SHARP MAXIMAL INEQUALITIES FOR STOCHASTIC INTEGRALS IN WHICH THE INTEGRATOR IS A SUBMARTINGALE

WILLIAM HAMMACK

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Abstract. We obtain sharp maximal inequalities for strong subordinates of real-valued submartingales. Analogous inequalities also hold for stochastic integrals in which the integrator is a submartingale. The impossibility of general moment inequalities is also demonstrated.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\) where \(\mathcal{F}_0\) contains all \(P\)-null sets. Suppose \(X\) is an adapted right-continuous real-valued submartingale with left limits and \(H\) is a predictable process with values in the closed unit ball of \(\mathbb{R}^\nu\), where \(\nu\) is a positive integer. Define an adapted right-continuous process \(Y\) with left limits by

\[ Y_t = H_0X_0 + \int_{[0,t]} H_s dX_s. \]

We will compare the size of \(Y\) with that of \(X\) by finding constants \(\beta\) such that for all \(\lambda > 0\),

\[ \lambda P(Y^* \geq \lambda) \leq \beta \|X\|_1 \]

where \(\|X\|_1 = \sup_{t \geq 0} \|X_t\|_1\) and \(Y^* = \sup_{t \geq 0} |Y_t|\). In this paper we will denote the Euclidean norm of \(y \in \mathbb{R}^\nu\) by \(|y|\) and the inner product of \(y, k \in \mathbb{R}^\nu\) by \(y \cdot k\).

If we restrict \(X\) to the class of martingales, it is known that the best constant satisfying (1.1) is \(\beta = 2\) \([2, 3]\). By the best constant we mean that for any \(\beta < 2\) there exist a martingale \(X\), a predictable process \(H\), and a \(\lambda > 0\) such that \(\lambda P(Y^* \geq \lambda) > \beta \|X\|_1\). It is also known \([5]\) that if we restrict \(X\) to the class of nonnegative submartingales, then the best constant satisfying (1.1) is \(\beta = 3\).

In this paper we will show that for the class of real-valued submartingales, the best constant in (1.1) is \(\beta = 6\). To do this we shall first prove the analogous inequality and more for discrete-time submartingales. In the last section of this
paper we shall show that there are no moment inequalities of the form \( \|Y\|_p \leq \beta \|X\|_p \) where \( 1 < p < \infty \) and \( \beta \) is finite and depends only on \( p \). In fact, we shall show that for any \( p \in [1, \infty) \), there is no finite \( \beta \) such that \( \|Y\|_1 \leq \beta \|X\|_p \). For the case \( p = \infty \), see [7] where it is shown that if \( \|X\|_\infty = 1 \), then there is a constant \( \gamma \) such that for \( \lambda > 4 \), \( P(Y^* \geq \lambda) \leq \gamma \exp(-\lambda/4) \), so, for any \( r \in [1, \infty) \), \( \|Y\|_r \) is bounded by some constant depending only on \( r \).

2. A MAXIMAL INEQUALITY FOR SUBMARTINGALES

Let \( f_0, f_1, \ldots \) be a real-valued submartingale relative to a filtration \( (\mathcal{F}_n)_{n \geq 0} \) on a probability space \((\Omega, \mathcal{F}, P)\) with difference sequence \( d_0, d_1, \ldots \) and \( g_0, g_1, \ldots \) an \( \mathbb{R}^\nu \)-valued process adapted to \( (\mathcal{F}_n)_{n \geq 0} \) with difference sequence \( e_0, e_1, \ldots \), where \( \nu \) is a positive integer. We say that \( g \) is strongly subordinate to \( f \) if \( g \) is both differentially subordinate and conditionally differentially subordinate to \( f \), i.e. for all \( n \geq 0 \), \( |e_n| \leq |d_n| \) and \( |E(e_n | \mathcal{F}_n)| \leq |E(d_{n+1} | \mathcal{F}_n)| \). Note that if for \( k \geq 0 \), \( e_k = h_k d_k \) where \( h_k : \Omega \rightarrow [-1, 1] \) is \( \mathcal{F}_{k-1} \)-measurable, then \( g \) is strongly subordinate to \( f \). In particular, if \( g \) is a \( \pm 1 \)-transform of \( f \), i.e. \( e_k = \epsilon_k d_k \) where \( \epsilon_k \in \{-1, 1\} \), then \( g \) is strongly subordinate to \( f \).

