_0 = 0$, as can be seen from Theorem 4.1 of [5] which shows in this case

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n\geq 0}(|f_n| + |g_n|) \geq \lambda) \leq 3 \|f\|_1 .$$

Proof. We will assume $\|f\|_1$ is finite. This is equivalent to saying $\sup_{n\geq 0} \mathbf{E} f_n^+$ is finite, as for all $n \geq 0$, $\mathbf{E} f_n^+ \leq \|f_n\|_1 \leq 2\mathbf{E} f_n^+ - \mathbf{E} f_0$. The first inequality is obvious, the second follows from $\mathbf{E} f_0 \leq \mathbf{E} f_n = \mathbf{E} f_n^+ - \mathbf{E} f_n^-$.

To show (2.1), it suffices to show that for $n \geq 0$,

$$\lambda P(|f_n| + |g_n| \geq \lambda) \leq 4\mathbf{E} f_n^+ - 2\mathbf{E} f_0 \tag{2.2}$$

since if (2.2) holds, then with $\tau = \inf\{n \geq 0 \ : \ |f_n| + |g_n| \geq \lambda\}$, τ is a stopping time, f^τ is a submartingale, and g^τ is strongly subordinate to f^τ, so by (2.2)

$$\lambda P(\sup_{m \leq n} (|f_m| + |g_m|) \geq \lambda) = \lambda P(|f_{\tau\wedge n}| + |g_{\tau\wedge n}| \geq \lambda) \leq 4\mathbf{E}f^+_{\tau\wedge n} - 2\mathbf{E}f_0.$$

Since $(f_n^+)_{n\geq 0}$ is a submartingale, it follows by Doob's optional sampling theorem that $\mathbf{E}f^+_{\tau\wedge n} \leq \mathbf{E}f_n^+$, thus implying (2.1).

By dividing by λ throughout in (2.2), we may assume $\lambda = 1$. Using the methods developed by Burkholder [2], we define V on $\mathbb{R} \times \mathbb{R}^\nu$ by

$$V(x,y) = \begin{cases} 1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\ -4x^+, & \text{if } |x| + |y| < 1. \end{cases}$$

Then (2.2) is equivalent to $\mathbf{E}V(f_n, g_n) \leq -2\mathbf{E}f_0$. Define U on $\mathbb{R} \times \mathbb{R}^\nu$ by

$$U(x,y) = \begin{cases} 1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\ |y|^2 - x^2 - 2x, & \text{if } |x| + |y| < 1. \end{cases}$$

Then $V \leq U$ (in the case of $|x| + |y| < 1$ this follows from $-4x^+ \leq -x^2 - 2x$ for $|x| < 1$) and $U(f_0, g_0) \leq -2f_0$ (recall that by assumption $|f_0| \geq |g_0|$).

Thus $\mathbf{E}V(f_n, g_n) \leq \mathbf{E}U(f_n, g_n)$ and $\mathbf{E}U(f_0, g_0) \leq -2\mathbf{E}f_0$. To show (2.2), it will suffice to show that for $1 \leq j \leq n$,

$$\mathbf{E}U(f_j, g_j) \leq \mathbf{E}U(f_{j-1}, g_{j-1}). \tag{2.3}$$

Define ϕ, ψ on $\mathbb{R} \times \mathbb{R}^\nu$ by

$$\phi(x,y) = \begin{cases} -4, & \text{if } |x| + |y| \geq 1 \text{ and } x \geq 0, \\ 0, & \text{if } |x| + |y| \geq 1 \text{ and } x < 0, \\ -2x - 2, & \text{if } |x| + |y| < 1, \end{cases}$$

$$\psi(x,y) = \begin{cases} 0, & \text{if } |x| + |y| \geq 1, \\ 2y, & \text{if } |x| + |y| < 1. \end{cases}$$

Then $U_x(x,y) = \phi(x,y)$ and $U_y(x,y) = \psi(x,y)$ for $|x| + |y| \neq 1$, $y \neq 0$, and $x \neq 0$ where $U_x(x,y)$ and $U_y(x,y)$ are the partials of U with respect to x and y respectively. Note that $|\psi| \leq -\phi$.

