PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

A CHARACTERIZATION OF REFLEXIVE BANACH SPACES

EVA MATOUŠKOVÁ AND CHARLES STEGALL

(Communicated by Dale Alspach)

ABSTRACT. A Banach space Z is not reflexive if and only if there exist a closed separable subspace X of Z and a convex closed subset Q of X with empty interior which contains translates of all compact sets in X. If, moreover, Z is separable, then it is possible to put $X = Z$.

We consider the following problem: When does a Banach space contain a closed convex set Q with empty interior which contains a translate of any compact set in X? The basic example of such a Banach space is the space $C(\mathcal{K})$ of continuous functions on a compact infinite space $\mathcal{K}$. Indeed, it is enough to choose a point $p \in \mathcal{K}$ which is not isolated and define Q as the set of all functions in $C(\mathcal{K})$ which attain their minima at p. Since p is not isolated, Q has empty interior. If K is a compact subset of $C(\mathcal{K})$, then by the Banach-Dieudonné theorem [3] there exists a sequence $\{f_n\}$ of functions in $C(\mathcal{K})$ converging to zero such that K is contained in its closed convex hull. If we define the function g by

$$g(t) := \sup\{|f_n(t) - f_n(p)| : \ n \in N\} \quad \text{for} \quad t \in \mathcal{K},$$

then it is easy to check that g is continuous and each function $g + f_n$ is in Q. Consequently, since Q is convex, the translate $g + K$ is contained in Q.

If a Banach space Z can be mapped linearly onto a Banach space X containing the required set Q, then Z also contains such a set. Namely, by the open mapping theorem, it is enough to take the preimage of Q. Therefore, for example, ℓ_1 contains the required set because it can be mapped onto any separable Banach space, in particular, $C[0,1]$.

In this note we show that, in fact, any separable nonreflexive Banach space X contains a closed convex set with empty interior which contains a translate of any compact set in X.

Borwein and Noll observed in [1] that there exist a convex continuous function on the space c_0 of null sequences and a closed subset Q of c_0 which is not a Haar null set so that f fails to be Fréchet differentiable on Q. They define f as the distance from the positive cone $Q := \{\{x_n\} \in c_0;\ x_n \geq 0,\ n = 1, 2, \dots\}$. As Q has no interior points, f fails to be Fréchet differentiable at all points of Q. The set Q contains a translate of any compact set in c_0, and, therefore, for any

Received by the editors May 24, 1994 and, in revised form, August 18, 1994.
1991 *Mathematics Subject Classification.* Primary 46B10; Secondary 46B20.
Key words and phrases. Banach spaces, reflexivity, convexity.
The first author was partially supported by a grant of the Őterreichische Akademische Austauschdienst.

probability Borel measure μ on c_0 there exists some $x \in c_0$ such that $\mu(Q+x) > 0$. Consequently, Q is not Haar null (for the definition see [2]). They conjecture in [1] that also in ℓ_2 there exists a closed convex set C with empty interior which contains a translate of any compact set. We show that this is not the case in any reflexive Banach space, but on the other hand every nonreflexive Banach space has a closed subspace containing such a set.

By B_X we denote the open unit ball of a Banach space X, and $B_X(x,r)$ is the usual notation for the open ball with center x and diameter r; the subscript will be often omitted. We denote the closure of a set A by $\bar{A}$ or cl$A$.

We will make use of the following variation of the Banach-Dieudonné theorem: Let X be a Banach space, K a compact subset of $B_X(0,c)$ and E a dense subset of $B_X(0,2c)$. Then there exists a sequence $\{F_n\}$ of finite sets in E so that

$$K \subset \mathrm{cl}\left(\sum_{n=1}^{\infty} 2^{-n} F_n\right). \tag{1}$$

This follows from the fact that there exist a sequence $\{F_n\}$ of finite sets in E and a sequence of compact sets $\{K_n\}$ in $\bar{B}_X(0,c)$ so that

$$K \subset \sum_{i=1}^{n} 2^{-i} F_i + 2^{-n} K_n \quad \text{for} \quad n \in N. \tag{2}$$

