PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 4, April 1996

A NOTE ON INTERPOLATION IN THE HARDY SPACES OF THE UNIT DISC

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(Communicated by Albert Baernstein II)

ABSTRACT. In this note we formulate and solve a natural interpolation problem for the Hardy spaces in the unit disc in terms of maximal functions and weighted summable sequences.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane. For $0 < p < \infty$, $H^p(\mathbb{D})$ denotes the Hardy space of holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_p^p = \sup_r \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(re^{i\theta})|^p \, d\theta < +\infty.$$

In this paper we are interested in the interpolating problem

$$f(z_n) = w_n, \qquad n = 1, 2, \dots, \tag{1}$$

where $Z = \{z_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{D}$ satisfying the Blaschke condition

$$\sum_n (1 - |z_n|) < +\infty.$$

In [2] and [3], this problem has already been studied, proving that the restriction operator

$$R\colon f \mapsto \{f(z_n)\}_{n=1}^{\infty}$$

maps H^p onto $\{w_n\colon \sum_{n=1}^{\infty} |w_n|^p(1 - |z_n|) < +\infty\}$ if and only if Z is uniformly separated, i.e.

$$\inf_n \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z_k} z_n} \right| \geq \delta > 0.$$

The starting point of this paper is the observation that the growth condition on the $\{w_n\}$,

$$\sum_n (1 - |z_n|)|w_n|^p < +\infty, \tag{2}$$

is not necessary for a general Blaschke sequence, and in this sense the Shapiro-Shields result is somewhat unnatural. Here (Section 2) we first obtain elementary

Received by the editors February 25, 1994 and, in revised form, October 13, 1994.
1991 *Mathematics Subject Classification.* Primary 30D55, 30D50; Secondary 46J15.
The first two authors were partially supported by DGICYT grant PB92-0804-C02-02, Spain.

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necessary conditions on the $\{w_n\}, \{z_n\}$ for the interpolation problem (1) to have a solution $f \in H^p$. These conditions are expressed in terms of kth-order hyperbolic divided differences $\Delta^k W$ of the sequence $W = \{w_n\}_{n=1}^{\infty}$ and a corresponding maximal function W_k^*. For $k = 0$ it is simply the statement that the maximal function

$$W_0^*(e^{i\theta}) = \sup\{|w_n| \colon z_n \in C_t(\theta)\},$$

where $C_t(\theta)$ is the Stolz angle at $e^{i\theta}$ of opening t, must be in $L^p(\mathbb{T})$. This of course follows from the maximal characterization of $H^p(\mathbb{D})$. We also obtain necessary conditions of type (2) for a general Blaschke sequence Z.

In Section 3 we pose and solve the corresponding interpolation problem, one for each k. That is, if

$$S_k^p(Z) = \{W = \{w_n\}_{n=1}^{\infty} \colon W_k^* \in L^p(\mathbb{T})\},$$

we prove

Theorem. *The restriction map R is onto from H^p to $S_k^p(Z)$ if and only if Z is the union of $k+1$ uniformly separated sequences.*

As R always maps $H^p(\mathbb{D})$ into $S_k^p(Z)$, for $k = 0$ this result might be called a "Shapiro-Shields theorem revisited".

Finally, we mention that our study has close connections with [4], where a similar result is obtained for H^∞ (the first named author thanks Professor Nikolskii for pointing this out to him).

2. Necessary conditions

2.1. We will denote by $M_\alpha f$ the maximal function

$$M_\alpha f(\theta) = \sup\{|f(z)|, z \in C_\alpha(\theta)\}$$

corresponding to the angle α. For $z, w \in \Delta$, we set

$$\rho(z,w) = \frac{w - z}{1 - \bar{z}w}$$

so that $|\rho(z,w)|$ is the pseudohyperbolic distance between z and w.