Claim: Given $h \in \mathbb{R}$ and $k \in \mathbb{R}^\nu$ with $|k| \leq |h|$, then for all $x \in \mathbb{R}$ and $y \in \mathbb{R}^\nu$

$$U(x+h, y+k) \leq U(x,y) + \phi(x,y)h + \psi(x,y) \cdot k. \tag{2.4}$$

This can be verified by checking the various cases:

For $|x| + |y| \geq 1$ and $x \geq 0$, we need to show $U(x+h, y+k) \leq 1 - 4(x+h)$. For $|x+h| + |y+k| \geq 1$ this is clear. For $|x+h| + |y+k| < 1$ it follows from

$$|y+k|^2 < (1 - |x+h|)^2 \leq 1 - 2(x+h) + (x+h)^2.$$

For $|x| + |y| \geq 1$ and $x < 0$, we need to show $U(x+h, y+k) \leq 1$. However $U(x,y) \leq 1$ for all x, y, this being obvious for $|x|+|y| \geq 1$. In the region $|x|+|y| < 1$, since $U_x(x,y) \leq 0$, it follows that $U(x,y) \leq |y|^2 - (|y|-1)^2 - 2(|y|-1) = 1$.

For the case $|x| + |y| < 1$, (2.4) is equivalent to showing

$$U(x+h, y+k) \le |y+k|^2 - (x+h)^2 - 2(x+h) - |k|^2 + h^2. \tag{2.5}$$

For $|x+h| + |y+k| < 1$, this follows from $|k| \le |h|$ and the definition of U. For $|x+h| + |y+k| \ge 1$, (2.5) can be rewritten as

$$(1 - |x+h|)^2 \le |y+k|^2 + h^2 - |k|^2 .$$

If $|x+h| \le 1$, this inequality follows from $|k| \le |h|$ and $|x+h| + |y+k| \ge 1$. If $|x+h| > 1$, then $(1-|x+h|)^2 \le (1-|x|-|h|)^2$ and it suffices to show

$$(1-|x|)^2 - 2|h|(1-|x|) \le |y|^2 - 2|y||k| .$$

Since $|h| \ge |k|$, it then suffices to show $(1-|x|)^2 \le |y|^2 + 2|h|(1-|x|-|y|)$, an inequality which follows from $|x| + |h| \ge 1$ and $0 \le |y|^2 - 2|y|(1-|x|) + (1-|x|)^2$, so that

$$(1-|x|)^2 \le |y|^2 + 2(1-|x|)(1-|x|-|y|) \le |y|^2 + 2|h|(1-|x|-|y|).$$

To prove (2.3), since $|e_j| \le |d_j|$, by (2.4) we have

$$U(f_j, g_j) \le U(f_{j-1}, g_{j-1}) + \phi(f_{j-1}, g_{j-1})d_j + \psi(f_{j-1}, g_{j-1}) \cdot e_j. \tag{2.6}$$

Since f is a submartingale, $\mathbf{E}(d_j|\mathcal{F}_{j-1}) \ge 0$. It then follows from $|\psi| \le -\phi$ and g being strongly subordinate to f that

$$\phi(f_{j-1}, g_{j-1})\mathbf{E}(d_j|\mathcal{F}_{j-1}) + \psi(f_{j-1}, g_{j-1}) \cdot \mathbf{E}(e_j|\mathcal{F}_{j-1}) \le 0.$$

Using this after taking the conditional expectations relative to $\mathcal{F}_{j-1}$ in (2.6) gives

$$\mathbf{E}(U(f_j, g_j)|\mathcal{F}_{j-1}) \le U(f_{j-1}, g_{j-1}).$$

Taking expectations of both sides gives (2.3) and completes the proof.

