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ON PROOFS OF THE C^0 GENERAL DENSITY THEOREM

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ABSTRACT. We show that if M is a compact manifold, then there is a residual subset $\mathcal{N}$ of the set of homeomorphisms on M with the property that if $f \in \mathcal{N}$, then the periodic points of f are dense in its chain recurrent set. This result was first announced in [4], but a flaw in that argument was noted in [1], where a different proof was given. It was recently noted in [5] that this new argument only serves to show that the density of periodic points in the chain recurrent set is generic in the closure of the set of diffeomorphisms. We show how to patch the original argument from [4] to prove the result.

Two fundamental results in the theory of smooth dynamical systems are the C^1 closing lemma and the associated C^1 general density theorem that were established by C. Pugh in [6]. The former shows that if p is a nonwandering point of a diffeomorphism f of a compact manifold M, then in any C^1 neighborhood of f there is a diffeomorphism g for which x is a periodic point. The latter shows that C^1 generically the nonwandering set of a diffeomorphism coincides with the closure of the set of its periodic points.

This paper addresses similar (but much simpler) questions in the space of homeomorphisms of a smooth manifold M (compact and without boundary) with the C^0 topology. In this setting the closing lemma is a 'folk theorem' whose proof can be found in many places; references are given below. In fact, these references show that any chain recurrent point of f is a periodic point for some g that is arbitrarily C^0 close to f. The argument that establishes this is much simpler than any known proof of the C^1 closing lemma. Somewhat surprisingly, however, the proof of the full C^0 general density theorem, that for a generic homeomorphism the chain recurrent set is the same as the closure of the periodic points, has been elusive. A claim of a general density theorem (with the nonwandering set in place of the chain recurrent set) was made in [4]; a flaw in that argument was later described in [1] where a different argument was given. It was recently pointed out in [5] that the argument in [1] does not quite establish the full general density theorem; instead it shows that C^0 generically in the C^0 closure of the set of diffeomorphisms on M the chain recurrent set is the closure of the set of periodic points. This does not establish the full C^0 general density theorem because in large dimensions it is known that the set of diffeomorphisms is not C^0 dense in the set of homeomorphisms. In the current paper we show how to patch the original argument from [4], and so establish the full C^0 general density theorem. At the end of the paper we also describe briefly how to

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use these ideas to obtain three further results: (a) the generic homeomorphism has no periodic attractors or repellers, (b) the generic homeomorphism has infinitely many periodic points of some finite period, (c) the generic (noninvertible) map on M has its periodic points dense in its chain recurrent set. (a) and (b) are among the results of [4], and (c) is noted in [1]. The arguments in those sources suffer from the same problems as the proofs there of the general density theorem.

In the argument below we use a result from [3] that requires the dimension of M to be at least 2; this restriction causes no dificulty because when $\dim(M) \leq 3$ it is known [2] that the diffeomorphisms on M are dense in $\mathcal{H}(M)$, so in low dimensions the proof of the C^0 general density theorem given in [1] is complete. For the rest of the paper we will assume that $\dim(M) \geq 2$.

The idea of the proof below is to combine the Brouwer fixed point theorem with the fact that in the C^0 topology, any periodic orbit of f is a periodic attractor for some C^0 small perturbation of f. This is essentially the same approach that was taken in [4], where the remainder of the argument was based on the assertion that the set of so-called 'permanent periodic orbits' of a homeomorphism varies lower semicontinuously with the homeomorphism; an attracting periodic point is the prototypical example of a permanent periodic orbit. An example was given in [1] showing that the assertion about lower semicontinuity of the permanent periodic orbits is false. We avoid the problem with the permanent periodic orbits by considering instead the set of points that are contained in 'small absorbing disks'. To make this precise, suppose that the dimension of M is n and let D^n denote the closed unit ball in R^n. Call $D \subset M$ a *disk* if D is homeomorphic to D^n. A closed subset X of M is called k-*absorbing* for a homeomorphism f on M if $f^k(X) \subset \operatorname{int}(X)$, and X is *absorbing* if it is k-absorbing for some $k \geq 1$. For $\epsilon > 0$ define $\mathcal{D}(\epsilon)$ to be the set of all disks in M that are absorbing and have diameter at most ϵ, and set

$$\mathcal{P}_\epsilon(f) = \overline{\bigcup_{D \in \mathcal{D}(\epsilon)} D}.$$

Note that if D is a k-absorbing disk, then by the Brouwer fixed point theorem D contains a point of period k. Because of this the following observation is immediate.