The following well-known lemma is an obvious consequence of the Cauchy formula:

Lemma 1. *Given $0 < \alpha < \beta < \pi$ there exists a constant $C = C(\alpha, \beta)$ such that for all holomorphic f and all k,*

$$\sup_{z \in C_\alpha(\theta)} (1 - |z|)^k |f^{(k)}(z)| \le Ck! M_\beta f(\theta).$$

For a holomorphic function f, we define

$$\begin{aligned} \Delta^0 f(z) &= f(z), \\ \Delta^1 f(z,w) &= \frac{f(w) - f(z)}{\rho(z,w)}, \qquad z, w \in \mathbb{D}, \end{aligned}$$

and, inductively, for $z_i \in \mathbb{D}$

$$\begin{aligned} &(\Delta^k f)(z_1, \ldots, z_{k-1}, z_k, z_{k+1}) \\ &\quad = \frac{(\Delta^{k-1} f)(z_1, \ldots, z_{k-1}, z_{k+1}) - (\Delta^{k-1} f)(z_1, \ldots, z_{k-1}, z_k)}{\rho(z_k, z_{k+1})}. \end{aligned}$$

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Lemma 2. *Given $0<\alpha<\beta<\pi$, there exists a constant $C=C(\alpha,\beta)$ such that for any holomorphic function f and $k\geq 1$, one has*

$$\sup_{z_1,\dots,z_{k+1}\in C_\alpha(\theta)} |(\Delta^k f)(z_1,\dots,z_{k+1})| \leq C \sup_{t_1,\dots,t_k\in C_\beta(\theta)} |(\Delta^{k-1} f)(t_1,\dots,t_k)|.$$

Proof. First, let us consider the case $k=1$. If $|\rho(z,w)|\geq\frac{1}{2}$, $|(\Delta^1 f)(z,w)|\leq 2(|f(z)|+|f(w)|)$, and if $|\rho(z,w)|<\frac{1}{2}$, $z,w\in C_\alpha(\theta)$, there exists an absolute constant A such that $|(\Delta^1 f)(z,w)|\leq A\sup\{(1-|z|)|f'(z)|\colon z\in C_\alpha(\theta)\}$. Hence

$$\sup_{z_1,z_2\in Z\cap C_\alpha(\theta)} |(\Delta^1 f)(z_1,z_2)| \leq 2M_\alpha f(\theta) + A\sup_{z\in C_\alpha(\theta)}(1-|z|)|f'(z)|$$

and Lemma 1 finishes the proof.

For $k>1$, fixed $z_1,\dots,z_k$, consider $F_k(z)=(\Delta^{k-1}f)(z_1,\dots,z_{k-1},z)$ as a holomorphic function of z. Writing

$$(\Delta^k f)(z_1,\dots,z_{k+1}) = (\Delta^1 F_k)(z_k,z_{k+1})$$

and applying the result for $k=1$, one finishes the proof. □

The maximal characterization of $H^p(\mathbb{D})$ gives the following result.

Theorem 1. *Let $f\in H^p$ and let $Z=\{z_n\}_{n=1}^\infty$ be a sequence of different points in $\mathbb{D}$. Then, for $k\geq 0$*

$$\sup_{\{z_{n_j}\}\subset Z\cap C_\alpha(\theta)} |(\Delta^k f)(z_{n_1},\dots,z_{n_{k+1}})| \in L^p(\mathbb{T}).$$

This result immediately gives a set of necessary conditions for the problem (1). Denoting, as before, $W=\{w_n\}_{n=1}^\infty$, we introduce

$$(\Delta^0 W)(w_n) = w_n, \qquad (\Delta^1 W)(w_n,w_k) = \frac{w_k-w_n}{\rho(z_n,z_k)},$$

$$(\Delta^k W)(w_{n_1},\dots,w_{n_{k-1}},w_{n_k},w_{n_{k+1}}) = \frac{(\Delta^{k-1}W)(w_{n_1},\dots,w_{n_{k-1}},w_{n_{k+1}}) - (\Delta^{k-1}W)(w_{n_1},\dots,w_{n_k})}{\rho(z_{n_k},z_{n_{k+1}})},$$

the maximal function

$$W_k^*(e^{i\theta}) = \sup_{z_{n_1},\dots,z_{n_{k+1}}\in Z\cap C_\alpha(\theta)} |(\Delta^k W)(z_{n_1},\dots,z_{n_{k+1}})|$$

and the sequence spaces

$$S_k^p(Z) = \{W\colon W_k^*\in L^p(\mathbb{T})\}$$

with norm

$$\|W\|_{p,0}^p = \|W_0^*\|_{L^p(\mathbb{T})}^p,$$
$$\|W\|_{p,k}^p = \|W_k^*\|_{L^p(\mathbb{T})}^p + \|W_{k-1}^*\|_{L^p(\mathbb{T})}^p.$$

Then, $W\in S_k^p(Z)$ is a necessary condition for (1), for all k.