Indeed, if $n = 1$, choose $F_1 \subset E$ so that $2^{-1}F_1$ is a $\frac{c}{2}$-net for K. Then the set

$$K_1 := 2\left((K - 2^{-1}F_1) \cap \bar{B}(0, \frac{c}{2})\right)$$

is a compact subset of $\bar{B}(0,c)$ and $K \subset 2^{-1}F_1 + 2^{-1}K_1$. Now we can continue by induction. Suppose that F_i and K_i for $i = 1, \ldots, n$ so that (2) holds have been already constructed. Choosing $F_{n+1} \subset E$ so that $2^{-1}F_{n+1}$ is a $\frac{c}{2}$-net for the set K_n and defining

$$K_{n+1} := 2\left((K_n - 2^{-1}F_{n+1}) \cap \bar{B}(0, \frac{c}{2})\right)$$

completes the proof. The following lemmata are possibly not the most efficient way to our main result, but we think that they may be of independent interest.

Lemma 1. *Let Z be a Banach space, U an open convex subset of Z and f a continuous real valued function defined on U. Then, either f is affine or the convex hull G of the graph of f has nonempty interior.*

Proof. Suppose that f is not affine. Then there exist x and y in U such that $1/2(f(x)+f(y)) \neq f((x+y)/2)$. Define $z_0 := (x+y)/2$ and $c := (f(x)+f(y))/2$. We can suppose by replacing f by $-f$ and adding a constant, if necessary, that

$$(f(x)+f(y))/2 - f(z_0) = \alpha > 0 \quad \text{and} \quad f(z_0) > 0.$$

Choose some $\varepsilon > 0$ so that $0 < f(v) < f(z_0) + \alpha/2$ for every $v \in Z$ for which $\|v - z_0\| < \varepsilon$. Clearly the interior of the cone cap

$$M := \{x_{z,t} = t(z,0) + (1-t)(z_0,c) : \ z \in Z, \ \|z - z_0\| < \varepsilon, \ 0 \le t \le \alpha/(2c)\} \subset Z \times R$$

is nonempty. Let some $x_{z,t} \in M$ be given, we will show that $x_{z,t} \in G$. Consider the function

$$g(s) := (1-s)c - f((1-s)z_0 + sz), \quad t \le s \le 1.$$

The function g is continuous, $g(t) > 0$, and $g(1) < 0$. Therefore, there exists some $r \in (t, 1)$ for which $g(r) = 0$. Hence, $x_{z,r}$ is contained in the graph of f and since

$$x_{z,t} = \frac{t}{r} x_{z,r} + (1 - \frac{t}{r})(z_0, c)$$

we have $x_{z,t} \in G$. □

We say that a convex subset Q of a Banach space X is spanning if it contains a line segment in every direction, that is $X = \bigcup_{t>0} t(Q - Q)$. Observe that if a convex set Q contains a translate of every finite subset of the unit ball, then Q is spanning. If Q contains translates of all compact sets in X (or, for that matter, of all line segments), then $X = Q - Q$. Indeed, if $x \in X$ is given, then there exists $z \in X$ so that $[z, z + x] \subset Q$, and $x = z + x - z \in Q - Q$.

Lemma 2. *Suppose that X is a Banach space and $Q \subseteq X$ is a bounded, closed and convex set with empty interior that is also spanning. Then for any compact subset K of X it follows that $Q + K$ also has empty interior.*