**Theorem 2.1.** If \( f = (f_n)_{n \geq 0} \) is a submartingale relative to a filtration \( (\mathcal{F}_n)_{n \geq 0} \) and \( g = (g_n)_{n \geq 0} \) is strongly subordinate to \( f \), then for all \( \lambda > 0 \),

\[
\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \sup_{n \geq 0} E f_n^+ - 2E f_0
\]

where \( g^* = \sup_{n \geq 0} |g_n| \).

**Remarks.** If \( f \) is a martingale, then \( E f_n^+ \) and \( E f_n^- \) are nondecreasing sequences. It then follows from \( E f_0 = E f_n^+ - E f_n^- \) that \( \|f\|_1 = 2 \sup_{n \geq 0} E f_n^+ - E f_0 \), where \( \|f\|_1 = \sup_{n \geq 0} \|f_n\|_1 \). Thus in the martingale case, (2.1) implies that

\[
\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 2 \|f\|_1
\]

which is Theorem 4.1 of [4]. If \( f \) is a nonnegative supermartingale, (2.1) implies

\[
\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 2E f_0
\]

which is Theorem 8.1 of [5]. Both results are shown to be sharp in the articles quoted. If \( f \) is a nonnegative submartingale with \( f_0 = 0 \), the resulting inequality is not sharp in the case \( f_0 = 0 \), as can be seen from Theorem 4.1 of [5] which shows in this case

\[
\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 3 \|f\|_1
\]

**Proof.** We will assume \( \|f\|_1 \) is finite. This is equivalent to saying \( \sup_{n \geq 0} E f_n^+ \) is finite, as for all \( n \geq 0 \), \( E f_n^+ \leq \|f_n\|_1 \leq 2E f_n^+ - E f_0 \). The first inequality is obvious, the second follows from \( E f_0 \leq E f_n = E f_n^+ - E f_n^- \).

To show (2.1), it suffices to show that for \( n \geq 0 \),

\[
\lambda P(|f_n| + |g_n| \geq \lambda) \leq 4E f_n^+ - 2E f_0
\]
Then $U$ can be verified by checking the various cases: 

$$\lambda P(\sup_{m\leq n}(|f_m| + |g_m|) \geq \lambda) = \lambda P(|f_{\tau \wedge n}| + |g_{\tau \wedge n}| \geq \lambda) \leq 4Ef^+_{\tau \wedge n} - 2E f_0.$$  

Since $(f^+_n)_{n \geq 0}$ is a submartingale, it follows by Doob’s optional sampling theorem that $E f^+_{\tau \wedge n} \leq Ef^+_0$, thus implying (2.1).

By dividing by $\lambda$ throughout in (2.2), we may assume $\lambda = 1$. Using the methods developed by Burkholder [2], we define $V$ on $\mathbb{R} \times \mathbb{R}^\nu$ by 

$$V(x, y) = \begin{cases} 1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\ -4x^+, & \text{if } |x| + |y| < 1. \end{cases}$$

Then (2.2) is equivalent to $E V(f_u, g_u) \leq -2E f_0$. Define $U$ on $\mathbb{R} \times \mathbb{R}^\nu$ by 

$$U(x, y) = \begin{cases} 1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\ |y|^2 - x^2 - 2x, & \text{if } |x| + |y| < 1. \end{cases}$$

Then $V \leq U$ (in the case of $|x| + |y| < 1$ this follows from $-4x^+ \leq -x^2 - 2x$ for $|x| < 1$) and $U(f_u, g_u) \leq -2f_0$ (recall that by assumption $|f_0| \geq |g_0|$).

