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RIGHT ADJOINT FOR THE SMASH PRODUCT FUNCTOR

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Abstract. The smash-product functor $(-) \wedge (X, x_0)$ in the category $\underline{\mathbf{Top}}_*$ of pointed topological spaces has a right adjoint for any choice of the base point x_0, if and only if the topological space X is quasi-locally compact, that is, if and only if the product functor $(-) \times X$ has a right adjoint in the category $\underline{\mathbf{Top}}$ of topological spaces.

Introduction

A space X is cartesian in the category of topological spaces and continuous maps if the product functor $(-) \times X$ has a right adjoint. This means that there exists a proper and admissible topology on the space of maps Y^X between X and Y (for any topological space Y) [D]. Cartesian objects in $\underline{\mathbf{Top}}$ were characterized by Day and Kelly [D-K]. They are the quasi-locally compact spaces [H-L].

The problem of the existence of a proper and admissible topology on the function space $(Y, y_0)^{(X,x_0)}$ consisting of the maps of Y^X preserving base points is related to the adjointness of the smash-product functor. It is known that this functor has a right adjoint whenever X is locally compact and Hausdorff; in this case the topology on $(Y, y_0)^{(X,x_0)}$ is the compact open topology [M].

In this paper, it is proved that the spaces (X, x_0) for which the functor $(-) \wedge (X, x_0)$ has a right adjoint are exactly the spaces X which are cartesian in $\underline{\mathbf{Top}}$, independently of the choice of x_0. That is, the existence of a proper and admissible topology on $(Y, y_0)^{(X,x_0)}$ for any (Y, y_0) is equivalent to the existence of a proper and admissible topology on the whole space of maps from X to Y, for any Y.

Smash-product and adjunction

We can consider, in $\underline{\mathbf{Top}}_*$, the endofunctor $(-) \wedge (X, x_0)$ and ask when it has a right adjoint. When it exists we will call it $\mathcal{G}_{(X,x_0)}$.

In the case of X cartesian in $\underline{\mathbf{Top}}$, we indicate by Y^X the power object and by $(Y, y_0)^{(X,x_0)}$ the subspace of Y^X given by $\{f \in Y^X | f(x_0) = y_0\}$ with base point the constant y_0-valued map.

Theorem 1. *If X is cartesian in $\underline{\mathbf{Top}}$, then $(-) \wedge (X, x_0)$ has a right adjoint, for every x_0 in X. Moreover $\mathcal{G}_{(X,x_0)}(Y, y_0) = (Y, y_0)^{(X,x_0)}$.*

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Proof. Suppose X is cartesian in $\underline{\mathbf{Top}}$ and take any space Y; let $\hat{e}\colon Y^X \times X \to Y$ be the evaluation map. Consider in Y^X the subspace $\mathcal{G}_{(X,x_0)}(Y,y_0) = \{f \in Y^X | f(x_0) = y_0\}$ and the restriction e_1 of $\hat{e}$ to $\mathcal{G}_{(X,x_0)}(Y,y_0) \times (X,x_0)$ which is a map in $\underline{\mathbf{Top}}_*$. The map e_1 is compatible with the quotient in the definition of the smash-product, and so we can consider the map $e\colon \mathcal{G}_{(X,x_0)}(Y,y_0) \wedge (X,x_0) \to (Y,y_0)$ induced by e_1. We note that $e(f,x) = f(x)$.

Let $f\colon (Z,z_0) \wedge (X,x_0) \to (Y,y_0)$ be a map in $\underline{\mathbf{Top}}_*$ and consider the quotient $p\colon (Z,z_0) \times (X,x_0) \to (Z,z_0) \wedge (X,x_0)$ which gives the smash-product. Since X is cartesian in $\underline{\mathbf{Top}}$, related to $fp\colon (Z,z_0) \times (X,x_0) \to (Y,y_0)$ there is an $(fp)_1\colon Z \to Y^X$ such that $\hat{e}((fp)_1 \times \mathrm{id}_X) = fp$. The map $(fp)_1$ preserves the base points and its image is a subspace of $\mathcal{G}_{(X,x_0)}(Y,y_0)$, so we can factor $(fp)_1$ through the inclusion of $\mathcal{G}_{(X,x_0)}(Y,y_0)$ in Y^X and consider the first factor $(fp)_2$ as a map in $\underline{\mathbf{Top}}_*$. In such a way we obtain $(fp)_2 \times \mathrm{id}_X\colon (Z,z_0) \times (X,x_0) \to (Y,y_0)$. By construction $(fp)_2 \times \mathrm{id}_X$ is compatible with the quotient p, and the proof is complete.