Proof. First, we show that $Q \cap H$ is nowhere dense in H if H is any closed hyperplane. Suppose that $x^* \neq 0$, $w \in H = \{x^* = a\}$, $\delta > 0$ and

$$B(w, \delta) \cap H \subseteq Q \cap H.$$

Choose some $y \in X$ such that $\langle x^*, y\rangle > 0$. Since Q is spanning there exist $t > 0$, u and v, both in Q, so that $t(u-v) = y$. It follows that $\langle x^*, u-v\rangle > 0$ and one of u or v, say u, is not in H. It is routine to check that the convex hull of $\{u\} \cup (B(w, \delta) \cap H)$ has an interior point relative to X (try $\frac{1}{2}(u + w)$) which contradicts the fact that Q has no interior. Suppose that $H \subseteq X$ is a closed hyperplane, $u \in X$, $x \notin H$ and suppose that $h^* \in H^*$. Then the set $\{y + \langle h^*, y\rangle x + u : y \in H\}$ is a hyperplane in X and the transformation $y \mapsto y + \langle h^*, y\rangle x + u$ is an affine homeomorphism. If $x \in X$, then the set $Q' = Q + [-x, x]$ is also bounded, closed, convex and spanning. We will show that it has empty interior. Suppose that the interior of Q' is nonempty. Then $x \neq 0$; choose $x^* \in X^*$ so that $\langle x^*, x\rangle > 0$. Let P be the projection on X whose image is the kernel H of x^* and whose kernel is the span of x. The open mapping theorem says that $P(Q) = P(Q')$ has nonempty interior in H. Suppose that $w \in H$, $\delta > 0$ and $B(w, \delta) \cap H \subseteq P(Q)$. For $z \in B(w, \delta) \cap H$ define

$$f(z) := \inf\{t : z + tx \in Q\}.$$

It is easy to see that f is bounded and convex, hence continuous. The mapping $(z, t) \mapsto z + tx$ is an isomorphism from $H \times R$ onto X which maps the graph of f onto the set $\{z + f(t)x : z \in B(w, \delta) \cap H\} \subset Q$. Because Q has empty interior, Lemma 1 implies that f must be affine, and we shall show that this leads to a contradiction. Since it is defined on an open convex subset of H, there exists an $h^* \in H^*$ and a real number b such that $f(z) = \langle h^*, z\rangle + b$. Finally,

$$\{z + \langle h^*, z\rangle x + bx : z \in B(w, \delta) \cap H\} \subseteq Q$$

and this means that Q contains a relatively open subset of a hyperplane, which is a contradiction. By induction, given $x_1, x_2, \ldots, x_n \in X$ we have that

$$Q + [-x_1, x_1] + \cdots + [-x_n, x_n]$$

has no interior point. The case of an arbitrary compact set K can be verified by an application of (1). We give a few details. Suppose the interior of $Q + K$ is nonempty. By translating $Q + K$ if necessary we can suppose that $B(0, r) \subset Q + K$

for some $r > 0$. Choose a sequence $\{F_n\}$ of finite subsets of a ball in X so that (1) holds. Choose $n_0 \in N$ so that

$$\sum_{i=n_0}^{\infty} 2^{-i} F_i \subset B(0, r/4). \tag{3}$$

Because the interior of the closed and convex set $Q_0 := Q + \sum_{i=1}^{n_0} 2^{-i} \mathrm{co} F_i$ is empty, there exists $v \in B_X(0, r)$ so that

$$\mathrm{dist}(v,\ Q_0) > r/2. \tag{4}$$

To see this choose a point $y \in B(0, r/4) \setminus Q_0$ and x^* in the unit sphere of X^* which separates y from Q_0, namely $r/4 \geq \langle x^*, y\rangle \geq \langle x^*, u\rangle$ for any $u \in Q_0$. Choose $x \in B(0, r)$ so that $\langle x^*, x\rangle > 3r/4$. Then $v = x$ satisfies the required inequality. From (3) and (4) follows that

$$\mathrm{dist}(v, Q + \sum_{i=1}^{\infty} 2^{-i} F_i) \geq r/4,$$

which is a contradiction. □

With the hypothesis above, observe that if $T : X \to Y$ is a surjective linear operator with finite-dimensional kernel F, then $T(Q)$ is a bounded, closed and convex set with empty interior that is also spanning; this is because $T^{-1}(T(X)) = Q + F$ is a first category set.