Thus $E V(f_u, g_u) \leq E U(f_u, g_u)$ and $E U(f_u, g_u) \leq -2E f_0$. To show (2.2), it will suffice to show that for $1 \leq j \leq n$, 

$$EU(f_j, g_j) \leq EU(f_{j-1}, g_{j-1}).$$

Define $\phi, \psi$ on $\mathbb{R} \times \mathbb{R}^\nu$ by 

$$\phi(x, y) = \begin{cases} -4, & \text{if } |x| + |y| \geq 1 \text{ and } x \geq 0, \\ 0, & \text{if } |x| + |y| \geq 1 \text{ and } x < 0, \\ -2x - 2, & \text{if } |x| + |y| < 1, \end{cases}$$

$$\psi(x, y) = \begin{cases} 0, & \text{if } |x| + |y| \geq 1, \\ 2y, & \text{if } |x| + |y| < 1. \end{cases}$$

Then $U_x(x, y) = \phi(x, y)$ and $U_y(x, y) = \psi(x, y)$ for $|x| + |y| \neq 1$, $y \neq 0$, and $x \neq 0$ where $U_x(x, y)$ and $U_y(x, y)$ are the partials of $U$ with respect to $x$ and $y$ respectively. Note that $|\psi| \leq -\phi$.

Claim: Given $h \in \mathbb{R}$ and $k \in \mathbb{R}^\nu$ with $|k| \leq |h|$, then for all $x \in \mathbb{R}$ and $y \in \mathbb{R}^\nu$

$$U(x + h, y + k) \leq U(x, y) + \phi(x, y)h + \psi(x, y) \cdot k.$$  

This can be verified by checking the various cases:

For $|x| + |y| \geq 1$ and $x \geq 0$, we need to show $U(x + h, y + k) \leq 1 - 4(x + h)$. For $|x + h| + |y + k| \geq 1$ this is clear. For $|x + h| + |y + k| < 1$ it follows from $|y + k|^2 < (1 - |x + h|)^2 \leq 1 - 2(x + h) + (x + h)^2$.

For $|x| + |y| \geq 1$ and $x < 0$, we need to show $U(x + h, y + k) \leq 1$. However $U(x, y) \leq 1$ for all $x, y$, this being obvious for $|x| + |y| \geq 1$. In the region $|x| + |y| < 1$, since $U_x(x, y) \leq 0$, it follows that $U(x, y) \leq |y|^2 - (|y| - 1)^2 - 2(|y| - 1) = 1$. 

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For the case $|x| + |y| < 1$, (2.4) is equivalent to showing
\[
(2.5) \quad U(x + h, y + k) \leq |y + k|^2 - (x + h)^2 - 2(x + h) - |k|^2 + h^2.
\]
For $|x + h| + |y + k| < 1$, this follows from $|k| \leq |h|$ and the definition of $U$. For $|x + h| + |y + k| \geq 1$, (2.5) can be rewritten as
\[
(1 - |x + h|)^2 \leq |y + k|^2 + h^2 - |k|^2.
\]
If $|x + h| \leq 1$, this inequality follows from $|k| \leq |h|$ and $|x + h| + |y + k| \geq 1$. If $|x + h| > 1$, then $(1 - |x + h|)^2 \leq (1 - |x| - |h|)^2$ and it suffices to show
\[
(1 - |x|)^2 - 2|h| (1 - |x|) \leq |y|^2 - 2|y| |k|.
\]
Since $|h| \geq |k|$, it then suffices to show $(1 - |x|)^2 \leq |y|^2 + 2|h| (1 - |x| - |y|)$, an inequality which follows from $|x + |h| \geq 1$ and $0 \leq |y|^2 - 2|y| (1 - |x|) + (1 - |x|)^2$, so that
\[
(1 - |x|)^2 \leq |y|^2 + 2(1 - |x|)(1 - |x| - |y|) \leq |y|^2 + 2|h| (1 - |x| - |y|).
\]
To prove (2.3), since $|e_j| \leq |d_j|$, by (2.4) we have
\[
(2.6) \quad U(f_j, g_j) \leq U(f_{j-1}, g_{j-1}) + \phi(f_{j-1}, g_{j-1}) d_j + \psi(f_{j-1}, g_{j-1}) \cdot e_j.
\]
Since $f$ is a submartingale, $E(d_j | \mathcal{F}_{j-1}) \geq 0$. It then follows from $|\psi| \leq -\phi$ and $g$ being strongly subordinate to $f$ that
\[
\phi(f_{j-1}, g_{j-1}) E(d_j | \mathcal{F}_{j-1}) + \psi(f_{j-1}, g_{j-1}) \cdot E(e_j | \mathcal{F}_{j-1}) \leq 0.
\]
Using this after taking the conditional expectations relative to $\mathcal{F}_{j-1}$ in (2.6) gives
\[
E(U(f_j, g_j) | \mathcal{F}_{j-1}) \leq U(f_{j-1}, g_{j-1}).
\]
Taking expectations of both sides gives (2.3) and completes the proof.

