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AN EXAMPLE OF FINITE DIMENSIONAL KAC ALGEBRAS OF KAC-PALJUTKIN TYPE

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Abstract. An example of finite dimensional Kac algebras of Kac-Paljutkin type is given.

1. Introduction

Motivated by Pontryagin-Tannaka-Krein-Stinespring-Tatsuuma duality for locally compact groups, M.Enock and J.M.Schwartz introduced the notion of a Kac algebra and extended the duality to that of the category of Kac algebras. Any locally compact group is realized as a commutative Kac algebra and a cocommutative one, and conversely, any commutative (resp. cocommutative) Kac algebra comes from a locally compact group. However, to the author's knowledge, except for recent works of Majid [7, 8, 9] based on the matched pair introduced by Takeuchi [16], a number of examples of non-commutative and non-cocommutative Kac algebras do not seem to be known.

Before the general theory of Kac algebras, G.I.Kac and V.G.Paljutkin [5] considered a finite dimensional Kac algebra of a certain type. (They used the terminology "ring group".) They showed that the problem of constructing examples of the type is reduced to that of finding a family of unitary matrices satisfying certain conditions and gave a non-commutative and non-cocommutative 8 dimensional Kac algebra. (See also Masuoka [10].)

In the theory for subfactors initiated by V.Jones [4], A.Ocneanu [11] introduced the concept of a paragroup, which can be regarded as a quantization of groups, to classify subfactors. By using the so-called standard invariant equivalent to the paragroup, S.Popa classified subfactors under more general conditions. (See [13, 14].) It is known that the finite dimensional Kac algebras are characterized as special paragroups. (See David [1], Longo [6], Szymanski [15], and also Yamanouchi [17].)

Motivated by constructing non-commutative and non-cocommutative finite dimensional Kac algebras (as examples of paragroups), we shall give an example of finite dimensional Kac algebras of Kac-Paljutkin type by using the representation theory of extra-special p-groups and compute the dual structure.

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2. Construction of finite dimensional Kac algebras

In order to fix notation, we first recall the definition of a finite dimensional Kac algebra. (We consider only the finite dimensional case.) For the general theory of Kac algebras, we refer to Enock and Schwartz [2, 3].

A finite dimensional Kac algebra $\mathbf{K}$ is a quadruple $(M, \Gamma, \kappa, \varphi)$ such that

(i) (M, Γ, κ) is an involutive finite dimensional Hopf-von Neumann algebra,
(ii) φ is a faithful positive trace on M satisfying

$$(\iota_M \otimes \varphi)((1 \otimes y^*)\Gamma(x)) = \kappa((\iota_M \otimes \varphi)(\Gamma(y^*)(1 \otimes x))),\ x, y \in M.$$

Here ι_M is the identity mapping of M. The linear functional φ is called a Haar weight of $\mathbf{K}$. Given a Kac algebra $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$, the dual Kac algebra $\hat{\mathbf{K}} = (\hat{M}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$ of $\mathbf{K}$ is canonically constructed as follows. M is assumed to be represented on the standard Hilbert space $L^2(\varphi)$. Let λ be the left regular representation of M on $L^2(\varphi)$ with respect to the convolution product in M, W the fundamental operator of $\mathbf{K}$, and Σ the flip on $L^2(\varphi) \otimes L^2(\varphi)$. Then $\hat{\mathbf{K}} = (\hat{M}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$ is defined by

$$\hat{M} = \lambda(M),$$
$$\hat{\Gamma}(\lambda(f)) = \Sigma\, W^* \,(\lambda(f) \otimes 1)\, W\, \Sigma,\ f \in M,$$
$$\hat{\kappa}(\lambda(f)) = \lambda(\kappa(f)),\ f \in M,$$
$$\hat{\varphi}(\lambda(f)) = \varphi(ef),\ f \in M,$$

where e is the unit of M with respect to the convolution product.

In the following, we shall give an example of finite dimensional Kac algebras by using Kac-Paljutkin's method. For details, see Kac and Paljutkin [5].

Let $k \in \mathbf{N}, k \geq 3$ be fixed and η a primitive k-th root of 1. We set

$$G = \mathbf{Z}_k \times \mathbf{Z}_k = \{(i, j) \mid i, j = 0, 1, \cdots, k-1\}$$

and define $k \times k$ unitary matrices $\{p_{(i,j)}\}, \{u_{(i,j)}\}$ and $\{v_{(i,j)}\}$ parametrized by G by

$$p_{(i,j)} = u_{(i,j)} = \sum_{m=1}^{k} \eta^{im}\, e_{m,m+j}, \quad v_{(i,j)} = \overline{u_{(i,j)}} = \sum_{m=1}^{k} \eta^{-im}\, e_{m,m+j}.$$

Here $\{e_{ij}\}_{i,j=1}^{k}$ means the usual matrix units in $M_k(\mathbf{C})$. It is easy to check that the above matrices satisfy the conditions in Kac-Paljutkin [5], hence we have a $2k^2$ dimensional Kac algebra $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$ whose structure is given by the following:

$$M = \bigoplus_{(i,j) \in G} \mathbf{C} e_{(i,j)} \,\oplus\, M_k(\mathbf{C}),$$

where $\{e_{(i,j)}\}_{(i,j)\in G}$ denotes the 1-dimensional central minimal projections,

