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COMPLETELY BOUNDED ISOMORPHISMS OF OPERATOR ALGEBRAS

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Abstract. In this paper the author proves that any two elements from one of the following classes of operators are completely isomorphic to each other.
1. $\{VN(F_n) : n \geq 2\}$. The II_1 factors generated by the left regular representation of the free group on n-generators.
2. $\{C^*_\lambda(F_n) : n \geq 2\}$. The reduced C^*-algebras of the free group on n-generators.
3. Some "non-commutative" analytic spaces introduced by G. Popescu in 1991.

The paper ends with some applications to Popescu's version of von Neumann's inequality.

1. Introduction and preliminaries

E. Christensen and A. M. Sinclair [CS] showed that any non-elementary injective von Neumann algebra on a separable Hilbert space is completely isomorphic to $B(H)$, and A. G. Robertson and S. Wassemann [RW] generalized the work of [CS] and proved that an infinite-dimensional injective operator system on a separable Hilbert space is completely isomorphic to either $B(H)$ or ℓ_∞.

The techniques in those papers depend on the injectivity of the spaces and do not extend to interesting non-injective von Neumann algebras or operator algebras. In the present note we address some of these examples. For instance, we prove that all the von Neumann factors of the free group on n-generators, $n \geq 2$, are completely isomorphic to each other. We prove the same result for the reduced C^*-algebras of the free group on n-generators, $n \geq 2$, and for some non-selfadjoint operator algebras introduced by G. Popescu [Po].

Let H be a Hilbert space and $B(H)$ the set of bounded linear operators on H. If we identify $M_n(B(H))$, the set of $n \times n$ matrices with entries from $B(H)$, with $B(\ell_2^n(H))$, we have a natural norm on $M_n(B(H))$. (Here $\ell_2^n(H)$ means $H \oplus H \oplus \cdots \oplus H$, n-times.)

An *operator space* X is a closed subspace of $B(H)$. Then considering $M_n(X)$ as a subspace of $M_n(B(H)) \equiv B(\ell_2^n(H))$, we have norms for $M_n(X)$, $n \geq 1$. (See [BP] and [ER] for more on the development of this recent theory.)

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Let X, Y be operator spaces and $u : X \to Y$ be a linear map. Define $u_n : M_n(X) \to M_n(Y)$ by

$$u_n\big[(x_{ij})\big] = \big[(u(x_{ij}))\big].$$

We say that u is completely bounded (cb in short) if

$$\|u\|_{cb} = \sup_{n\geq 1} \|u_n\| < \infty.$$

If u_n is an isometry for every $n \geq 1$, then u is a complete isometry. Finally, X and Y are completely isomorphic if there exists $u : X \to Y$ such that u and u^{-1} are completely bounded.

Let $X \subset B(H)$, $Y \subset B(K)$ be two operator spaces. The spatial tensor product of X and Y, $X \otimes Y$, is the completion of the algebraic tensor product of X and Y with the norm induced by $B(H \otimes_2 K)$. With this notation, $M_n(X) = M_n \otimes X$.

One of the main features of operator spaces is that the scalars are replaced by matrices (see [E]). To see this concretely consider X a finite-dimensional operator space with basis $\{e_1, \cdots, e_n\}$. A canonical element in X looks like $\sum_{i=1}^n a_i e_i$ for some $a_i \in \mathbb{C}$; whereas a canonical element in $M_n(X) = M_n \otimes X$ looks like $\sum_{i=1}^n A_i \otimes e_i$, for some $A_i \in M_n$.

Two of the most important operator spaces are the row and column Hilbert spaces. In $B(\ell_2)$ define $C = \overline{\text{span}}\{e_{i1} : i \in \mathbb{N}\}$, the column Hilbert space, and $R = \overline{\text{span}}\{e_{1i} : i \in \mathbb{N}\}$, the row Hilbert space. Both spaces are Banach spaces isometric to ℓ_2, but have very different operator space structure. If $\sum_i e_{i1} \otimes T_i \in C \otimes B(H)$, then

$$\Big\|\sum_i e_{i1} \otimes T_i\Big\| = \Big\|\sum_i T_i^* T_i\Big\|^{\frac{1}{2}},$$

and if $\sum_i e_{1i} \otimes T_i \in R \otimes B(H)$, then

$$\Big\|\sum_i e_{1i} \otimes T_i\Big\| = \Big\|\sum_i T_i T_i^*\Big\|^{\frac{1}{2}}.$$

In this paper we will prove that several operator algebras are completely isomorphic to each other. The tool that we use is Pełczyński's decomposition method.

This method says that if X and Y are Banach spaces such that X embeds complementably into Y, Y embeds complementably into X, and X and Y satisfy one of the following conditions:

1. $X \approx X \oplus X$ and $Y \approx Y \oplus Y$, or
2. $X \approx (\sum_{i=1}^\infty X)_p$, $1 \leq p \leq \infty$,

then X and Y are isomorphic. Moreover, if the embeddings and projections are completely bounded, the isomorphism is a complete isomorphism and then X and Y are completely isomorphic.

We will also use a variant of condition 2.

The main examples in this paper will be the C^*-algebras generated by the left regular representation of the free group on n-generators, $\lambda : F_n \to B(\ell_2(F_n))$.