Lemma 1. *For each $\epsilon > 0$, $\mathcal{P}_\epsilon(f) \subset B_{2\epsilon}(\overline{Per(f)})$ where $B_\epsilon(\overline{Per(f)})$ denotes the ϵ neighborhood of the closure of the set of periodic points of f.*

The next result is also easy. Let $\mathcal{H}(M)$ denote the set of homeomorphisms on M, topologized by the metric

$$\rho(f,g) = \max_{x \in M} \left[d(f(x), g(x)) + d(f^{-1}(x), g^{-1}(x)) \right],$$

and let M^* denote the set of closed nonempty subsets of M topologized by the Hausdorff metric.

Lemma 2. *For each $\epsilon > 0$ the map $\mathcal{P}_\epsilon \colon \mathcal{H}(M) \to M^*$ is lower semicontinuous.*

Proof. Suppose $f \in \mathcal{H}(M)$, $x \in \mathcal{P}_\epsilon(f)$, and $\delta > 0$. By definition there is a k-absorbing disk D of diameter at most ϵ that is within δ of x. Compactness and the fact that $f^k(D) \subset \operatorname{int}(D)$ ensure that D is k-absorbing for g provided g is sufficiently close to f, so $D \subset \mathcal{P}_\epsilon(g)$. This means that for each g that is close enough to f there are points of $\mathcal{P}_\epsilon(g)$ that are within δ of x; by compactness of $\mathcal{P}_\epsilon(f)$, this implies the asserted lower semicontinuity of $\mathcal{P}_\epsilon$. $\square$

Let $CR(f)$ denote the set of chain recurrent points of f. A point $p \in M$ is an element of $CR(f)$ if for each $\epsilon > 0$ there is a finite sequence $x_0, x_1, \dots, x_k$ with $k \geq 1$, $x_0 = x_k = p$, and $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \leq i < k$. We say that a point $q \in M$ is a *periodic attracting point* for f if there is a k-absorbing disk D with $\bigcap_{j \geq 0} f^{jk}(D) = \{q\}$. Note that each image $f^{jk}(D)$ is a k-absorbing disk, and the diameters of these disks go to 0 as j goes to infinity, so it is clear that any periodic attracting point of f is contained in $\mathcal{P}_\epsilon(f)$ for all $\epsilon > 0$.

Proposition 3. *If $q \in CR(f)$ and $\eta > 0$, then there is $g \in \mathcal{H}(M)$ with $\rho(f,g) < \eta$ and such that q is a periodic attracting point for g.*

Proof. We proceed in two steps, showing first that there is $g_0 \in \mathcal{H}(M)$ with $\rho(f, g_0) < \eta/2$ which has q as a periodic point, and then we show there is g within $\eta/2$ of g_0 which has q as an attracting periodic point.

To accomplish the first step let $\delta = \eta/4\pi$ and select a δ-chain $q_0, q_1, \dots, q_k$ with $q_0 = q_k = q$. If two of the q_i's with $i < k$ are equal, then we can delete one of them and all the q_j's between them to get a shorter δ-chain from q to q, so there is no loss in generality in assuming that $q_i \neq q_j$ for $0 < i < j \leq k$. For $0 \leq i < k$ let $p_{i+1} = f(q_i)$, so all the p_i's are distinct and $d(q_i, p_i) < \delta$ for $1 \leq i \leq k$. By Lemma 13 of [3] there is a diffeomorphism ϕ that is uniformly $2\pi\delta$ close to the identity such that $\phi(p_i) = q_i$ for each i, so setting $g_0 = \phi \circ f$ completes step one.