In connection with the next theorem observe that the positive cone of ℓ_2 is a closed convex set with empty interior which contains a translate of any finite subset F of ℓ_2. (Indeed, if for $x = \{x_n\} \in \ell_2$ we define $x^- = \{x_n^-\}$ so that $x_n^- = -x_n$ if $x_n < 0$ and $x_n^- = 0$ otherwise, then the set $F + \sum_{x \in F} x^-$ is contained in the positive cone.) However, as we will see later, because ℓ_2 is reflexive it does not contain a closed convex set with empty interior containing a translate of every compact set. Hence the boundedness hypothesis in (iv) of the next theorem is needed.

Theorem 3. *Let X be a Banach space. Then the following are equivalent*:

(i) *there exists a convex and closed subset Q of X with empty interior which contains translates of all compact sets in X; i.e. whenever K is a compact subset of X there exists $x_K \in X$ so that $K + x_K \subset Q$*;

(ii) *there exists a convex and closed subset P of X with empty interior such that if K is a compact subset of the unit ball of X, then there exists $x_K \in X$ so that $K + x_K \subset P$*;

(iii) *there exists a convex, closed and bounded subset C of X with empty interior such that if K is a compact subset of the unit ball of X, then there exists $x_K \in X$ so that $K + x_K \subset C$; and*

(iv) *there exist a dense subset E of the unit ball of X and a convex, closed and bounded subset D of X with empty interior so that whenever F is a finite set contained in E, there exists $x_F \in X$ so that $F + x_F \subset D$.*

Proof. Clearly (i) implies (ii). To prove that (ii) implies (iii), it is enough to show that there exists $1 \geq r > 0$ and $c > 0$ so that for any compact set $K \subset \bar{B}(0, r)$ there exist $z_K \in B(0, c)$ so that $K + z_K \subset P$, for then we may define

$$C := \frac{1}{r}\left(P \cap \bar{B}(0, r + c)\right).$$

For a contradiction, suppose that for every $n \in N$ there exists a compact set $K_n \subset \bar{B}(0, 1/n)$ so that

$$\text{if } K_n + x \subset P, \quad \text{then} \quad \|x\| \geq n. \tag{5}$$

Define

$$K := \bigcup_{n=1}^{\infty} K_n \cup \{0\}.$$

The set K is a compact subset of the unit ball, hence there exists $z \in X$ such that $K + z \subset P$. Because $K_n \subset K$ for $n \in N$, we have $\|z\| \geq n$ for all n, which is nonsense.

Let us show now that (iii) implies (i). We can suppose that zero is not contained in C and define

$$Q := \bigcup_{\lambda \geq 0} \lambda C.$$

The set Q is convex and contains translates of all compact sets in X. To show that Q is closed, let $z \in X$, $x_n \in C$ and $\lambda_n \geq 0$ such that $\lim_{n\to\infty} \lambda_n x_n = z$ be given. Because the sequence $\{x_n\}$ is bounded away from zero, the sequence $\{\lambda_n\}$ is bounded, and consequently it has a converging subsequence $\lambda_{n_k} \to \lambda \geq 0$. If $\lambda = 0$, then from the boundedness of C it follows that $z = 0 \in Q$. Otherwise the sequence $\{x_{n_k}\}$ converges to z/λ. Because C is closed we get that $z = \lim_{k\to\infty} \lambda_{n_k} x_{n_k} = z \in \lambda C$. Finally, let us show that the set Q has empty interior. Choose some $z \in C$. The set $\tilde{C} := C + [-z, 0]$ is closed and convex, and because C is spanning $\tilde{C}$ has empty interior by Lemma 2. Since

$$Q = \bigcup_{\lambda \geq 0} \lambda C \subset \bigcup_{n \in N} n\tilde{C},$$

it follows from the Baire theorem that the interior of Q is empty.