Let F_n be the free group on n-generators, $\ell_2(F_n)$ the Hilbert space with orthonormal basis $\{e_x : x \in F_n\}$, and $\mathcal{L}$ (or $\mathcal{L}_n$ to avoid confusion) the linear span of the basis; i.e.,

$$\mathcal{L} = \Big\{\sum_{i=1}^k a_i e_{x_i} : k \in \mathbb{N}, a_i \in \mathbb{C}, x_i \in F_n\Big\}.$$

$\mathcal{L}$ is an algebra if we multiply the elements in the natural way: i.e., $e_x e_y = e_{xy}$. We think of $\mathcal{L}$ as the Laurent polynomials on n non-commutative coordinates. We use two norms on $\mathcal{L}$: The $\|\cdot\|_2$-norm, induced by $\ell_2(G)$, and the $\|\cdot\|$-norm defined as

$$\|p\| = \sup\{\|pq\|_2 : q \in \mathcal{L}, \|q\|_2 \leq 1\}.$$

Notice that $p \in \mathcal{L}$ induces a left multiplication map on $\ell_2(F_n)$ and $\|p\|$ equals the operator norm of that map. $C^*_\lambda(F_n)$ is the closure of $\mathcal{L}$ in the norm topology of $B(\ell_2(F_n))$, and $VN(F_n)$ is the closure of $\mathcal{L}$ in the strong operator topology of $B(\ell_2(F_n))$.

The following fact is well known (see [FP], Chapter 1). We sketch the proof to emphasize an argument that appears repeatedly in the paper.

Proposition 1. *Let $n, m = 2, 3, \cdots, \infty$. Then $C^*_\lambda(F_n)$ is contained completely isometrically in $C^*_\lambda(F_m)$ and there is a completely contractive projection onto $C^*_\lambda(F_n)$. The same is true for $VN(F_n)$ and $VN(F_m)$.*

Proof. If $n \leq m$, then the formal identity from $C^*_\lambda(F_n)$ into $C^*_\lambda(F_m)$ is a complete isometry. We claim that the orthogonal projection onto $\ell_2(F_n)$ is a complete contraction from $C^*_\lambda(F_m)$ onto $C^*_\lambda(F_n)$. Let $p \in \mathcal{L}_m$ and decompose it as $p = r + s$ where $r \in \mathcal{L}_n$ and $s \in (\mathcal{L}_n)^\perp$. If $q \in \mathcal{L}_n$, then $rq \in \mathcal{L}_n$ and $sq \in (\mathcal{L}_n)^\perp$. Therefore,

$$\|r\| = \sup_{\substack{q \in \mathcal{L}_n \\ \|q\|_2 \leq 1}} \|rq\|_2 \leq \sup_{\substack{q \in \mathcal{L}_n \\ \|q\|_2 \leq 1}} \|pq\|_2 \leq \|p\|.$$

The completely bounded part is very similar.

To complete the circle we need to show that $C^*_\lambda(F_\infty)$ embeds completely complemented into $C^*_\lambda(F_2)$. Assume that F_2 is generated by a, b and let G be the subgroup generated by aba^{-1}, a^2ba^{-2}, $a^3ba^{-3}, \cdots$. It is easy to see that G is isomorphic to F_∞, $C^*_\lambda(G)$ is completely isometric to $C^*_\lambda(F_\infty)$ and the orthogonal projection onto $\ell_2(G)$ is a complete contraction from $C^*_\lambda(F_2)$ onto $C^*_\lambda(G)$.

The proof for the $VN(F_n)$'s is very similar. □

2. Isomorphisms of $C^*_\lambda(F_n)$, $n \geq 2$

In this section we will prove that

Theorem 2. *$C^*_\lambda(F_n)$ is completely isomorphic to $C^*_\lambda(F_\infty)$ when $n = 2, 3, 4, \cdots$.*

Theorem 3. *$VN(F_n)$ is completely isomorphic to $VN(F_\infty)$ when $n = 2, 3, 4, \cdots$.*

Remark. It is known that the $C^*_\lambda(F_n)$'s are not $*$-isomorphic for different n's (see [PV]); however, it is still not known if the $VN(F_n)$'s are $*$-isomorphic to each other for $n \geq 2$ (see [S], Problem 4.4.44).

The proof of Theorem 2 follows from Propositions 5 and 6. It is simple to go from there to Theorem 3.

We need some notation. Divide the generators of F_∞ into $\alpha_1, \alpha_2 \cdots; e_1, e_2, \cdots$, and denote by F_α the subgroup generated by the α's. F_α is isomorphic to F_∞ of course.

Let $K = \bigcup_{j=0}^\infty e_j F_\alpha$. Denote $\mathcal{L}_K = \operatorname{span}\{e_x : x \in K\}$, and let $\ell_2(K)$ be the closure of $\mathcal{L}_K$ in the $\|\cdot\|_2$-norm of $\ell_2(F_\infty)$, $C^*_\lambda(K)$ the closure of $\mathcal{L}_K$ in the $\|\cdot\|$-norm of $C^*_\lambda(F_\infty)$, and $VN(K)$ the closure of $\mathcal{L}_K$ in the strong operator topology of $B(\ell_2(F_\infty))$.