3. Discrete-time sharp maximal inequalities

**Theorem 3.1.** If $f$ is a submartingale relative to a filtration $(\mathcal{F}_n)_{n \geq 0}$ and $g$ is strongly subordinate to $f$, then for all $\lambda > 0$
\[
(3.1) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \|f\|_1 - 2E f_0.
\]
Thus if $f_0 \equiv 0$, then
\[
(3.2) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \|f\|_1,
\]
while in general
\[
(3.3) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 6 \|f\|_1.
\]
The constants 4 and 6 are the best possible in (3.2) and (3.3) respectively, even in the case $\nu = 1$ and $g$ is a $\pm 1$-transform of $f$.

**Proof.** The inequalities follow immediately from Theorem 2.1. For the sharpness, first consider the following example:
Example 3.1. Fix $3 < \beta < 4$ and let $\alpha = (4 - \beta)/4$, so that $\beta < 4 - 2\alpha$. On the Lebesgue interval $[0,1]$, let $f_0 = g_0 \equiv 0$,

$$f_1 = 1_{[0, \alpha)} - \frac{\alpha}{1 - \alpha} 1_{[\alpha,1]}, \quad g_1 = f_1,$$

$$f_2 = f_1 - 1_{[\alpha,2\alpha-\alpha^2]} + \frac{\alpha}{1 - \alpha} 1_{[2\alpha-\alpha^2,1]}, \quad g_2 = g_1 + 1_{[\alpha,2\alpha-\alpha^2]} - \frac{\alpha}{1 - \alpha} 1_{[2\alpha-\alpha^2,1]},$$

$$f_3 = f_2 + \frac{1}{1 - \alpha} 1_{[\alpha,2\alpha-\alpha^2]}, \quad g_3 = g_2 + \frac{1}{1 - \alpha} 1_{[\alpha,2\alpha-\alpha^2]}.$$

Then $f = (f_0,f_1,f_2,f_3)$ is a submartingale and $g = (g_0,g_1,g_2,g_3)$ is a $\pm 1$-transform of $f$. Note that $g_3 = 1_{[0,\alpha]} + 2 1_{[\alpha,2\alpha-\alpha^2]} - (2\alpha/(1 - \alpha)) 1_{[2\alpha-\alpha^2,1]}$ and $f_3 = 1_{[0,\alpha]}$. Thus

$$2P(f_3 + g_3 \geq 2) = (4 - 2\alpha)\alpha > \beta \sup_{0 \leq j \leq 3} E f_j^+.$$

Now let $\tilde{f}_0 = \tilde{g}_0 \equiv 0$ and for $j \geq 0$, $1 \leq k \leq 3$, and $s \in [0,1]$, let

$$\tilde{f}_{3j+k}(s) = \tilde{f}_{3j}(s) + 1_{[1-2^{-j}1-2^{-j-1}]}(s) f_{3k}(2^{j+1}(s - 1 + 2^{-j})), \quad \tilde{g}_{3j+k}(s) = \tilde{g}_{3j}(s) + 1_{[1-2^{-j}1-2^{-j-1}]}(s) g_{3k}(2^{j+1}(s - 1 + 2^{-j})).$$

By induction on $j \geq 0$, we have

$$P(\tilde{f}_{3j} = 1, \tilde{g}_{3j} = 1) = (1 - 2^{-j})\alpha, \quad P(\tilde{f}_{3j} = 0, \tilde{g}_{3j} = 2) = (1 - 2^{-j})(\alpha - \alpha^2),$$

$$P(\tilde{f}_{3j} = 0, \tilde{g}_{3j} = -2\alpha/(1 - \alpha)) = (1 - 2^{-j})(1 - \alpha)^2,$$

and, for $k \leq 3j$, $\sup_{s \in [0,1]} \tilde{f}_k \subseteq [0, 1 - 2^{-j}]$.