3. Discrete-time sharp maximal inequalities

Theorem 3.1. *If f is a submartingale relative to a filtration $(\mathcal{F}_n)_{n\ge0}$ and g is strongly subordinate to f, then for all $\lambda > 0$*

$$\lambda P(g^* \ge \lambda) \le \lambda P(\sup_{n\ge0} (|f_n| + |g_n|) \ge \lambda) \le 4\|f\|_1 - 2\mathbf{E}f_0. \tag{3.1}$$

Thus if $f_0 \equiv 0$, then

$$\lambda P(g^* \ge \lambda) \le \lambda P(\sup_{n\ge0} (|f_n| + |g_n|) \ge \lambda) \le 4\|f\|_1 , \tag{3.2}$$

while in general

$$\lambda P(g^* \ge \lambda) \le \lambda P(\sup_{n\ge0} (|f_n| + |g_n|) \ge \lambda) \le 6\|f\|_1 . \tag{3.3}$$

The constants 4 and 6 are the best possible in (3.2) and (3.3) respectively, even in the case $\nu = 1$ and g is a ±1-transform of f.

Proof. The inequalities follow immediately from Theorem 2.1. For the sharpness, first consider the following example:

Example 3.1. Fix $3 < \beta < 4$ and let $\alpha = (4-\beta)/4$, so that $\beta < 4 - 2\alpha$. On the Lebesgue interval $[0,1]$, let $f_0 = g_0 \equiv 0$,

$$f_1 = \mathbf{1}_{[0,\alpha)} - \frac{\alpha}{1-\alpha}\mathbf{1}_{[\alpha,1]}, \qquad g_1 = f_1,$$

$$f_2 = f_1 - \mathbf{1}_{[\alpha,2\alpha-\alpha^2)} + \frac{\alpha}{1-\alpha}\mathbf{1}_{[2\alpha-\alpha^2,1]}, \qquad g_2 = g_1 + \mathbf{1}_{[\alpha,2\alpha-\alpha^2)} - \frac{\alpha}{1-\alpha}\mathbf{1}_{[2\alpha-\alpha^2,1]},$$

$$f_3 = f_2 + \frac{1}{1-\alpha}\mathbf{1}_{[\alpha,2\alpha-\alpha^2)}, \qquad g_3 = g_2 + \frac{1}{1-\alpha}\mathbf{1}_{[\alpha,2\alpha-\alpha^2)}.$$

Then $f = (f_0, f_1, f_2, f_3)$ is a submartingale and $g = (g_0, g_1, g_2, g_3)$ is a ± 1-transform of f. Note that $g_3 = \mathbf{1}_{[0,\alpha)} + 2\mathbf{1}_{[\alpha,2\alpha-\alpha^2)} - (2\alpha/(1-\alpha))\mathbf{1}_{[2\alpha-\alpha^2,1]}$ and $f_3 = \mathbf{1}_{[0,\alpha)}$. Thus

$$2P(f_3 + g_3 \geq 2) = (4-2\alpha)\alpha > \beta \sup_{0\leq j\leq 3} \mathbf{E} f_j^+.$$

Now let $\tilde{f}_0 = \tilde{g}_0 \equiv 0$ and for $j \geq 0$, $1 \leq k \leq 3$, and $s \in [0,1]$, let

$$\tilde{f}_{3j+k}(s) = \tilde{f}_{3j}(s) + \mathbf{1}_{[1-2^{-j},1-2^{-j-1}]}(s) f_k\big(2^{j+1}(s-1+2^{-j})\big),$$

$$\tilde{g}_{3j+k}(s) = \tilde{g}_{3j}(s) + \mathbf{1}_{[1-2^{-j},1-2^{-j-1}]}(s) g_k\big(2^{j+1}(s-1+2^{-j})\big).$$

By induction on $j \geq 0$, we have

$$P(\tilde{f}_{3j} = 1, \tilde{g}_{3j} = 1) = (1-2^{-j})\alpha, \qquad P(\tilde{f}_{3j} = 0, \tilde{g}_{3j} = 2) = (1-2^{-j})(\alpha - \alpha^2),$$

$$P(\tilde{f}_{3j} = 0, \tilde{g}_{3j} = \frac{-2\alpha}{1-\alpha}) = (1-2^{-j})(1-\alpha)^2,$$

and, for $k \leq 3j$, $\mathrm{supp}\tilde{f}_k \subseteq [0, 1-2^{-j}]$.