Step two is a folk theorem, a proof of which is outlined in Lemma 3.1 of [4]. Here is a slightly more detailed version of that argument. We will define a homeomorphism α that is supported in a small neighborhood of q, fixes q, and is a strong contraction toward q. We will show that $g = \alpha \circ g_0$ has q as an attracting point of period k. The perturbation α will be supported on a coordinate neighborhood U containing q; to make the construction easier to describe we will identify U with R^n. We will assume that U has diameter less than $\eta/2$ and is small enough that $g_0^j(U) \cap U = \emptyset$ for $0 < j < k$. The fact α is the identity on the complement of U in M ensures that $\rho(g, g_0) < \eta/2$. We will use the following preliminary result.

Lemma 4. *Suppose that $w\colon (0,\infty) \to [0,\infty)$ is continuous, nonincreasing, $w(t) \to \infty$ as t decreases to 0, and that $w(t) = 0$ for all $t \geq 2$. Define $\alpha(y) = e^{-w(|y|)}y$ for $y \neq 0$, and $\alpha(0) = 0$. Then α is a homeomorphism on R^n, and $\alpha(y) = y$ whenever $|y| \geq 2$.*

Proof. Continuity at 0 is easy to check, and since α takes each ray from the origin into itself, it is easy to see that is is onto. To verify that it is injective, use the fact that each ray is invariant to see that if $\alpha(y) = \alpha(z)$, then $y = \beta z$ for some positive constant β, which we may assume is bigger than 1. This means that $|y| > |z|$ so that $w(y) \leq w(z)$, which leads to the following contradiction to the presumed equality of $\alpha(y)$ and $\alpha(z)$:

$$|\alpha(y)| = e^{-w(|y|)}|y| = e^{-w(|y|)}\beta|z| > e^{-w(|z|)}|z| = |\alpha(z)|.$$

Finally, since α is the identity on the complement of a compact set, α^{-1} is automatically continuous. □

Now we return to the proof of the proposition. We continue to identify a neighborhood U of q with R^n, the identification taking q to 0. In the following we consider $r > 0$ small enough that $|y| \leq r$ implies $g_0^k(y) \in U$, and we define

- $m_1(r) = \min\{|y| : |g_0^k(y)| \geq r\}$; note that $m_1(|g_0^k(y)|) \leq |y|$. The intuition is that if g_0^k is pushing points away from q, then $m_1(r)$ should be small compared to r.
- $m_2(r) = \gamma(r)m_1(r)\}$where γ is a positive function satisfying $\gamma(r) = 1$ if $r \leq 1$, γ is constant on $[2, \infty)$ and is chosen so that $m_2(2) = 4$.
- $m_3(r) = r/m_2(r)$ for $r \leq 2$ and $m_3(r) = 1/2$ for $r \geq 2$; note that m_3 is continuous.
- $m(r) = \max\{m_3(s) \mid s \geq r\}$, so that $m(r) \geq m_3(r)$ and m is nonincreasing.

Finally define $w(r) = \ln 2m(r)$; it is clear that w is continuous, nonincreasing, and equal to 0 for $r \geq 2$. There are two cases, depending upon whether $w(r) \to \infty$ as $r \to 0$. If it does not, then for small enough $\epsilon > 0$, 0 will be an attracting fixed point for the map $z \to \epsilon g_0^k(z)$ (i.e., in local coordinates our perturbation α is just $\alpha(y) = \epsilon \cdot y$). In the remaining case w satisfies all the hypotheses of the lemma; let α be given by the lemma and define $g = \alpha \circ g_0$ so that for small $|y|$

$$|g^k(y)| = |\alpha(g_0^k(y))| = e^{-w(|g_0^k(y)|)}|g_0^k(y)| = \frac{|g_0^k(y)|}{2m(|g_0^k(y)|)} \leq \frac{|g_0^k(y)|}{2m_3(|g_0^k(y)|)}$$
$$= \frac{|g_0^k(y)| \cdot m_2(|g_0^k(y)|)}{2|g_0^k(y)|} = m_1(|g_0^k(y)|)/2 \leq \frac{|y|}{2},$$

so 0 is an attracting period-k point for g. This completes the proof of the proposition. □

Lemma 5. *If f is a continuity point of $\mathcal{P}_\epsilon$, then $CR(f) \subset \mathcal{P}_\epsilon(f)$.*