It follows that $\tilde{f}$ is a submartingale, $\tilde{g}$ is a $\pm 1$-transform of $f$, and, for $j \geq 0$, $1 \leq k \leq 3$, $\|\tilde{f}_{3j+k}\|_1 = \|\tilde{f}_{3j}\|_1 + 2^{-j-1} \|f_{3k}\|_1$. Since $\|\tilde{f}_{3j}\|_1 = (1 - 2^{-j})\alpha$ and $\|f_{3j}\|_1 = \|f_3\|_1 = 2\alpha$, we have that $\|\tilde{f}_{3j+k}\|_1 \leq \alpha = E f_3$. Thus, with $\lambda = 2$,

$$\lim_{j \to \infty} \lambda P(\tilde{f}_{3j} + \tilde{g}_{3j} \geq \lambda) = \lambda P(f_3 + g_3 \geq \lambda) > \beta E f_3 \geq \beta \sup_{k \geq 0} \|f_k\|_1.$$

Since we are assuming a strict inequality, there exists an $n$ such that

$$\lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda > \beta \sup_{j \geq 0} \|\tilde{f}_j\|_1 \geq \beta \sup_{0 \leq j \leq n} \|\tilde{f}_j\|_1).$$

Now let $(r_j)_{j \geq 1}$ be a sequence of independent identically distributed random variables such that $P(r_1 = 1) = P(r_1 = -1) = \frac{1}{2}$ and the $(r_j)$ are independent from both the $(f_j)$ and the $(\tilde{g}_j)$.

For $j \geq 0$, let $\tilde{f}_{n+j+1} = \tilde{f}_{n+j} + \tilde{f}_{n+1} r_{j+1}$ and $\tilde{g}_{n+j+1} = \tilde{g}_{n+j} - \tilde{f}_{n+1} r_{j+1}$. By this sequence of double or nothings, we have that for $j \geq n$, $\|\tilde{f}_j\|_1 = \|\tilde{f}_n\|_1$, and

$$\lim_{j \to \infty} \lambda P(\tilde{g}_n \geq \lambda) = \lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \|\tilde{f}\|_1.$$
and since we are assuming a strict inequality, we can choose an \( m > n \) that satisfies

\[
\lambda P(\tilde{g}_m \geq \lambda) > \beta \|\tilde{f}\|_1.
\]

This immediately implies the sharpness in (3.2). To show the sharpness in (3.3), it suffices to use \( \tilde{f} \) and \( \tilde{g} \) to construct a submartingale \( F \) with a \( \pm 1 \)-transform \( G \) such that

\[
(3.6) \quad \lambda P(\sup_{j \geq 0} G_j \geq \lambda) > \frac{3}{2} \beta \|F\|_1.
\]

Let \( \alpha = P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda) \) so that \( \alpha > 0 \) and let \( \delta = (4\|\tilde{f}\|_1 - \lambda \alpha)/(6 - 6\alpha) \) (in the case \( \alpha = 1 \), let \( \delta = 0 \)). By (3.2), \( \lambda \alpha \leq 4\|\tilde{f}\|_1 \), hence \( \delta \geq 0 \).

Let \( s \) and \( t \) be independent random variables, independent from the \( (\tilde{f}_j) \) such that \( P(s = \lambda/6) = \alpha \) and \( P(s = \delta) = 1 - \alpha \), while \( P(t = -1) = 2/3 \) and \( P(t = 2) = 1/3 \). Note that \( \mathbb{E}s \leq 2\|\tilde{f}\|_1/3 \).

Let \( F_0 = -s, \ G_0 = s, \ F_1 = F_0 + tF_0, \) and \( G_1 = G_0 - tF_0 \). We then have that \( \|F_1\|_1 = \|F_0\|_1 = \mathbb{E}s \).