Clearly (iii) implies (iv), so let us show that the opposite implication also holds. Let K be a compact subset of $B_X(0, 2^{-1})$. We will show that K can be translated into D. Then $C := 2D$ will satisfy (iii). Let $\{F_n\}$ be a sequence of finite sets in E so that (1) holds. Choose $z_n \in X$ so that $z_n + F_n \subset D$. Because D is bounded, the sequence $\{z_n\}$ is bounded. If we define $z := \sum_{n=1}^{\infty} (1/2^n) z_n$, we get

$$z + K \subset z + \mathrm{cl} \sum_{n=1}^{\infty} 2^{-n} F_n \subset \mathrm{cl} \left(\sum_{n=1}^{\infty} 2^{-n} z_n + 2^{-n} F_n \right) \subset D,$$

where the last inclusion follows from the fact that D is convex and closed. □

It should be remarked here that from the proof of equivalence of (i) and (iii) of the previous theorem it follows that if a Banach space X contains a closed and convex set with empty interior containing the translates of all compacts, then X contains a closed and convex cone Q with empty interior which contains the translates of all compacts.

Corollary 4. *Let Z be a Banach space and Y be a separable subspace of Z. Let Z contain a convex closed set Q with empty interior which contains translates of all compact sets in Z. Then there exist a closed, separable and linear subspace X of Z containing Y and a convex closed subset P of X with empty interior which contains translates of all compact sets in X.*

Proof. By Theorem 3 there exists a convex closed bounded subset C of Z with empty interior which contains translates of all compact subsets of B_Z. Using induction we construct an increasing sequence $\{X_n\}$ of closed separable subspaces of Z. Define $X_1 := Y$ and choose a dense countable subset S_1 of the unit ball of X_1. Choose a countable set $T_1 \subset Z$ such that whenever F is a finite subset of S_1 there exists $v \in T_1$ for which $v + F \subset C$. Choose a countable set $C_1 \subset Z \setminus C$ such that $\overline{C_1} \supset C \cap X_1$. Suppose X_n, S_n, T_n and C_n for some $n \in N$ have been already constructed. Define

$$X_{n+1} := \overline{\operatorname{span}}(X_n \cup T_n \cup C_n),$$

and choose a countable dense subset $S_{n+1} \supset S_n$ of the unit ball of X_{n+1}. Choose a countable set $T_{n+1} \subset Z$ such that whenever F is a finite subset of S_{n+1} there exists $v \in T_{n+1}$ for which $v + F \subset C$. Choose a countable set $C_{n+1} \subset Z \setminus C$ such that $\overline{C_{n+1}} \supset C \cap X_{n+1}$. Define

$$X := \overline{\bigcup_{n=1}^{\infty} X_n} \quad \text{and} \quad D := \overline{\bigcup_{n=1}^{\infty} (X_n \cap C)}.$$

The set $E := \bigcup_{n=1}^{\infty} S_n$ is dense in $\bar{B}_X$ and from the construction it follows that any finite set contained in E can be translated into D. The set D is closed and convex, and it has empty interior because $\bigcup_{n=1}^{\infty} C_n \subset X \setminus D$ and $\overline{\bigcup_{n=1}^{\infty} C_n} \supset D$. An application of Theorem 3 completes the proof. □

The following lemma is essentially due to James [4].

Lemma 5. *Let X be a nonreflexive Banach space. Then there exists a sequence $\{x_n\}$ in the unit ball of X and $\varepsilon > 0$ so that for any finite-dimensional subspace Y of X there exists $n \in N$ so that*

$$\operatorname{dist}(Y,\ \operatorname{co}\{x_i\}_{i=n}^{\infty}) > \varepsilon.$$

Proof. The unit ball $\bar{B}_X$ of X is not weakly compact, therefore by the Gantmacher-Smulyan theorem [3] there exists a decreasing sequence $\{C_n\}$ of nonempty, closed and convex subsets of $\bar{B}_X$ such that $\bigcap_{n=1}^{\infty} C_n = \emptyset$. We will show that there exist $\varepsilon > 0$ and a decreasing sequence of convex nonempty sets $\{D_n\}$ so that $D_n \subset C_n$ for $n \in N$ and for any compact set $K \subset X$ there exists $m \in N$ such that

$$(K + B(0, \varepsilon)) \cap D_m = \emptyset.$$

Suppose for a contradiction that the required sequence $\{D_n\}$ does not exist. Let $C_{1,n} := C_n$ for $n \in N$. There exists a compact convex set K_1 so that

$$C_{1,n} \cap (K_1 + B(0, 2^{-1})) \neq \emptyset \quad \text{for} \quad n \in N.$$