Proposition 4. *$C^*_\lambda(K)$ is 2-cb-complemented in $C^*_\lambda(F_\infty)$. Moreover, the orthogonal projection onto $\ell_2(K)$ is completely bounded from $C^*_\lambda(F_\infty)$ onto $C^*_\lambda(K)$.*

We will present the proof of Proposition 4 after the proof of Theorem 2.

Proposition 5. *$C^*_\lambda(K) \approx C^*_\lambda(F_\infty) \approx C^*_\lambda(F_\infty) \oplus C^*_\lambda(F_\infty)$. Moreover, the isomorphisms are completely bounded.*

Proof. Decompose $K = K_1 \cup K_2$ where $K_1 = \bigcup_{j=0}^\infty e_{2j}F_\alpha$ and $K_2 = \bigcup_{j=0}^\infty e_{2j+1}F_\alpha$. It is clear that $C^*_\lambda(K_1)$ and $C^*_\lambda(K_2)$ are completely isometric to $C^*_\lambda(K)$. Moreover, Proposition 4 applied to K_1 and K_2 implies that they are cb-complemented in $C^*_\lambda(F_\infty)$ by the orthogonal projection. Therefore they are complemented in $C^*_\lambda(K)$ and we have

$$C^*_\lambda(K) = C^*_\lambda(K_1) \oplus C^*_\lambda(K_2) \approx C^*_\lambda(K) \oplus C^*_\lambda(K).$$

Similarly, decompose $K = K_3 \cup K_4$, where $K_3 = e_1F_\alpha$ and $K_4 = \bigcup_{j=2}^\infty e_jF_\alpha$, and apply the previous argument to conclude that $C^*_\lambda(K) \oplus C^*_\lambda(F_\infty) \approx C^*_\lambda(K)$.

Proposition 4 tells us that $C^*_\lambda(F_\infty) = C^*_\lambda(K) \oplus Z$ for some Z. Then

$$C^*_\lambda(F_\infty) \approx C^*_\lambda(K) \oplus Z \approx C^*_\lambda(K) \oplus C^*_\lambda(K) \oplus Z \approx C^*_\lambda(K) \oplus C^*_\lambda(F_\infty) \approx C^*_\lambda(K). \square$$

Proposition 6. *$C^*_\lambda(F_k) \approx C^*_\lambda(F_k) \oplus C^*_\lambda(F_k)$, for $k = 2, 3, \cdots$.*

Proof. Divide the generators of F_∞ into $\beta_1, \cdots, \beta_k; e_1, e_2, \cdots$, and denote by F_β the subgroup generated by the β's; F_β is isomorphic to F_k. Let $K_\beta = \bigcup_{j=0}^\infty e_jF_\beta$.

The proof of Proposition 4 works and we get that $C^*_\lambda(K_\beta)$ is 2-cb complemented in $C^*_\lambda(F_\infty)$; hence, by Proposition 1, it is 2-cb-complemented in $C^*_\lambda(F_k)$ also. It is clear that $C^*_\lambda(K_\beta) \approx C^*_\lambda(K_\beta) \oplus C^*_\lambda(K_\beta)$ and that $C^*_\lambda(K_\beta) \oplus C^*_\lambda(F_k) \approx C^*_\lambda(K_\beta)$. Hence the proof of Proposition 5 applies and we get the result. $\square$

We will present the proof of Theorem 2 for completeness. This is a standard version of Pełczyński's decomposition method.

Proof of Theorem 2. Proposition 1 tells us that $C^*_\lambda(F_k) \approx C^*_\lambda(F_\infty) \oplus Y$ for some Y, and that $C^*_\lambda(F_\infty) \approx C^*_\lambda(F_k) \oplus Z$, for some Z. Then Propositions 5 and 6 give

$$C^*_\lambda(F_k) \approx C^*_\lambda(F_\infty) \oplus Y \approx C^*_\lambda(F_\infty) \oplus C^*_\lambda(F_\infty) \oplus Y \approx C^*_\lambda(F_\infty) \oplus C^*_\lambda(F_k).$$

On the other hand

$$C^*_\lambda(F_\infty) \approx C^*_\lambda(F_k) \oplus Z \approx C^*_\lambda(F_k) \oplus C^*_\lambda(F_k) \oplus Z \approx C^*_\lambda(F_k) \oplus C^*_\lambda(F_\infty). \qquad \square$$

The first step for the proof of Proposition 4 is to understand how to norm the elements in $M_n(C^*_\lambda(K))$. The typical element in $\mathcal{L}_K$ looks like: $\sum_{i\le k} e_ip_i$, where $p_i \in \mathcal{L}_\alpha$; i.e., $p_i = \sum_j a_{ij}e_{x_j}$, for some $x_j \in F_\alpha$. When we consider operator spaces, we replace the scalars by matrices, so the canonical element of $M_n(C^*_\lambda(K))$ looks like

$$\sum_{i\le k}(I \otimes e_i)A_i, \quad \text{for some } A_i \in M_n(\mathcal{L}_\alpha), \tag{1}$$

where I is the identity in M_n and

$$A_i = \sum_j A_{ij} \otimes e_{x_j}, \quad \text{for some } A_{ij} \in M_n \text{ and } x_j \in F_\alpha. \tag{2}$$

We use the fact (see [HP]) that, as elements of $B(\ell_2(F_\infty))$,

$$e_i = P_i e_i + e_i P_{-i},$$

where P_i is the orthogonal projection onto the set of reduced words starting from a positive power of e_i and P_{-i} is the orthogonal projection onto the set of reduced words starting from a negative power of e_i. To simplify the notation we set $(e_i)^{-1} = e_{-i}$.