It follows that $\tilde{f}$ is a submartingale, $\tilde{g}$ is a ± 1-transform of f, and, for $j \geq 0$, $1 \leq k \leq 3$, $\|\tilde{f}_{3j+k}\|_1 = \|\tilde{f}_{3j}\|_1 + 2^{-j-1}\|f_k\|_1$. Since $\|\tilde{f}_{3j}\|_1 = (1-2^{-j})\alpha$ and $\|f_1\|_1 = \|f_2\|_1 = 2\alpha$, we have that $\|\tilde{f}_{3j+k}\|_1 \leq \alpha = \mathbf{E} f_3$. Thus, with $\lambda = 2$,

$$\lim_{j\to\infty} \lambda P(\tilde{f}_{3j} + \tilde{g}_{3j} \geq \lambda) = \lambda P(f_3 + g_3 \geq \lambda) > \beta \mathbf{E} f_3 \geq \beta \sup_{k\geq 0} \|\tilde{f}_k\|_1.$$

Since we are assuming a strict inequality, there exists an n such

$$\lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \sup_{j\geq 0} \|\tilde{f}_j\|_1 \geq \beta \sup_{0\leq j\leq n} \|\tilde{f}_j\|_1. \tag{3.4}$$

Now let $(r_j)_{j\geq 1}$ be a sequence of independent identically distributed random variables such that $P(r_1 = 1) = P(r_1 = -1) = \frac{1}{2}$ and the (r_j) are independent from both the $(\tilde{f}_j)$ and the $(\tilde{g}_j)$

For $j \geq 0$, let $\tilde{f}_{n+j+1} = \tilde{f}_{n+j} + \tilde{f}_{n+j} r_{j+1}$ and $\tilde{g}_{n+j+1} = \tilde{g}_{n+j} - \tilde{f}_{n+j} r_{j+1}$. By this sequence of double or nothings we have that for $j \geq n$, $\|\tilde{f}_j\|_1 = \|\tilde{f}_n\|_1$, yet

$$\lim_{m\to\infty} \lambda P(\tilde{g}_m \geq \lambda) = \lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \|\tilde{f}\|_1$$

and since we are assuming a strict inequality, we can choose an $m > n$ that satisfies

$$\lambda P(\tilde{g}_m \geq \lambda) > \beta \|\tilde{f}\|_1.$$

This immediately implies the sharpness in (3.2). To show the sharpness in (3.3), it suffices to use $\tilde{f}$ and $\tilde{g}$ to construct a submartingale F with a ± 1-transform G such that

$$\lambda P(\sup_{j\geq 0} G_j \geq \lambda) > \frac{3}{2}\beta \|F\|_1. \tag{3.6}$$

Let $\alpha = P(\sup_{0\leq j\leq m} \tilde{g}_j \geq \lambda)$ so that $\alpha > 0$ and let $\delta = (4\|\tilde{f}\|_1 - \lambda\alpha)/(6-6\alpha)$ (in the case $\alpha = 1$, let $\delta = 0$). By (3.2), $\lambda\alpha \leq 4\|\tilde{f}\|_1$, hence $\delta \geq 0$.

Let s and t be independent random variables, independent from the $(\tilde{f}_j)$ such that $P(s = \lambda/6) = \alpha$ and $P(s = \delta) = 1 - \alpha$, while $P(t = -1) = 2/3$ and $P(t = 2) = 1/3$. Note that $\mathbf{E}s \leq 2\|\tilde{f}\|_1/3$.

Let $F_0 = -s$, $G_0 = s$, $F_1 = F_0 + tF_0$, and $G_1 = G_0 - tF_0$. We then have that $\|F_1\|_1 = \|F_0\|_1 = \mathbf{E}s$.