Let \( F_2 = F_1 - F_0 \) and \( G_2 = G_1 - F_1 \). Thus \( F_2 = 0 \) a.s. while \( G_2 = 6s \) on the set \( \{t = 2\} \) and \( G_2 = 0 \) on the set \( \{t = -1\} \). We then have that

\[
P(F_2 = 0, G_2 = \lambda) = \alpha/3, \quad P(F_2 = 0, G_2 = 6\delta) = (1 - \alpha)/3,
\]

\[
P(F_2 = 0, G_2 = 0) = 2/3.
\]

Let \( A = \{G_2 = 0\} \) and, for \( j \geq 1 \), let \( F_{2+j} = 1_A \tilde{f}_j \) and \( G_{2+j} = G_2 + 1_A \tilde{g}_j \). Then by the independence of \( t \) and the \( (\tilde{f}_j) \), \( F \) is a submartingale, \( G \) is a \( \pm 1 \)-transform of \( F \), and for \( j \geq 1 \) we have that \( \|F_{2+j}\|_1 = \|\tilde{f}_j\|_1/3 \), while

\[
P(\sup_{0 \leq j \leq m+2} G_j \geq \lambda) = \left(1 + \frac{2}{3}\right)P(\sup_{0 \leq j \leq 2} \tilde{g}_j \geq \lambda)
\]

\[
\geq \frac{1}{3} \alpha + \frac{2}{3}P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda) = P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda),
\]

so that

\[
\lambda P(\sup_{0 \leq j \leq m+2} G_j \geq \lambda) \geq \lambda P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda) \geq \beta \|\tilde{f}\|_1 \geq \frac{3}{2} \beta \|F\|_1.
\]

4. Applications to stochastic integrals

**Theorem 4.1.** With \( (\Omega, \mathcal{F}, P) \) and \( (\mathcal{F}_t)_{t \geq 0} \) as in Section 1, suppose \( X \) is an adapted right-continuous submartingale with left limits such that \( \mathbb{E}X_0 \) is finite and \( H \) is a predictable process with values in the closed unit ball of \( \mathbb{R}^p \). Then with \( Y \) defined by \( Y_t = H_0X_0 + \int_{(0,t]} H_s dX_s \), we have that, for \( \lambda > 0 \),

\[
(4.1) \quad \lambda P(Y^* \geq \lambda) \leq 4 \sup_{t \geq 0} \mathbb{E}X_t^+ - 2 \mathbb{E}X_0.
\]

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so that

\[ \lambda P(Y^* \geq \lambda) \leq 6 \|X\|_1 \]  

and if \( X_0 \equiv 0 \), then

\[ \lambda P(Y^* \geq \lambda) \leq 4 \|X\|_1. \]

The constants 6 and 4 in (4.2) and (4.3) respectively are the best possible.

**Proof.** As in Theorem 2.1 we have

\[ E(X + t) \leq \|Xt\|_1 \leq 2E(X^+ - EX), \]

hence we can assume the finiteness of \( \|X\|_1 \). The proof follows in the same way as the proof of Theorem 5.1 of [5], except that we use Theorem 2.1 above to show that \( X \) is an \( L^{1,\infty} \)-integrator in the sense of [1].

The sharpness in (4.2) and (4.3) follow from the sharpness in (3.8) and (3.7) holding even for \( \pm 1 \)-transforms.

5. Lack of \( L^p \) inequalities

Fix \( p \in [1, \infty) \) and \( \beta > 1 \). We shall construct a discrete time submartingale \( F = (F_0, F_1, \ldots) \) with \( F_0 = 0 \) and a \( \pm 1 \)-transform of \( F \), \( G = (G_0, G_1, \ldots) \) such that

\[ \|G\|_1 > \beta \|F\|_p. \]

To do this, we will first construct a finite length submartingale \( f = (f_0, f_1, \ldots, f_N) \) with \( f_0 = 0 \), \( f_N \geq 0 \) together with a \( \pm 1 \)-transform of \( f \), \( g = (g_0, \ldots, g_N) \) such that

\[ \|g\|_1 > \beta \|f^+\|_p, \]

where \( \|f^+\|_p = \sup_{0 \leq N} \mathbb{E}(f^+_N)^p \). Let \( M > 4 \beta \) and let \( (r_1, \ldots, r_{2M}) \) be a sequence of independent random variables such that for \( j = 1, 2 \), \( P(r_j = 1) = P(r_j = -1) = 1/2 \) and for \( 2 \leq j \leq M \),

\[ P(r_{2j-1} = 1) = P(r_{2j-1} = -1) = \frac{1}{2}; \]

\[ P(r_{2j} = -1) = \frac{1}{3}, \quad P(r_{2j} = \frac{1}{2}) = \frac{2}{3}. \]