We also use that $\sum_i (P_i e_i)(P_i e_i)^* = \sum_i P_i e_i e_{-i} P_i = \sum_i P_i \leq I$, the identity on $B(\ell_2(F_\infty))$, and if $T_i, S_i \in B(\ell_2)$, then $\| \sum_i T_i S_i \| \leq \| \sum_i T_i T_i^* \|^{\frac{1}{2}} \| \sum_i S_i^* S_i \|^{\frac{1}{2}}$.

We need the following technical lemma.

Lemma 7. *Let $x_1, x_2 \in F_\alpha$, $z_1, z_2 \in F_\infty$ and suppose that (as elements of $\ell_2(F_\infty)$) $e_{x_1} P_{-i} e_{-i} e_{z_1} = e_{x_2} P_{-j} e_{-j} e_{z_2}$. Then either they are equal to zero, or $i = j$, $x_1 = x_2$, $z_1 = z_2$ and $e_{x_1} P_{-i} e_{-i} e_{z_1} = e_{x_1} e_{-i} e_{z_1}$.*

Proof. If z_1 starts from e_i, $P_{-i} e_{-i} e_{z_1} = 0$, so we assume that the reduced words of z_1 and z_2 do not start from e_i or e_j respectively. Then we have that $x_1 e_{-i} e_{z_1} = x_2 e_{-j} e_{z_2}$. Since we have no cancellation on the z's and e_{-i} and e_{-j} are the first non-F_α elements of the words, then $e_i = e_j$, $x_1 = x_2$ and $z_1 = z_2$. □

Proposition 8. *Let $T \in M_n(\mathcal{L}_K)$. Then*

$$\begin{aligned}
&\max\Bigg\{ \sup_{\substack{q \in \ell_2^n(\mathcal{L}_\alpha) \\ \|q\|_2 \leq 1}} \|Tq\|_2, \ \sup_{\substack{b \in \ell_2^n \\ \|b\|_2 \leq 1}} \|T^* b \otimes e_0\|_2 \Bigg\} \\
&\quad \leq \|T\| \leq \sup_{\substack{q \in \ell_2^n(\mathcal{L}_\alpha) \\ \|q\|_2 \leq 1}} \|Tq\|_2 + \sup_{\substack{b \in \ell_2^n \\ \|b\|_2 \leq 1}} \|T^* b \otimes e_0\|_2.
\end{aligned}$$

Proof. The left inequality is trivially true. For the other one take $T \in M_n(\mathcal{L}_K)$ as in (1)

$$T = \sum_{i \leq k} (I \otimes e_i) A_i = \sum_{i \leq k} (I \otimes P_i e_i) A_i + \sum_{i \leq k} (I \otimes e_i P_{-i}) A_i.$$

Then

$$\begin{aligned}
\Bigg\| \sum_{i \leq k} (I \otimes P_i e_i) A_i \Bigg\| &\leq \Bigg\| \sum_{i \leq k} (I \otimes P_i e_i)(I \otimes P_i e_i)^* \Bigg\|^{\frac{1}{2}} \Bigg\| \sum_{i \leq k} A_i^* A_i \Bigg\|^{\frac{1}{2}} \\
&\leq \Bigg\| \sum_{i \leq k} A_i^* A_i \Bigg\|^{\frac{1}{2}} \\
&= \sup_{\substack{q \in \ell_2^n(\mathcal{L}_\alpha) \\ \|q\|_2 \leq 1}} \sqrt{\sum_{i \leq k} \|A_i q\|_2^2} \\
&= \sup_{\substack{q \in \ell_2^n(\mathcal{L}_\alpha) \\ \|q\|_2 \leq 1}} \Bigg\| \sum_{i \leq k} (I \otimes e_i) A_i q \Bigg\|_2 \\
&= \sup_{\substack{q \in \ell_2^n(\mathcal{L}_\alpha) \\ \|q\|_2 \leq 1}} \|Tq\|_2.
\end{aligned}$$

On the other hand, $\| \sum_{i\le k} (I\otimes e_i P_{-i})A_i\| = \|\sum_{i\le k} A_i^*(I\otimes P_{-i}e_{-i})\|$. To norm the latter one, take $q\in \ell_2^n(\ell_2(F_\infty))$, $\|q\|_2\le 1$, and decompose it as

$$q=\sum_l b_l\otimes e_{z_l} \quad \text{where } b_l\in \ell_2^n, z_l\in F_\infty. \tag{3}$$

Using (2) and (3) we have

$$\sum_{i\le k} A_i^*(I\otimes P_{-i}e_{-i})q=\sum_{i\le k}\sum_j\sum_l A_{ij}^* b_l\otimes e_{-x_j}P_{-i}e_{-i}e_{z_l}.$$