Let $C_{2,n} := C_{1,n} \cap (K_1 + B(0, 2^{-1}))$ for $n \in N$. In general, if the sequence $\{C_{k,n}\}$ and the compact convex set K_k have been already constructed, define

$$C_{k+1,n} := C_{k,n} \cap (K_k + B(0, 2^{-k})) \quad \text{for} \quad n \in N,$$

and choose a compact convex set K_{k+1} so that

$$C_{k+1,n} \cap (K_{k+1} + B(0, 2^{-(k+1)})) \neq \emptyset \quad \text{for} \quad n \in N.$$

Then $C_{k+1,n} \subset C_{k,n}$, and by induction $C_{k,n+1} \subset C_{k,n}$. In particular if we define $G_n := C_{n,n}$, then the sequence $\{G_n\}$ is decreasing, $G_n \subset C_n$ and

$$G_{n+1} \subset K_n + B(0, 2^{-n}).$$

Choose some $y_n \in G_n$. The sequence $\{y_n\}$ has a finite δ-net for any $\delta > 0$. Therefore it has a converging subsequence. The limit point of this subsequence is contained in $\bigcap_{n=1}^{\infty} C_n$, which is a contradiction. Now that we have shown the existence of the sequence $\{D_n\}$, to finish the proof simply choose any $x_n \in D_n$. □

Theorem 6. *Let Z be a Banach space. The following are equivalent*:

(i) *Z is not reflexive*;

(ii) *there exist a nontrivial closed subspace X of Z and a convex closed subset Q of X with empty interior which contains translates of all compact sets in X, i.e. whenever K is a compact subset of X there exists $x_K \in X$ so that $K + x_K \subset Q$.*

Moreover, if Z is separable, then (ii) *holds with $X = Z$.*

Proof. To show that (i) implies (ii) choose any separable nonreflexive subspace X of Z; such a space exists by the Eberlein-Smulyan theorem. If Z is separable let $X := Z$. Choose an increasing sequence $\{X_n\}$ of finite-dimensional subspaces of X so that $X = \overline{\bigcup_{n=1}^{\infty} X_n}$. Choose a sequence $\{x_n\}$ in the unit ball of X and $\varepsilon > 0$ as in Lemma 5. By passing to a subsequence of $\{x_n\}$ if necessary we may suppose that

$$\operatorname{dist}\left(\operatorname{span}(X_n \cup \{x_i\}_{i=1}^{n}),\ \operatorname{co}\{x_i\}_{i=n+1}^{\infty}\right) > \varepsilon \quad \text{for} \quad n \in N. \tag{6}$$

Put $K_n := X_n \cap B_Z$, and define

$$D := \overline{\operatorname{co}} \bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i).$$

The convex, closed and bounded set $\tilde{D} := (4/\varepsilon)D$ contains a translate of any finite subset of $\bar{B}_X \cap \bigcup_{n=1}^{\infty} X_n$. By Theorem 3, it only remains to show that the interior of $\tilde{D}$ is empty. For a contradiction, suppose that the interior of D is nonempty. Because $\operatorname{co} \bigcup_{i=1}^{\infty} (x_i + (\varepsilon/4)K_i)$ is dense in D there exist $n \in N$, $\alpha_i \geq 0$ and $u_i \in (\varepsilon/4)K_i$, $i = 1, \dots n$, so that $\sum_{i=1}^{n} \alpha_i = 1$ and the point $z := \sum_{i=1}^{n} \alpha_i (x_i + u_i)$ is contained in the interior of D. From (6) it follows that there exists a point x^* in the unit sphere of X^* so that

$$\begin{array}{rcll} \langle x^*, x \rangle & = & 0 & \text{for} \quad x \in X_n, \\ \langle x^*, x \rangle & \leq & -\varepsilon/2 & \text{for} \quad x \in \operatorname{co}\{x_i\}_{i=n+1}^{\infty}. \end{array}$$