Theorem 2. *If the functor $(-)\wedge(X,x_0)$ has a right adjoint, then $\mathcal{G}_{(X,x_0)}(Y,y_0)$ is a space whose underlying set is in natural bijective correspondence with $(Y,y_0)^{(X,x_0)}$, the counit of the adjunction is the map $e\colon \mathcal{G}_{(X,x_0)}(Y,y_0) \wedge (X,x_0) \to (Y,y_0)$ such that $e(f,x) = f(x)$ and the base point corresponds to the constant function valued at y_0.*

Proof. Let D_2 be the space with two points a, b and the discrete topology. By the adjunction, there is a bijection between $(Y,y_0)^{(D_2,a)\wedge(X,x_0)}$ and $(\mathcal{G}_{(X,x_0)}(Y,y_0))^{(D_2,a)}$, and on the other side $(D_2,a) \wedge (X,x_0)$ is homeomorphic to (X,x_0) and $(G_{(X,x_0)}(Y,y_0))^{(D_2,a)}$ is in bijection with $\mathcal{G}_{(X,x_0)}(Y,y_0)$; so the first part of the theorem is proved.

Any map $f\colon (X,x_0) \to (Y,y_0)$ can be considered as a map from $(D_2,a) \wedge (X,x_0)$ into (Y,y_0). As a consequence, by the adjunction, for any f, there is an $f_1\colon (D_2,a) \to \mathcal{G}_{(X,x_0)}(Y,y_0)$ such that $e(f_1 \wedge \mathrm{id}_X) = f$; so $e(f,x) = f(x)$. Finally, given the one point space $\bullet$, and the map $h : \bullet \wedge (X,x_0) \to (Y,y_0)$, there is a map $h_1 : \bullet \to \mathcal{G}_{X,x_0)}(Y,y_0)$ such that $h_1(\bullet)$ is the base point of $\mathcal{G}_{(X,x_0)}(Y,y_0)$. This completes the proof.

We denote by S the Sierpinski space with the two points 0 and 1 and $\{0\}$ the nontrivial open set. If the functor $(-)\wedge(X,x_0)$ has a right adjoint, as a consequence of Theorem 2, $\mathcal{G}_{(X,x_0)}(S,0)$ can be identified with the set of the open sets U of X such that $x_0 \in U$ and base point the open set X. On the other hand, $\mathcal{G}_{(X,x_0)}(S,1)$ can be identified with the set of the open sets U of X such that $x_0 \notin U$ and base point the empty set.

The following Lemma characterizes convergent nets of the spaces $\mathcal{G}_{(X,x_0)}(S,0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S,1)$), while Lemma 4 proves that the open sets of these spaces are Scott-open [H-L].

Lemma 3. *Suppose $(-)\wedge(X,x_0)$ admits a right adjoint. A net U_i converges to U in $\mathcal{G}_{(X,x_0)}(S,0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S,1)$) if and only if:*

$(*)$ *for each $x \in U$ and for each net x_λ converging to x in X, there is an i' and a λ' such that $x_\lambda \in U_i$, for every $i > i'$ and $\lambda > \lambda'$.*

Proof. Let U_i converge to U in $\mathcal{G}_{(X,x_0)}(S,0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S,1)$), $x \in U$ and x_λ converge to x in X. Consider the counit of the adjunction $e\colon \mathcal{G}_{(X,x_0)}(S,0) \wedge$

$(X, x_0) \to (S, 0)$ (respectively, $(S, 1)$) and the quotient map $p: \mathcal{G}_{(X,x_0)}(S, 0) \times (X, x_0) \to \mathcal{G}_{(X,x_0)}(S, 0) \wedge (X, x_0)$ (respectively, $(S, 1)$). Since (U_i, x_λ) converges to (U, x) and the map ep is continuous with $ep(U, x) = 0$, then $ep(U_i, x_\lambda)$ converges to 0. In S the nets converging to 0 are eventually constant, so there is an i' and a λ' such that, for every $i > i'$ and $\lambda > \lambda'$, $ep(U_i, x_\lambda) = 0$, that is $x_\lambda \in U_i$.