Lemma 7 tells us that all those terms are orthogonal to each other or zero. Hence,

$$\begin{aligned}
\left\|\sum_{i\le k} A_i^*(I\otimes P_{-i}e_{-i})q\right\|_2^2 &\le \sum_{i\le k}\sum_j\sum_l \|A_{ij}^* b_l\|_2^2\\
&=\sum_l\left[\sum_{i\le k}\sum_j\left\|A_{ij}^*\frac{b_l}{\|b_l\|_2}\right\|_2^2\right]\|b_l\|_2^2\\
&\le \sup_{\substack{b\in\ell_2^n\\ \|b\|_2\le 1}}\sum_{i\le k}\sum_j\|A_{ij}^* b\|_2^2\\
&=\sup_{\substack{b\in\ell_2^n\\ \|b\|_2\le 1}}\|T^*b\otimes e_0\|_2^2. \qquad \square
\end{aligned}$$

Proof of Proposition 4. Let $T\in M_n(\mathcal{L}_\infty)$. Write it as $T=T_1+T_2$, where $T_1\in M_n(\mathcal{L}_K)$ and $T_2\in M_n((\mathcal{L}_K)^\perp)$. Notice that if $q\in \ell_2^n(\mathcal{L}_\alpha)$, then

$$T_1 q\in \ell_2^n(\mathcal{L}_K) \quad\text{and}\quad T_2 q\in\ell_2^n((\mathcal{L}_K)^\perp).$$

Hence,

$$\sup_{\substack{q\in\ell_2^n(\mathcal{L}_\alpha)\\ \|q\|_2\le 1}}\|T_1 q\|_2\le \sup_{\substack{q\in\ell_2^n(\mathcal{L}_\alpha)\\ \|q\|_2\le 1}}\|Tq\|_2\le\|T\|.$$

Moreover, it is clear that given $b\in\ell_2^n$, we have that

$$\|T_1^* b\otimes e_0\|_2\le \|T^*b\otimes e_0\|_2\le\|T^*\|=\|T\|.$$

Therefore, by Proposition 8, $\|T_1\|\le 2\|T\|$. $\square$

Propositions 5 and 8 give a representation of $C_\lambda^*(F_\infty)$ in terms of row and column Hilbert spaces.

Let $T=\sum_{i\le k}(I\otimes e_i)A_i\in M_n(\mathcal{L}_K)$, $A_i\in M_n(\mathcal{L}_\alpha)$. Use (2) to write $T=\sum_{i\le k}\sum_j A_{ij}\otimes e_i e_{x_j}$, for some $A_{ij}\in M_n$. Then we have

$$\sup_{\substack{q\in\ell_2^n(\mathcal{L}_\alpha)\\ \|q\|_2\le 1}}\|Tq\|_2=\left\|\sum_{i\le k}A_i^*A_i\right\|^{\frac12},$$

$$\sup_{\substack{b\in\ell_2^n\\ \|b\|_2\le 1}}\|T^*b\otimes e_0\|_2=\left\|\sum_{i\le k}\sum_j A_{ij}A_{ij}^*\right\|^{\frac12}.$$

We see that the first term is the norm of T in $M_n(\,C\otimes C_\lambda^*(\mathcal{L}_\alpha)\,)$, and the second one is the norm of T in $M_n(R\otimes R(F_\alpha)\,)$. Here $R(F_\alpha)$ is $\ell_2(F_\alpha)$ with the row operator space structure.

Using the notation of interpolation theory of operator spaces (see [P]) we conclude

Proposition 9. $C^*_\lambda(F_\infty) \overset{c.b.}{\approx} [\, C \otimes C^*_\lambda(F_\infty)\,] \,\cap\, [\, R \otimes R(F_\infty)\,]$.

Remark. If we are interested only in the Banach space structure, we have that $C^*_\lambda(F_\infty)$ is isomorphic (but probably not completely isomorphic) to $C \otimes C^*_\lambda(F_\infty)$.

3. Isomorphisms of non-commutative analytic algebras

In this section we will consider only the words consisting of positive powers of the generators of F_k, and the identity. We denote this set by $P_k \subset F_k$, and let $\ell_2(P_k)$ be the Hilbert space with orthonormal basis $\{e_x : x \in P_k\}$. This Hilbert space is also denoted by $\mathbb{F}^2(H_k)$, the full Fock space on k-generators, see [Po]. Let $\mathcal{P}$ (or $\mathcal{P}_k$ to avoid confusion) be the linear span of the basic elements, and consider two norms on $\mathcal{P}$: The $\|\cdot\|_2$-norm, induced by $\ell_2(P_k)$, and the $\|\cdot\|_\infty$-norm, defined as

$$\|p\|_\infty = \sup\{\|pq\|_2 : q \in \mathcal{P}_k, \|q\|_2 \le 1\}.$$

Notice that $p \in \mathcal{P}_k$ induces a left multiplication operator on $\ell_2(P_k)$ and $\|p\|_\infty$ equals the operator norm of that map.

Remark. If $p \in \mathcal{P}_k$, then the $\|p\|_\infty$-norm does not coincide with the $\|p\|$-norm as an element of $C^*_\lambda(F_k)$. In fact, if Q is the orthogonal projection onto $\ell_2(P_k)$, then $\|p\|_\infty = \|QpQ\|$. We always have that $\|p\|_\infty \le \|p\|$ and sometimes the inequality is strict (see Proposition 17).