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2.2. Now we look for necessary conditions on $W = \{w_n\}_{n=1}^{\infty}$ for the problem (1) of the type of (2). The following lemma was proved in [1].

Lemma 3. *If $h \in H^{\infty}(\mathbb{D})$ and $\varepsilon > 0$, the measure*

$$\frac{|h'(z)|^2}{|h(z)|^{2-\varepsilon}}(1-|z|)\,dV(z)$$

is a Carleson measure with constant $C\|h\|_\infty/\varepsilon^2$, that is, for all $f \in H^p(\mathbb{D})$

$$\int_{\mathbb{D}} |f(z)|^p \frac{|h'(z)|^2}{|h(z)|^{2-\varepsilon}}(1-|z|)\,dV(z) \le \frac{C}{\varepsilon^2}\|f\|_p \|h\|_\infty.$$

Let us apply this last inequality to $h = B$, the Blaschke product with zeros in Z. We use the notation

$$B_n(z) = \prod_{k\neq n} \frac{\overline{-z_k}}{|z_k|}\frac{z-z_k}{1-\bar{z}_k z}, \qquad \mu_n = \inf_{k\neq n}|\rho(z_n, z_k)|,$$

i.e. z_n is at hyperbolic distance μ_n from the other points in Z. We denote by D_n the hyperbolic disc centered at z_n of radius $\mu_n/2$. As these are disjoint,

$$\begin{aligned}\frac{C}{\varepsilon^2}\|f\|_p &\ge \int_{\mathbb{D}} |f(z)|^p \frac{|B'(z)|^2}{|B(z)|^{2-\varepsilon}}(1-|z|)\,dV(z)\\ &\ge \sum_n \int_{D_n} |f(z)|^p \frac{|B'(z)|^2}{|B(z)|^{2-\varepsilon}}(1-|z|)\,dV(z).\end{aligned}$$

In $D_n, 1-|z| \simeq 1-|z_n|$ and

$$|B(z)| = |B_n(z)|\left|\frac{z-z_n}{1-\bar{z}_n z}\right| \simeq \frac{|B_n(z)|\,|z-z_n|}{1-|z_n|}.$$

Hence

$$\frac{C}{\varepsilon^2}\|f\|_p \ge \sum_n (1-|z_n|)^{3-\varepsilon}\int_{D_n}|f(z)|^p \frac{|B'(z)|^2}{|B_n(z)|^{2-\varepsilon}|z-z_n|^{2-\varepsilon}}\,dV(z).$$

We may think that D_n is a euclidean disk centered at z_n of radius $\mu_n(1-|z_n|)$. Using polar coordinates in D_n, this last integral equals

$$\int_0^{\mu_n(1-|z_n|)} r^{\varepsilon-1}\left\{\int_0^{2\pi}|f(z_n+re^{i\theta})|^p \frac{|B'(z_n+re^{i\theta})|^2}{|B_n(z_n+re^{i\theta})|^{2-\varepsilon}}\,d\theta\right\}dr.$$

In D_n, B_n does not vanish, hence by subharmonicity the integral in θ dominates

$$|f(z_n)|^p \frac{|B'(z_n)|^2}{|B_n(z_n)|^{2-\varepsilon}} = |f(z_n)|^p \frac{|B_n(z_n)|^{\varepsilon}}{(1-|z_n|^2)^2}.$$

Thus we obtain

$$\frac{C}{\varepsilon}\|f\|_p \ge \sum_n (1-|z_n|)(|B_n(z_n)|\mu_n)^{\varepsilon}|f(z_n)|^p. \tag{3}$$

We have therefore proved

Theorem 2. *For a Blaschke sequence $\{z_n\}_{n=1}^{\infty}$, the measure*

$$\sum_n (1-|z_n|)(|B_n(z_n)|\mu_n)^{\varepsilon}\delta_{z_n}$$

is a Carleson measure with constant $C/\varepsilon, \varepsilon > 0$.