Choose a point w in the unit sphere of X for which

$$\langle x^*, w \rangle \geq 1/2.$$

Since z is an interior point of D, there exists an $r > 0$ so that $z + rw \in D$. Consequently, there exist $m \in N$, $m > n$, $\beta_i \geq 0$ and $v_i \in (\varepsilon/4)K_i$, $i = 1, \dots m$, so that $\sum_{i=1}^{m} \beta_i = 1$ and if we define $y := \sum_{i=1}^{m} \beta_i (x_i + v_i)$, then

$$\|z + rw - y\| < r/2.$$

From the definition of x^* it follows that

$$\begin{aligned} &\langle rw + z - y, x^* \rangle \\ &= r\langle w, x^* \rangle + \langle \textstyle\sum_{i=1}^{n} \alpha_i (x_i + u_i) - \beta_i (x_i + v_i), x^* \rangle - \langle \textstyle\sum_{i=n+1}^{m} \beta_i (x_i + v_i), x^* \rangle \\ &\geq r/2 + 0 - \textstyle\sum_{i=n+1}^{m} \beta_i (\langle x_i, x^* \rangle + \langle v_i, x^* \rangle) \\ &\geq r/2 - \textstyle\sum_{i=n+1}^{m} \beta_i (-\varepsilon/2 + \varepsilon/4) \\ &\geq r/2, \end{aligned}$$

which is a contradiction.

Now, let us prove that (ii) implies (i). By Corollary 4, we may suppose that X is separable. We will show that X is nonreflexive and therefore Z is also nonreflexive. For a contradiction suppose that X is reflexive. Choose a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ that is dense in the unit sphere of X. Denote

$$K_n := \operatorname{span}\{x_i\}_{i=1}^{n} \cap \bar{B}_X.$$

Clearly $\{K_n\}$ is an increasing sequence of compact subsets of the unit ball of X for which

$$\overline{\bigcup_{n=1}^{\infty} K_n} = \bar{B}_X. \tag{7}$$

By Theorem 3 there exists a closed, convex and bounded subset C of X with empty interior which contains translates of all compact subsets of the unit ball of X. For $n \in N$ choose $z_n \in X$ so that $z_n + K_n \subset C$. The sequence $\{z_n\}$ is bounded, therefore it has a weakly converging subsequence $\{z_{n_k}\}$. Denote $z := w\text{-}\lim_{k\to\infty} z_{n_k}$. Because the set C is convex and closed, it is also weakly closed. Consequently, because the sets K_n are increasing, if there exists a $k \in N$ so that $y \in K_{n_k}$, then $y + z \in C$. Hence,

$$z + \bar{B}_X = z + \overline{\bigcup_{k=1}^{\infty} K_{n_k}} \subset C, \tag{8}$$

which, of course, means that the interior of C is nonempty, which is a contradiction. □

Acknowledgment

The authors thank the participants of the semiannual k&k analysis meeting and namely L. Zajíček for helpful discussions. We would also like to thank the referee for his close reading of the first submitted version of this paper.

References

1. J.M. Borwein and D. Noll, *Second order differentiability of convex functions in Banach spaces*, Trans. Amer. Math. Soc. **342** (1994), 43–81. MR **94e:**46076
2. J.P.R. Christensen, *Topology and Borel structure*, North-Holland, Amsterdam, 1974. MR **50:**1221
3. M.M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1973. MR **49:**9588
4. R.C. James, *Weak compactness and reflexivity*, Israel J. Math. **2** (1964), 101–119. MR **31:**585

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ-18600 Prague, Czech Republic

E-mail address: eva@csmat.karlin.mff.cuni.cz

Institut für Mathematik, Johannes Kepler Universität, Altenbergerstrasse, A-4040 Linz, Austria

E-mail address: stegall@caddo.bayou.uni-linz.ac.at