Let $\mathbb{A}(k)$ be the closure of $\mathcal{P}_k$ in the norm topology of $B(\ell_2(P_k))$, and $\mathbb{F}^\infty(k)$ the closure in the strong operator topology. These spaces are studied in [Po], where he calls them non-commutative analogues of the disk algebra and H^∞. When $k = 1$ they coincide with the classical definitions.

The main results of this section are

Theorem 10. *$\mathbb{A}(k)$ is completely isomorphic to $\mathbb{A}(\infty)$ when $n = 2, 3, 4, \cdots$.*

Theorem 11. *$\mathbb{F}^\infty(k)$ is completely isomorphic to $\mathbb{F}^\infty(\infty)$ when $n = 2, 3, 4, \cdots$.*

As in the previous section we will only present the proof of the first one, the other one is essentially the same. The proof of Theorem 10 will follow from Propositions 12, 13 and 14.

Proposition 12. *Let $n \le m$, and let $\Phi : \mathbb{A}(n) \to \mathbb{A}(m)$ be the formal identity. Then Φ is a complete isometry. Moreover, the orthogonal projection onto $\ell_2(P_n)$ is completely contractive from $\mathbb{A}(m)$ onto $\mathbb{A}(n)$.*

Proof. Let $E \subset P_m$ be the set of all $y \in P_m$ whose first letter does not start from $e_1, \cdots, e_n$. It is easy to see that $\{P_n y : y \in E\}$ forms a partition of P_m. Then $\ell_2(P_m) = \sum_{j=1}^\infty \oplus \ell_2(P_n y_j)$, where $E = \{y_j : j \in \mathbb{N}\}$.

Let $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$, $\|q\|_2 \le 1$. Use the previous partition to decompose q as $q = \sum_j r_j e_{y_j}$, for some $r_j \in \mathcal{P}_n$ and $y_j \in E$. Then

$$\|pq\|_2^2 = \sum_{i=0}^\infty \|pr_i\|_2^2 = \sum_{i=0}^\infty \left\| p \frac{r_i}{\|r_i\|_2} \right\|_2^2 \|r_i\|_2^2 \le \sup_i \left\| p \frac{r_i}{\|r_i\|_2} \right\|_2^2 \le \|p\|_\infty^2.$$

This tells us that $\|p\|_{\mathbb{A}(n)} = \|p\|_{\mathbb{A}(m)}$. Moreover, if $p \in \mathbb{A}(n) \subset \mathbb{A}(m)$, we norm it with elements from $\mathcal{P}_n$.

This is the fact that we use for the complementation. Given $p \in \mathcal{P}_m$, write it as $p = p_1 + p_2$, where $p_1 \in \mathcal{P}_n$ and $p_2 \in (\mathcal{P}_n)^\perp$. Then if $q \in \mathcal{P}_n$, we have that $p_1 q \in \mathcal{P}_n$ and $p_2 q \in (\mathcal{P}_n)^\perp$. Hence,

$$\|p_1\|_\infty = \sup_{\substack{q \in \mathcal{P}_n \\ \|q\|_2}} \|p_1 q\|_2 \leq \sup_{\substack{q \in \mathcal{P}_n \\ \|q\|_2}} \|pq\|_2 \leq \|p\|_\infty.$$

The completely bounded part is very similar. $\square$

Proposition 13. *There exists a subspace of* $\mathbb{A}(2)$ *completely isometric to* $\mathbb{A}(\infty)$ *and completely complemented by the orthogonal projection.*

Proof. Let a, b be the generators of P_2. Let P_α be the set of all words generated by $ab, a^2b, a^3b, a^4b, \cdots$. P_α is clearly isomorphic to P_∞. Let $E \subset P_2$ be the set of all words in P_2 such that no initial segment belongs to P_α. Then it is easy to see that $\{P_\alpha y : y \in E\}$ forms a partition of P_2, and the proof is like that of the previous proposition. $\square$

Proposition 14. $\mathbb{A}(\infty)$ *is completely isomorphic to* $C \otimes \mathbb{A}(\infty)$, *where* C *is the column Hilbert space.*

Proof. Divide the generators of P_∞ into $\alpha_1, \alpha_2 \cdots; e_1, e_2, \cdots$, and let P_α be the set of words generated by the α's. Let $K = \bigcup_{j=1}^\infty e_j P_\alpha \subset P_\infty$, and let $\mathcal{P}(K)$ be the span of the basic elements in K. Denote the closure of $\mathcal{P}(K)$ in the ℓ_2-norm by $\ell_2(K)$, and in the $\|\cdot\|_\infty$-norm by $\mathbb{A}(K)$.

The canonical element of $\mathcal{P}(K)$ looks like $\sum_{i \leq k} e_i p_i$, for some $k \in \mathbb{N}$ and $p_\alpha \in \mathcal{P}_\alpha$. Given any $q \in \mathcal{P}$ the $e_i p_i q$'s are orthogonal. Then we have

$$\left\| \sum_{i \leq k} e_i p_i \right\|_\infty = \sup_{\|q\|_2 \leq 1} \left\| \sum_{i \leq k} e_i p_i q \right\|_2 = \sup_{\|q\|_2 \leq 1} \sqrt{\sum_{i \leq k} \|p_i q\|_2^2} = \left\| \sum_{i \leq k} p_i^* p_i \right\|_\infty^{\frac{1}{2}}.$$

Since $\mathbb{A}(P_\alpha)$ is isometric to $\mathbb{A}(\infty)$, we conclude that $\mathbb{A}(K)$ is isometric to $C \otimes \mathbb{A}(\infty)$. Moreover, the elements in $\mathbb{A}(k)$ are normed by elements in $\ell_2(P_\alpha)$.