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If $\{z_n\}_{n=1}^{\infty}$ is a uniformly separated sequence, this result recaptures the well-known fact that

$$\sum_n (1-|z_n|)\delta_{z_n}$$

is a Carleson measure.

Of course, Theorem 2 gives as a necessary condition on $W=\{w_n\}$ for (1), namely

$$\sum_n (1-|z_n|)(|B_n(z_n)|\mu_n)^{\varepsilon}|w_n|^p < +\infty, \qquad \varepsilon > 0, \tag{4}$$

a Shapiro-Shields type condition. We point out that (4) is already captured by the statement $W \in S_0^p(Z)$. This follows from the fact that Carleson measures boundedly operate on (nonnecessarily holomorphic) functions having maximal function in $L^p(\mathbb{T})$ (in this case the function equals w_n on z_n and 0 elsewhere).

Theorem 2 can be improved, in the sense that $\varphi(t)=t^{\varepsilon}$ can be replaced by a function φ satisfying a Dini-type condition. For instance, multiplying both terms of (3) by ε^{β} and integrating in ε, one obtains

$$\sum_n (1-|z_n|)(|\log(|B_n(z_n)|\mu_n)|)^{-1-\beta}|f(z_n)|^p \le \frac{C}{\beta}, \qquad \beta > 0,$$

which can be integrated again, and so on. This leads to improvements of (4), all of them included in the statement $W \in S_0^p(Z)$. In fact, it is an interesting question to obtain conditions like (4) from $W \in S_0^p(Z)$ using only the geometry of the sequence Z.

3. Sufficient conditions

Let $Z=\{z_n\}$ be a Blaschke sequence. In section 2.1 it has been shown that the restriction operator

$$R\colon f \to \{f(z_n)\}_{n=1}^{\infty}$$

maps H^p into $S_k^p(Z), k=0,1,2,\ldots$.

Theorem 3. *Let $Z=\{z_n\}$ be a Blaschke sequence and $k \ge 0$. The restriction operator R maps H^p onto $S_k^p(Z)$ if and only if Z is the union of $k+1$ uniformly separated sequences.*

Proof. Assume R is onto. Consider $W=\{w_n\}$, $w_n=\delta_{n,m}$, i.e. $w_n=0$ if $n\neq m$ and $w_m=1$. An easy inductive argument shows

$$W_k^*(e^{i\theta}) \le \frac{2^k}{|\rho(z_m,z_{m_1})\cdots\rho(z_m,z_{m_k})|}, \qquad z_m \in C_\alpha(\theta),$$

and hence

$$\|W\|_{p,k} \le \frac{2^k(1-|z_m|)^{1/p}}{|\rho(z_m,z_{m_1})\cdots\rho(z_m,z_{m_k})|},$$

where $\{z_{m_j}: j=1,\ldots,k\}$ are the k points in $\{z_n\}$ closest in the pseudohyperbolic distance to z_m. Now, since R is onto, by the open mapping theorem there exists $f_m \in H^p$, $f_m(z_n)=w_n$, $\|f_m\|_p \le C\|W\|_{p,k}$ where C is a constant independent of m.

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Hence, $f_m = B_m \cdot g_m$ and

$$|B_m(z_m)|^{-1} = |g_m(z_m)| \le C_1 \frac{\|g_m\|_p}{(1-|z_m|)^{1/p}} \le \frac{C_1 C 2^k}{|\rho(z_m, z_{m_1}) \cdots \rho(z_m, z_{m_k})|}.$$

So,

$$|B_m(z_m)| \ge A |\rho(z_m, z_{m_1}) \cdots \rho(z_m, z_{m_k})|. \tag{5}$$

We will show that (5) implies that Z is the union of $k+1$ uniformly separated sequences. By Zorn's lemma, there exists a maximal subset Z_1 of Z such that if $z_r, z_s \in Z_1$ one has $|\rho(z_r, z_s)| > 2^{-1}A$. Do the same for Z replaced by $Z \backslash Z_1$ and repeat the process to obtain $Z_1, \dots, Z_{k+1}$. By (5) these sequences are uniformly separated. Now let us show

$$Z = \bigcup_{j=1}^{k+1} Z_j.$$

If this were not true, there exists $z_m \in Z \backslash \bigcup_{j=1}^{k+1} Z_j$. By the maximality of each Z_j, there exists $z_{m,j} \in Z_j$ such that $|\rho(z_m, z_{m,j})| < 2^{-1}A$. Hence, there exist $k+1$ points in Z at pseudohyperbolic distance from z_m less than $2^{-1}A$. This contradicts (5).