Vice versa, suppose U_i is a net in $\mathcal{G}_{(X,x_0)}(S, 0)$ and $U \in \mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$) fulfilling condition $(*)$. Consider the space $I \cup \{\bullet\}$, where I is the direct set of the net U_i, $\bullet$ is a maximum point whose base-neighborhoods are the sets of the form $I_j = \{\bullet\} \cup \{i \in I \mid i \geq j\},\ j \in I$, and the points of I are isolated. We can assume, without changing the nature of the net U_i, that there exists a point $i_0 \in I$, such that $U_{i_0} = X$ (respectively, $U_{i_0} = \emptyset$). Moreover, we consider the map $\alpha: (I \cup \{\bullet\}, i_0) \to \mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$) so defined by $\alpha(i) = U_i$, $\alpha(\bullet) = U$. We prove that the map α is continuous, which implies the convergence of the net U_i to U. By the existence of the right adjoint, α is continuous if and only if the corresponding map $e(\alpha \wedge \mathrm{Id}_X) = \overline{\alpha}: (I \cup \{\bullet\}, i_0) \wedge (X, x_0) \to (S, 0)$ (respectively, $(S, 1)$) is continuous; so we will prove the continuity of $\overline{\alpha}$. To this aim we can consider the quotient map $q: (I \cup \{\bullet\}, i_0) \times (X, x_0) \to (I \cup \{\bullet\}, i_0) \wedge (X, x_0)$ induced by the smash-product and prove the continuity of $\overline{\alpha}q$. By the adjunction $(\overline{\alpha}q)^{-1}(0) = \{(i, x) \mid x \in U_i,\ i \in I\} \cup \{(\bullet, x) \mid x \in U\}$. Since every point of I is isolated and each U_i is open, the set $\{(i, x) \mid x \in U_i,\ i \in I\}$ is open since it is union of open sets, therefore each (i, x) of $(\overline{\alpha}q)^{-1}(0)$ belongs to its interior. Take now $(\bullet, x) \in (\overline{\alpha}q)^{-1}(0)$ (that is $x \in U$); the topology defined on $(I \cup \{\bullet\})$ and condition $(*)$ implies that any net converging to $(\bullet, x)$ in the product space $(I \cup \{\bullet\}, i_0) \times (X, x_0)$ is eventually in $(\bar{\alpha}q)^{-1}(0)$, that is, $(\bullet, x)$ belongs to the interior of $(\overline{\alpha}q)^{-1}(0)$. We can conclude that $(\bar{\alpha}q)^{-1}(0)$ is open and so, α is continuous.

Lemma 4. *Suppose $(-) \wedge (X, x_0)$ admits a right adjoint, and suppose H is open in $\mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$); then H is Scott-open* [H-L], *i.e.*:

(a) *If U, U' are open in $X, U \in H$ and $U' \supseteq U$ (with $x_0 \notin U'$ when H is open in $\mathcal{G}_{(X,x_0)}(S, 1)$), then $U' \in H$.*

(b) *If $V = \{U_i \mid i \in I\}$ is a family of open subsets of X and $U = \bigcup\{U_i \mid i \in I\} \in H$, there exists a finite subfamily of V whose union belongs to H.*

Proof. (a) Suppose U, U' open in $X, U \in H$ and $U' \supseteq U$ ($x_0 \notin U'$ in the case when H is open in $\mathcal{G}_{(X,x_0)}(S, 1)$). The constant sequence whose value is U' converges to U in $\mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$) because it fulfils the condition $(*)$ in Lemma 3. Since H is open and U belongs to H, there must be an element of the constant sequence belonging to H; consequently $U' \in H$.