We will now see that $\mathbb{A}(K)$ is complemented in $\mathbb{A}(\infty)$. Let $p \in \mathcal{P}_\infty$ and decompose it as $p = p_1 + p_2$, where $p_1 \in \mathcal{P}(K)$ and $p_2 \in (\mathcal{P}(K))^\perp$. If $q \in \mathcal{P}_\alpha$, then $p_1 q \in \mathcal{P}(K)$ and $p_2 q \in (\mathcal{P}(K))^\perp$. Hence, $\|p_1 q\|_2 \leq \|pq\|_2$, and $\|p_1\|_\infty \leq \|p\|_\infty$.

As in the proof of Proposition 5 it is clear that $\mathbb{A}(K)$ is isomorphic to its square, and then isomorphic to $\mathbb{A}(\infty)$.

The completely bounded part follows in the same way after we replace the scalars by matrices. $\square$

Proof of Theorem 10. By Propositions 12 and 13 we have that $\mathbb{A}(k) = \mathbb{A}(\infty) \oplus Y$ for some Y. Since $\mathbb{A}(\infty) \approx \mathbb{A}(\infty) \oplus \mathbb{A}(\infty)$, we have

$$\mathbb{A}(k) = \mathbb{A}(\infty) \oplus Y \approx \mathbb{A}(\infty) \oplus \mathbb{A}(\infty) \oplus Y \approx \mathbb{A}(\infty) \oplus \mathbb{A}(k).$$

On the other hand, $\mathbb{A}(\infty) = \mathbb{A}(k) \oplus Z$, for some Z. If $Q : \mathbb{A}(\infty) \to \mathbb{A}(k)$ is that projection and $I : C \to C$ is the identity on C, $I \otimes Q$ decomposes $C \otimes \mathbb{A}(\infty) = [C \otimes \mathbb{A}(k)] \oplus [C \otimes Z]$ because Q is completely bounded. Hence,

$$\begin{aligned}
\mathbb{A}(\infty) &\approx C \otimes \mathbb{A}(\infty) \\
&= [C \otimes \mathbb{A}(k)] \oplus [C \otimes Z] \\
&\approx \mathbb{A}(k) \oplus [C \otimes \mathbb{A}(k)] \oplus [C \otimes Z] \\
&\approx \mathbb{A}(k) \oplus [C \otimes \mathbb{A}(\infty)] \approx \mathbb{A}(k) \oplus \mathbb{A}(\infty). \qquad \square
\end{aligned}$$

4. Applications to von Neumann's inequality

Fix k for this section and let P_k be the positive words generated by $e_1, \cdots, e_k$. As in the previous section, $\mathbb{F}^2(H_k) = \ell_2(P_k)$ is the full Fock space on H_k, a k-dimensional Hilbert space; $\mathbb{A}(k)$ and $\mathbb{F}^\infty(k)$ have the same meaning.

In [Po] G. Popescu proved that if $T_1, \cdots, T_k \in B(\ell_2)$ are such that $\| \sum_{i \le k} T_i T_i^* \| \le 1$ (i.e., $\|[T_1 \cdots T_k]\| \le 1$), then any $p(e_1, \cdots, e_k) \in \mathbb{A}(k)$ satisfies

$$\|p(T_1, \cdots, T_k)\| \le \|p\|_{\mathbb{A}(k)}. \tag{4}$$

When $k = 1$, this is the classical von Neumann's inequality.

In this section we prove that $\mathbb{A}(k)$ and $\mathbb{F}^\infty(k)$ contain many complemented Hilbertian subspaces. Hence we can easily compute $\|p\|_{\mathbb{A}(k)}$ whenever p belongs to one of those subspaces, and use $\|p\|_{\mathbb{A}(k)}$ in Popescu's inequality (4). (See [AP] for more examples and connections with inner functions).

We start with the following elementary lemma.

Lemma 15. *Let $p = \sum_i a_i e_{x_i}, q = \sum_j b_j e_{y_j} \in \mathcal{P}$ be such that $x_i y_j = x_{i'} y_{j'}$ if and only if $i = i'$ and $j = j'$ (that is, we cannot have any cancellation); then $\|pq\|_2 = \|p\|_2 \|q\|_2$.*

Proof. We have that $pq = \sum_i \sum_j a_i b_j e_{x_i y_j}$. Since all the $x_i y_j$-terms are different, the $e_{x_i y_j}$'s are orthogonal. Hence,

$$\|pq\|_2 = \sqrt{\sum_i \sum_j |a_i b_j|^2} = \sqrt{\sum_i |a_i|^2} \sqrt{\sum_j |b_j|^2} = \|p\|_2 \|q\|_2. \qquad \square$$

Remark. The lemma extends to the M_n-case just as easily. If $T = \sum_i A_i \otimes e_{x_i} \in M_n(\mathcal{P})$ and $q = \sum_j b_j \otimes e_{y_j} \in \ell_2^n(\mathcal{P})$, then $Tq = \sum_i \sum_j A_i b_j \otimes e_{x_i y_j}$ and $\|Tq\|_2 = \sqrt{\sum_i \sum_j \|A_i b_j\|_2^2}$.