Proof. Let $x \in CR(f)$. By the last lemma there is a sequence of homeomorphisms g_i converging to f in $\mathcal{H}(M)$ with $x \in \mathcal{P}_\epsilon(g_i)$ for every i. □

It is easy to verify that ρ is a complete metric on $\mathcal{H}(M)$, so that $\mathcal{H}(M)$ is a Baire space, and it follows from the semicontinuity of $\mathcal{P}_\epsilon$ that the set of continuity points of $\mathcal{P}_\epsilon$ is residual in $\mathcal{H}(M)$ for each $\epsilon > 0$; see [7]. Let $\mathcal{N}$ denote the set of $f \in \mathcal{H}(M)$ that are continuity points of $\mathcal{P}_{1/m}$ for every $m = 1, 2, \ldots$, so that $\mathcal{N}$ is also residual in $\mathcal{H}(M)$.

Theorem 6. *Let $\mathcal{N}$ be the residual subset of $\mathcal{H}(M)$ defined above. If $f \in \mathcal{N}$, then the periodic points of f are dense in $CR(f)$.*

Proof. Combining the last lemma with Lemma 1 we see that if $f \in \mathcal{N}$, then

$$CR(f) \subset \mathcal{P}_{1/m}(f) \subset B_{2/m}(Per(f)) \text{ for each } m \geq 1. \quad \square$$

Remark . In [4] the general density theorem was claimed as part (e) of Theorem 1. Parts (f) and (g) of that theorem are the assertions that, generically, a homeomorphism has no periodic sinks or sources, and has infinitely many periodic points of some finite period. The arguments supporting these assertions rely on the alleged semicontinuity of the map sending f to the set of its permanent periodic orbits. It is not hard to modify the ideas above to obtain a proof of (f) and (g). Given a positive integer k and an open subset $V \subset M$, let $C(f; V, k) \in 0, 1, \ldots, \infty$ denote the largest number of pairwise disjoint disks, each of which is j-absorbing for some $1 \leq j \leq k$. The proof of Lemma 2 shows that each map $C(\,\cdot\,; V, k)$ is lower semicontinuous. Lemma 3.2 of [4] shows that if V contains a period k point of f, then for g arbitrarily close to f $C(g; V, k) > C(f; V, k)$. It follows that if f is a continuity point of $C(\,\cdot\,; V, k)$, then (i) $C(f; V, k)$ is either 0 or ∞, and (ii) if V contains a periodic point of f of period at most k, then $C(f; V, k) = \infty$. Let $\mathcal{N}_0$ be the set of homeomorphisms that are continuity points of every map $C(\,\cdot\,; V, k)$ as k varies over the positive integers and V varies over the elements of some countable basis for the topology of M, and let $\mathcal{N}_1 = \mathcal{N}_0 \cap \mathcal{N}$. It now follows (using the Brouwer theorem) that if $f \in \mathcal{N}_1$, then for each k the set of its periodic points of periods at most k has no isolated points, which establishes (f) and (g) as in [4].

Remark . Essentially the same set of arguments shows that the density of the periodic points in the chain recurrent set is a generic property of continuous maps of M to itself (not necessarily invertible). There are only two places in the argument above where we made use of the invertibility of f. The first was in the last sentence before the statement of Proposition 3, but this was just for convenience. Clearly if q is a period k point of a continuous map h and if there are local coordinates centered at q for which h^k is a contraction (in the sense that $|h^k(y)| < |y|$ for any $y \neq q$ that is close to q), then $q \in \mathcal{P}_\epsilon(h)$ for all $\epsilon > 0$. Proposition 3 produces periodic points with this property so the proofs of Lemma 5 and Theorem 6 carry over.

The other place where we relied on the invertibility of f was in step one of the proof of Proposition 3, where the fact that the points $p_{i+1} \equiv f(q_i)$ are all distinct follows from the fact that the points q_i are distinct and the fact that f is injective. If two of the points p_i are the same, say $p_j = p_{j+k}$, then we can truncate the δ-chain $\{q_i\}$ by deleting q_j through q_{j+k-1}. Doing this as many times as necessary will eventually give us a δ-chain from q to q with all of its points distinct, and with all of their images under f distinct, so that we may apply the result from [3] as before.

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