(b) Let $U = \bigcup\{U_i \mid i \in I\} \in H$. If H is open in $\mathcal{G}_{(X,x_0)}(S, 0)$, it follows that $x_0 \in U$; therefore there exists an i' such that $x_0 \in U_{i'}$. Consider in $\mathcal{G}_{(X,x_0)}(S, 0)$ the net whose direct set is $\{(i_1, i_2, \ldots, i_n) \mid i_1, i_2, \ldots, i_n \in I\}$ with the relation $(i_1, i_2, \ldots, i_n) \succ (j_1, j_2, \ldots, j_m)$ if $U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n} \supseteq U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_m}$ and image of $(i_1, i_2, \ldots, i_n)$ equal to $U_{i'} \cup U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$. This net, according to $(*)$, converges to U in $\mathcal{G}_{(X,x_0)}(S, 0)$; as a consequence, since H is an open set which contains the limit of the net, there exists $(i_1, i_2, \ldots, i_n)$ such that $U_{i'} \cup U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n} \in H$. When H is open in $\mathcal{G}_{(X,x_0)}(S, 1), x_0 \notin U$ and therefore, $x_0 \notin U_i$, for each $i \in I$. In this case, we can consider the net with the same direct set as above and with the image of $(i_1, i_2, \ldots, i_n)$ equal to $U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$. The same argument of the first case proves that there is $(i_1, i_2, \ldots, i_n)$ such that $U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n} \in H$.

Definition 5. A space X is *quasi-locally* compact if for every x in X and for every neighbourhood U of x there is a neighbourhood of $V \subseteq U$ of x such that every open cover of U has a finite subcover of V.

Theorem 6. *Let X be a topological space. The following statements are equivalent:*

(1) *There exists x_0 in X such that the functor $(-) \wedge (X, x_0)\colon \underline{\mathbf{Top}}_* \to \underline{\mathbf{Top}}_*$ has a right adjoint.*

(2) *X is cartesian in $\underline{\mathbf{Top}}$, that is X is quasi-locally compact.*

(3) *For any x_0 in X, the functor $(-) \wedge (X, x_0)\colon \underline{\mathbf{Top}}_* \to \underline{\mathbf{Top}}_*$ has a right adjoint.*

Proof. (1) $\to$ (2) Since the continuity of the counit of the adjunction $e\colon \mathcal{G}_{(X,x_0)}(S,0) \wedge (X, x_0) \to (S, 0)$ implies the continuity of the evaluation map $e'\colon \mathcal{G}_{(X,x_0)}(S,0) \times (X, x_0) \to (S, 0)$, it follows that $(e')^{-1}(0)$ is an open set with $e'(W, x_0) = 0$, for any $W \in \mathcal{G}_{(X,x_0)}(S,0)$.

Take a point $x \in X$ and fix U open in X with $x \in U$.

Suppose $x \in \mathrm{cl}\{x_0\}$; then $x_0 \in U$. Consequently U belongs to $\mathcal{G}_{(X,x_0)}(S,0)$ and $e'(U, x) = 0$. Since $(e')^{-1}(0)$ is open, there is an H, open in $\mathcal{G}_{(X,x_0)}(S,0)$, with $U \in H$ and a neighbourhood V of x such that $(e')^{-1}(0) \supseteq H \times V$. Since $e'(W, y) = 0$ when $y \in W$, any element of H contains V. Now, consider an open cover of U, $\{U_i \mid i \in I\}$ and consider $U' = \bigcup\{U_i \mid i \in I\}$. Since $U' \supseteq U$, by Lemma 4a), $U' \in H$ and by Lemma 4b) there are $i_1, i_2, \dots, i_n \in I$ such that $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \in H$ and this implies that $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \supseteq V$.

Suppose now $x \notin \mathrm{cl}\{x_0\}$. Therefore, there is an open A of X, such that $x \in A$ and $x_0 \notin A$. The open set $U \cap A \in \mathcal{G}_{(X,x_0)}(S,1)$; the map $e\colon \mathcal{G}_{(X,x_0)}(S,1) \wedge (X, x_0) \to (S, 1)$ is continuous and $e(U \cap A, x) = 0$. Replacing $U' = \bigcup\{U_i \mid i \in I\}$ by $U' = \bigcup\{U_i \cap A \mid i \in I\}$ in the argument used before proves that there exists a neighbourhood V of x with $U \cap A \supseteq V$, such that any open cover of U (and then of $U \cap A$), admits a finite subcover for V.

(2) $\to$ (3) Theorem 1.

(3) $\to$ (1) Trivial.

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