Let \( f_j = \sum_{k=0}^j d_k \) where \( d_0 = 0 \), \( d_1 = r_1/2 \), \( d_2 = r_2/2 \), and \( d_j = 1_{\{f_j-1 \leq 0\}} r_j f_{j-1} \) for \( j > 2 \). By the independence of the \( r_j \), \( (f_j)_{1 \leq j \leq 2M} \) forms a martingale. Note that for \( j \geq 1 \),

\[ P(f_{2j} = 1) = \frac{1}{3}, \quad P(f_{2j} = -3^{j-1}) = \frac{1}{4} \left( \frac{1}{3} \right)^{j-1}, \]

\[ P(f_{2j} = 0) = \frac{3}{4} - \frac{1}{4} \left( \frac{1}{3} \right)^{j-1}. \]
For $0 \leq j \leq 2M$, let $g_j = \sum_{k=0}^{j} (-1)^k d_k$. Then for $2 \leq j \leq 2M$, $\| f_j^+ \|_p = f_j^+ \|_1 = 1/4$, while for $j \geq 1$,

$$\| g_{2j+2} \|_1 = \| g_{2j+1} \|_1 = \| g_{2j} \|_1 + 1/4.$$  

Since $\| g_2 \|_1 = 1/2$, it follows that $\| g_{2M} \|_1 = (M + 1)/4$. Now let $N = 2M + 1$, $f_N = f_{2M}$, and $g_N = g_{2M} + 1_{\{ f_{2M} < 0 \}} |f_{2M}|$. Then $f = (f_0, \ldots, f_N)$ forms a submartingale, $\| f^+ \|_p = 1/4$, and, since $f_{2M} < 0$ implies $g_{2M} = 0$, $\| g_N \|_1 \geq \| g_{2M} \|_1 - \| f_{2M} \|_1 = M/4 > \beta f^+ \|_p$ by our choice of $M$.

To construct $F$ and $G$, we will work with only a small portion of the probability space at a time in order to keep $\| F \|_p$ close to that of $\| f^+ \|_p$. More explicitly, by enriching the probability space if necessary, let $A_1, \ldots, A_K$ be a partition of the space such that $\sigma(A_1, \ldots, A_K)$ is independent of $\sigma(f_0, \ldots, f_N)$ and, for $1 \leq j \leq K$, $P(A_j) \leq \epsilon/3^{p\theta}$, where $\epsilon$ satisfies $\beta_p(\| f^+ \|_p + \epsilon) < \| g_N \|_1$.

Let $F_0 = G_0 = 0$ and for $1 \leq k \leq K$ and $1 \leq n \leq N$, let

$$F_{(k-1)N+n} = F_{(k-1)N} + A_k f_n, \quad G_{(k-1)N+n} = G_{(k-1)N} + 1_{A_k} g_n.$$  

Then $F$ is a submartingale and $G$ is a $\pm 1$-transform of $F$. Since $A_1, \ldots, A_N$ partition the space, $G_{KN} = g_N$ and for $1 \leq k \leq K$ and $1 \leq n \leq N$, the disjointness of the $A_j$ gives us

$$\| F_{(k-1)N+n} \|_p = f_n 1_{\bigcup_{j=1}^{k-1} A_j} \|_p + \| f_n 1_{A_k} \|_p.$$  

Since $f_N \geq 0$ a.s. and the $f_j$ are bounded in absolute value by $3^M$, we have that

$$\| F_{(k-1)N+n} \|_p \leq \| f_N \|_p + 3^{p\theta} P(A_k) \leq \| f^+ \|_p + \epsilon$$  

which gives us (5.1) by our choice of $\epsilon$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA V6T 1Z2

E-mail address: hammack@math.ubc.ca