Let $F_2 = F_1 - F_1$ and $G_2 = G_1 - F_1$. Thus $F_2 = 0$ a.s. while $G_2 = 6s$ on the set $\{t = 2\}$ and $G_2 = 0$ on the set $\{t = -1\}$. We then have that

$$P(F_2 = 0, G_2 = \lambda) = \alpha/3, \qquad P(F_2 = 0, G_2 = 6\delta) = (1-\alpha)/3,$$

$$P(F_2 = 0, G_2 = 0) = 2/3.$$

Let $A = \{G_2 = 0\}$ and, for $j \geq 1$, let $F_{2+j} = \mathbf{1}_A \tilde{f}_j$ and $G_{2+j} = G_2 + \mathbf{1}_A \tilde{g}_j$. Then by the independence of t and the $(\tilde{f}_j)$, F is a submartingale, G is a ± 1-transfrom of F, and for $j \geq 1$ we have that $\|F_{2+j}\|_1 = 2\|\tilde{f}_j\|_1/3$, while

$$\begin{aligned} P(\sup_{0\leq j\leq m+2} G_j \geq \lambda) &= P(\sup_{0\leq j\leq 2} G_j \geq \lambda) + \frac{2}{3} P(\sup_{0<j\leq m} \tilde{g}_j \geq \lambda) \\ &\geq \frac{1}{3}\alpha + \frac{2}{3} P(\sup_{0<j\leq m} \tilde{g}_j \geq \lambda) = P(\sup_{0\leq j\leq m} \tilde{g}_j \geq \lambda), \end{aligned}$$

so that

$$\lambda P(\sup_{0\leq j\leq m+2} G_j \geq \lambda) \geq \lambda P(\sup_{0\leq j\leq m} \tilde{g}_j \geq \lambda) > \beta\|\tilde{f}\|_1 \geq \frac{3\beta}{2}\|F\|_1.$$

4. Applications to stochastic integrals

Theorem 4.1. *With $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geq 0}$ as in Section 1, suppose X is an adapted right-continuous submartingale with left limits such that $\mathbf{E}X_0$ is finite and H is a predictable process with values in the closed unit ball of $\mathbb{R}^\nu$. Then with Y defined by $Y_t = H_0 X_0 + \int_{(0,t]} H_s \, dX_s$, we have that, for $\lambda > 0$,*

$$\lambda P(Y^* \geq \lambda) \leq 4 \sup_{t\geq 0} \mathbf{E}X_t^+ - 2\mathbf{E}X_0, \tag{4.1}$$

so that

$$\lambda P(Y^* \geq \lambda) \leq 6 \left\| X \right\|_1 \tag{4.2}$$

and if $X_0 \equiv 0$*, then*

$$\lambda P(Y^* \geq \lambda) \leq 4 \left\| X \right\|_1 . \tag{4.3}$$

The constants 6 and 4 in (4.2) and (4.3) respectively are the best possible.

Proof. As in Theorem 2.1 we have $\mathbf{E}X_t^+ \leq \|X_t\|_1 \leq 2\mathbf{E}X_t^+ - \mathbf{E}X_0$, hence we can assume the finiteness of $\|X\|_1$. The proof follows in the same way as the proof of Theorem 5.1 of [5], except that we use Theorem 2.1 above to show that X is an $L^{1,\infty}$-integrator in the sense of [1].

The sharpness in (4.2) and (4.3) follow from the sharpness in (3.8) and (3.7) holding even for ± 1-transforms.

5. Lack of L^p inequalities

Fix $p \in [1, \infty)$ and $\beta > 1$. We shall construct a discrete time submartingale $F = (F_0, F_1, \dots)$ with $F_0 = 0$ and a ± 1-transform of F, $G = (G_0, G_1, \dots)$ such that

$$\left\| G \right\|_1 > \beta \left\| F \right\|_p . \tag{5.1}$$

To do this, we will first construct a finite length submartingale $f = (f_0, f_1, \dots, f_N)$ with $f_0 = 0$, and $f_N \geq 0$ together with a ± 1-transform of f, $g = (g_0, \dots, g_N)$ such that

$$\left\| g \right\|_1 > \beta \left\| f^+ \right\|_p$$

where $\|f^+\|_p = \sup_{0 \leq n \leq N} \mathbf{E}(f_n^+)^p$. Let $M > 4\beta$ and let $(r_1, \dots, r_{2M})$ be a sequence of independent random variables such that for $j = 1, 2$, $P(r_j = 1) = P(r_j = -1) = 1/2$ and for $2 \leq j \leq M$,