To prove the converse, consider first the case $k = 0$, that is, $Z = \{z_n\}$ a uniformly separated sequence and $W = \{w_n\} \in S_0^p(Z)$, i.e. $W_0^*(e^{i\theta}) = \sup\{|w_n| \colon z_n \in C_\alpha(\theta)\} \in L^p(\mathbb{T})$. Since Carleson measures boundedly operate on functions having maximal function in $L^p(\mathbb{T})$, (2) is satisfied and the Shapiro-Shields result gives $f \in H^p(\mathbb{D})$, $f(z_n) = w_n,\ n = 1, 2, \dots$. However, using that $W \in S_0^p(Z)$ we can give a more elementary proof.

By normal families, the result will be proved if we show that there exists $C > 0$ such that for any N, there is $f_N \in H^p(\mathbb{D})$, satisfying $f_N(z_i) = w_i,\ i = 1, \dots, N$, and $\|f_N\|_p \le C$.

Take $\delta > 0$ such that $\mathbb{D}_n = \{z \colon |\rho(z, z_n)| \le 2\delta\}$ are pairwise disjoint. Let $H = H_N$ be a C^∞ in $\mathbb{D}$, $H(z) = w_n$ if $|\rho(z, z_n)| \le \delta$, $H = 0$ or $\mathbb{D} \backslash \bigcup_{n=1}^N D_n$ and $|H(z)| \le |w_n|$ for $z \in D_n$. It is clear that $\|M_\beta(H)\|_p \le \|W\|_{p,0}$ for some $\beta < \alpha$. Let B be the Blaschke product with zero set Z. We look for solutions of (1) of the form $H - BG$, where

$$\bar\partial(G) = B^{-1}\bar\partial(H), \qquad \|G\|_{L^p(\mathbb{T})} \le C \tag{6}$$

and C is a constant independent on N.

Since $Z = \{z_n\}$ is uniformly separated, one has $|B(z)| \ge C \inf_n |\rho(z, z_n)|$. Hence,

$$\begin{aligned} |B(z)^{-1}\bar\partial H(z)|\, dm(z) &\le C(\delta) \sum_n |w_n| (1-|z_n|)^{-1}\, dm_{\mathbb{D}_n} \\ &\le C(\delta)|H(z)| \sum_n (1-|z_n|)^{-1}\, dm_{\mathbb{D}_n}. \end{aligned}$$

Observe that $\mu = \sum_n (1-|z_n|)^{-1}\, dm_{\mathbb{D}_n}$ is a Carleson measure. Now, the function

$$G(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{1-|\xi|^2}{(\xi - z)(1-\bar\xi z)} B(\xi)^{-1} \bar\partial H(\xi)\, dm(\xi)$$

satisfies $\bar\partial G = B^{-1}\bar\partial H$. We estimate $\|G\|_p$ by duality.

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Let $A \in L^q(\mathbb{T}), p^{-1} + q^{-1} = 1$ and denote by $P[A](\xi)$ the Poisson integral of A at the point ξ. One has

$$\left|\int_0^{2\pi} G(e^{i\theta})A(e^{i\theta})\,d\theta\right| \leq \int_{\mathbb{D}} |P[|A|](\xi)|\,|B(\xi)|^{-1}|\bar\partial H(\xi)|\,dm(\xi)$$
$$\leq C(\delta)\int_{\mathbb{D}} |P[|A|](\xi)|\,|H(\xi)|\,d\mu(\xi) \leq C(\delta)C_1\|A\|_{L^q(\mathbb{T})},$$

where C_1 is independent on N, because $P[|A|](\xi)\cdot H(\xi)$ has maximal function in $L^1(\mathbb{T})$, so the function G satisfies (6) and this finishes the proof for $k=0$. ☐

Assume the proof is completed for k and let us show it for $k+1$, that is, assume Z is the union of $k+1$ uniformly separated sequences. One can split the sequence $Z = Z_1 \cup Z_2$, where $Z_1 = \{\alpha_n\}$ is the union of k uniformly separated sequences and $Z_2 = \{z_n\}$ is uniformly separated.