Proposition 16. *Let $W_n \subset P_k$ be the set of all words in P_k having n-letters, and let $X_n = span\{e_x : x \in W_n\} \subset \mathbb{A}(k)$. Then X_n is completely isometric to C_{n^k}, the column Hilbert space of the same dimension. Moreover, X_n is completely complemented in $\mathbb{A}(k)$.*

Proof. Let $p \in X_n$ and $q \in \mathcal{P}_k$, $\|q\|_2 \le 1$. Since $\mathbb{F}^2(H_k) = \sum_{m=0}^\infty \oplus X_m$, we write q as $q = \sum_m r_m$, where $r_m \in X_m$. Notice that $pr_m \in X_{n+m}$ and hence all of them are orthogonal. Moreover, if $x_1, x_2 \in X_n$, $y_1, y_2 \in X_m$ and $x_1 y_1 = x_2 y_2$, then it is necessary that $x_1 = x_2$ and $y_1 = y_2$. This implies that there is no cancellation in pr_m and hence, by the previous lemma, $\|pr_m\|_2 = \|p\|_2 \|r_m\|_2$. Therefore,

$$\|pq\|_2 = \sqrt{\sum_{m=0}^\infty \|pr_m\|_2^2} = \sqrt{\sum_{m=0}^\infty \|p\|_2^2 \|r_m\|_2^2} = \|p\|_2 \|q\|_2,$$

and $\|p\|_\infty = \|p\|_2$.

The completely bounded case is very similar. A canonical element of $M_{n_0}(X_n)$ looks like $T = \sum_{i \le k} A_i \otimes e_{x_i}$, where $A_i \in M_{n_0}$ and $e_{x_i} \in X_n$. A canonical element of $\ell_2^{n_0}(X_m)$ looks like $q = \sum_j b_j \otimes e_{y_j}$, where $b_j \in \ell_2^{n_0}$ and $e_{y_j} \in X_m$. Then $Tq = \sum_i \sum_j A_i b_j \otimes e_{x_i} e_{y_j} \in \ell_2^{n_0}(X_{n+m})$ and all of those terms are orthogonal to each other. Then the proof proceeds as those of section 2. The complementation part is very easy: If $p \in X_n$, then $\|p\|_\infty = \|pe_0\|_2$. $\square$

Multiplication from the left by any one of the e_i's in $\mathbb{A}(k)$ or $\mathbb{F}^\infty(k)$ is an isometry (i.e., $\|p\|_\infty = \|e_i p\|_\infty$). However, multiplication from the right does not have to be like that.

Proposition 17. *Let $p \in \mathbb{A}(k-1) \subset \mathbb{A}(k)$. Then $\|pe_k\|_\infty = \|pe_k\|_2 = \|p\|_2$.*

Proof. Let $p \in \mathcal{P}_{k-1}$, $p = \sum_i a_i e_{x_i}$ where $x_i \in P_{k-1}$, and $q \in \mathcal{P}_k$, $q = \sum_j b_j e_{y_j}$ where $y_j \in P_k$. Then

$$pe_k q = \sum_i \sum_j a_i b_j e_{x_i} e_k e_{y_j}.$$

Since $e_{x_i} e_k e_{y_j} = e_{x_{i'}} e_k e_{y_{j'}}$ iff $x_i = x_{i'}$ and $y_j = y_{j'}$, then Lemma 15 applies and we have that $\|pe_k q\|_2 = \|p\|_2 \|e_k q\|_2 = \|p\|_2 \|q\|_2$. $\square$

We conclude with the following two applications of the previous propositions.

1. Let $p(e_1, e_2, \cdots, e_k) \in \mathcal{P}$ be a non-commutative homogeneous polynomial of degree n; i.e., $p(\lambda e_1, \cdots, \lambda e_k) = \lambda^n p(e_1, \cdots, e_k)$ (or $p \in X_n$ with the notation of Proposition 16). If $\| \sum_{i \leq k} T_i T_i^* \| \leq 1$, then

$$\|p(T_1, T_2, \cdots, T_k)\| \leq \|p\|_2.$$

2. Let $T_1, T_2 \in B(\ell_2)$ be such that $\|[T_1\ T_2]\| \leq 1$ (i.e., $\|T_1 T_1^* + T_2 T_2^*\| \leq 1$) and let $p(t)$ be a polynomial in one variable. The classical von Neumann's inequality states that $\|p(T_1)\| \leq \|p\|_\infty$. Therefore, using the Banach algebra properties of $B(\ell_2)$ we get that

$$\|p(T_1)T_2\| \leq \sup_{t \in \mathbb{T}} |p(t)|,$$

but if we apply Proposition 17 to Popescu's inequality we get

$$\|p(T_1)T_2\| \leq \sqrt{\int_{\mathbb{T}} |p(t)|^2 dm(t)}.$$

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