$$P(r_{2j-1} = 1) = P(r_{2j-1} = -1) = \frac{1}{2},$$

$$P(r_{2j} = -1) = \frac{1}{3}, \qquad P(r_{2j} = \frac{1}{2}) = \frac{2}{3}.$$

Let $f_j = \sum_{k=0}^{j} d_k$ where $d_0 = 0$, $d_1 = r_1/2$, $d_2 = r_2/2$, and $d_j = \mathbf{1}_{\{f_{j-1} < 0\}} r_j f_{j-1}$ for $j > 2$. By the independence of the r_j, $(f_j)_{j \leq 2M}$ forms a martingale. Note that for $j \geq 1$,

$$P(f_{2j} = 1) = \frac{1}{4}, \quad P(f_{2j} = -3^{j-1}) = \frac{1}{4}\left(\frac{1}{3}\right)^{j-1},$$
$$P(f_{2j} = 0) = \frac{3}{4} - \frac{1}{4}\left(\frac{1}{3}\right)^{j-1}.$$

For $0 \le j \le 2M$, let $g_j = \sum_{k=0}^{j}(-1)^k d_k$. Then for $2 \le j \le 2M$, $\left\|f_j^+\right\|_p^p = \left\|f_j^-\right\|_1 = 1/4$, while for $j \ge 1$,

$$\|g_{2j+2}\|_1 = \|g_{2j+1}\|_1 = \|g_{2j}\|_1 + 1/4.$$

Since $\|g_2\|_1 = 1/2$, it follows that $\|g_{2M}\|_1 = (M+1)/4$. Now let $N = 2M+1$, $f_N = f_{2M}^+$, and $g_N = g_{2M} + \mathbf{1}_{\{f_{2M}<0\}}|f_{2M}|$. Then $f = (f_0, \dots, f_N)$ forms a submartingale, $\|f^+\|_p^p = 1/4$, and, since $f_{2M} < 0$ implies $g_{2M} = 0$, $\|g_N\|_1 \ge \|g_{2M}\|_1 - \left\|f_{2M}^-\right\|_1 = M/4 > \beta\|f^+\|_p$ by our choice of M.

To construct F and G, we will work with only a small portion of the probability space at a time in order to keep $\|F\|_p$ close to that of $\|f^+\|_p$. More explicitly, by enriching the probability space if necessary, let $A_1, \dots, A_K$ be a partition of the space such that $\sigma(A_1, \dots, A_K)$ is independent of $\sigma(f_0, \dots, f_N)$ and, for $1 \le j \le k$, $P(A_j) \le \epsilon/3^{Mp}$, where ϵ satisfies $\beta^p(\|f^+\|_p^p + \epsilon) < \|g_N\|_1^p$.

Let $F_0 = G_0 = 0$ and for $1 \le k \le K$ and $1 \le n \le N$, let

$$F_{(k-1)N+n} = F_{(k-1)N} + \mathbf{1}_{A_k} f_n, \qquad G_{(k-1)N+n} = G_{(k-1)N} + \mathbf{1}_{A_k} g_n.$$

Then F is a submartingale and G is a ±1-transform of F. Since $A_1, \dots, A_N$ partition the space, $G_{KN} = g_N$ and for $1 \le k \le K$ and $1 \le n \le N$, the disjointness of the A_j gives us

$$\left\|F_{(k-1)N+n}\right\|_p^p = \left\|f_N \mathbf{1}\Big(\bigcup_{j=1}^{k-1} A_j\Big)\right\|_p^p + \|f_n \mathbf{1}_{A_k}\|_p^p.$$

Since $f_N \ge 0$ a.s. and the f_j are bounded in absolute value by 3^M, we have that

$$\left\|F_{(k-1)N+n}\right\|_p^p \le \left\|f_N^+\right\|_p^p + 3^{Mp}P(A_k) \le \left\|f^+\right\|_p^p + \epsilon$$

which gives us (5.1) by our choice of ϵ.

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