Let $W \in S^p_{k+1}(Z)$. The previous splitting for Z gives $W = W_1 \cup W_2$, $W_1 = \{s_n\}$, $W_2 = \{w_n\}$. Applying the result for $k=0$, one gets $f_2 \in H^p(\mathbb{D})$, $f_2(z_n) = w_n,\ n = 1,2,\ldots$. Let B_2 be the Blaschke product with zero sequence Z_2. Now we look for a function $f \in H^p(\mathbb{D})$ such that

$$f(\alpha_n) = \frac{s_n - f_2(\alpha_n)}{B_2(\alpha_n)}, \qquad n = 1,2,\ldots, \tag{7}$$

because $f_2 + B_2 f$ will interpolate W at the points Z. By induction, (7) is solvable if and only if

$$\{(s_n - f_2(\alpha_n))B_2(\alpha_n)^{-1}\} \in S^p_k(Z_1).$$

Let $z_{k(n)}$ be the closest point, in the pseudohyperbolic metric, in Z_2 to α_n. Then,

$$(s_n - f_2(\alpha_n))B_2(\alpha_n)^{-1} = \frac{s_n - w_{k(n)}}{\rho(\alpha_n, z_{k(n)})}\frac{\rho(\alpha_n, z_{k(n)})}{B_2(\alpha_n)}$$
$$+ \frac{f_2(z_{k(n)}) - f_2(\alpha_n)}{\rho(\alpha_n, z_{k(n)})}\frac{\rho(\alpha_n, z_{k(n)})}{B_2(\alpha_n)}.$$

Now, since $W \in S^p_{k+1}(Z)$ and $f_2 \in H^p(\mathbb{D})$, one has

$$\left\{\frac{s_n - w_{k(n)}}{\rho(\alpha_n, z_{k(n)})}\right\} \in S^p_k(Z_1), \qquad \left\{\frac{f_2(z_{k(n)}) - f_2(\alpha_n)}{\rho(\alpha_n, z_{k(n)})}\right\} \in S^p_k(Z_1).$$

Hence in order to finish the proof it is sufficient to show the following two auxiliary results.

Lemma 4. *Let Z be a Blaschke sequence, $W = \{w_n\}$ and $A = \{a_n\}$ two sequences of complex numbers and denote by WA the sequence $\{w_n a_n\}$. Then for $k \geq 0$,*

$$(\Delta^k(WA))(w_{n_1}a_{n_1}, \ldots, w_{n_{k+1}}a_{n_{k+1}})$$
$$= \sum_{j=0}^{k} (\Delta^j W)(w_{n_1}, \ldots, w_{n_{j+1}}) \cdot (\Delta^{k-j}A)(a_{n_{j+1}}, \ldots, a_{n_{k+1}}).$$

Lemma 5. *Let $Z = \{z_n\}$ be a uniformly separated sequence, B the Blaschke product with zero set Z and $\delta > 0$ such that the discs $D_n = \{z : |\rho(z, z_n)| \leq \delta\}$ are pairwise disjoint. Consider $\Omega = \bigcup_n D_n$ and $\varphi\colon \Omega \to \mathbb{C}, \varphi(a) = B_{b(a)}(a)^{-1}$ where $b(a) = z_n$ if $a \in D_n$. Let $A = \{a_n\} \in \Omega$ and $\varphi(A) = \{\varphi(a_n)\}$. Then $\varphi(A) \in S^\infty_k(A)$, for any $k \geq 0$.*

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Lemma 4 follows from a simple inductive argument. The case $k = 0$ of Lemma 5 follows from the fact that Z is a uniformly separated sequence. For $k > 0$, one shows by induction that

$$z \to \Delta^m(a_{n_1}, \ldots, a_{n_m}, z)$$

is a bounded analytic function in Ω.

Finally, concerning the necessary condition (4), since it is captured from the fact $W \in S_0^p(Z)$, Theorem 3 shows

$$R(H^p(\mathbb{D})) = \{W \colon W \text{ satisfies } (4)\}$$

if and only if Z is a uniformly separated sequence.

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