$$\begin{aligned}\Gamma(e_{(i,j)}) &= \sum_{(m,n)\in G} e_{(m,n)} \otimes e_{(i-m,j-n)} \\ &\quad + \frac{1}{k} \sum_{m,n,s,t=1}^{k} (u_{(i,j)})_{m,s} \overline{(u_{(i,j)})_{n,t}}\, e_{mn} \otimes e_{st},\ (i,j)\in G, \\ \Gamma(X) &= \sum_{(i,j)\in G} e_{(-i,-j)} \otimes u_{(i,j)}\, X\, {u_{(i,j)}}^* \\ &\quad + \sum_{(i,j)\in G} v_{(i,j)}\, X\, {v_{(i,j)}}^* \otimes e_{(i,j)},\ X \in M_k(\mathbf{C}), \\ \kappa(e_{(i,j)}) &= e_{(-i,-j)},\ (i,j)\in G, \\ \kappa(X) &= {}^tX,\ X\in M_k(\mathbf{C}),\end{aligned}$$

where ${}^tX$ means the transpose of X,

$$\varphi\left(\sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k}\right) = \frac{1}{2k^2}\left(\sum_{(i,j)\in G} x_{(i,j)} + k\sum_{i=1}^{k} x_{ii}\right).$$

3. The dual Kac algebra

In this section, we shall compute the dual of the Kac algebra $\mathbf{K} = (M,\Gamma,\kappa,\varphi)$ constructed in the previous section. It is sufficient to calculate the convolution structure in M.

Let $*$ be the convolution product of M and $^{\#}$ the involution of M, that is, they are defined by the following relations :

$$\begin{aligned}\langle a, x*y\rangle &= \langle \Gamma(a), x\otimes y\rangle \\ &= (\varphi\otimes\varphi)(\Gamma(a)(x\otimes y)),\ x,y,a\in M,\end{aligned}$$

$$x^{\#} = \kappa(x^*) = \kappa(x)^*,\ x\in M.$$

Lemma 1. *Let $A=(a_{ij}),\ B=(b_{ij})\in M_k(\mathbf{C})$. Then we have*

(i) $e_{(i,j)} * e_{(i',j')} = \dfrac{1}{2k^2}\, e_{(i+i',j+j')}$,

(ii) $e_{(i,j)} * A = \dfrac{1}{2k^2}\displaystyle\sum_{m,n=1}^{k} \eta^{i(m-n)}\, a_{m+j,n+j}\, e_{mn}$,

(iii) $A * e_{(i,j)} = \dfrac{1}{2k^2}\displaystyle\sum_{m,n=1}^{k} \eta^{i(m-n)}\, a_{m-j,n-j}\, e_{mn}$,

(iv) $A * B = \displaystyle\sum_{(i,j)\in G} c_{(i,j)}\, e_{(i,j)}$,

where

$$c_{(i,j)} = \frac{1}{2k}\sum_{m,n=1}^{k} \eta^{-i(m-n)} a_{m,n}\, b_{m+j,n+j}\,.$$

Hence, for $X = \sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k}$ *and* $Y = \sum_{(i,j)\in G} y_{(i,j)}\, e_{(i,j)} + (y_{ij})_{i,j=1}^{k}$,
we get

$$X * Y = \sum_{(i,j)\in G} z_{(i,j)}\, e_{(i,j)} + (z_{ij})_{i,j=1}^{k},$$

where

$$z_{(i,j)} = \frac{1}{2k^2} \sum_{(m,n)\in G} x_{(i-m,j-n)}\, y_{(m,n)} + \frac{1}{2k} \sum_{m,n=1}^{k} \eta^{-i(m-n)}\, x_{m,n}\, y_{m+j,n+j}\,,$$

$$z_{mn} = \frac{1}{2k^2} \left(\sum_{(i,j)\in G} \eta^{i(m-n)} \left(x_{(i,j)}\, y_{m+j,n+j} + y_{(i,j)}\, x_{m-j,n-j} \right) \right).$$

Proof. Let $X = \sum_{(m,n)\in G} x_{(m,n)}\, e_{(m,n)} + (x_{mn})_{m,n=1}^{k} \in M$.

(i) We compute

$$\begin{aligned} \langle X, e_{(i,j)} * e_{(i',j')} \rangle &= \langle \Gamma(X), e_{(i,j)} \otimes e_{(i',j')} \rangle \\ &= (\varphi \otimes \varphi) \left(\Gamma(X)(e_{(i,j)} \otimes e_{(i',j')}) \right) \\ &= \left(\frac{1}{2k^2} \right)^2 x_{(i+i',j+j')} \\ &= \frac{1}{2k^2}\, \varphi \left(X\, e_{(i+i',j+j')} \right) \\ &= \left\langle X, \frac{1}{2k^2}\, e_{(i+i',j+j')} \right\rangle. \end{aligned}$$

Hence

$$e_{(i,j)} * e_{(i',j')} = \frac{1}{2k^2}\, e_{(i+i',j+j')},\ (i,j),(i',j') \in G.$$

(ii) Since

$$\begin{aligned} \langle X, e_{(i,j)} * A \rangle &= (\varphi \otimes \varphi) \left(\Gamma(X)\, (e_{(i,j)} \otimes A) \right) \\ &= \frac{1}{2k^2}\, \varphi \left(u_{(-i,-j)}\, (x_{mn})\, u_{(-i,-j)}^{*}\, A \right) \\ &= \frac{1}{2k^2}\, \varphi \left((x_{mn})\, u_{(-i,-j)}^{*}\, A\, u_{(-i,-j)} \right) \\ &= \left\langle X, \frac{1}{2k^2}\, u_{(-i,-j)}^{*}\, A\, u_{(-i,-j)} \right\rangle \\ &= \left\langle X, \frac{1}{2k^2}\, u_{(i,j)}\, A\, u_{(i,j)}^{*} \right\rangle, \end{aligned}$$

we get

$$\begin{aligned} e_{(i,j)} * A &= \frac{1}{2k^2}\, u_{(i,j)}\, A\, u_{(i,j)}^{*} \\ &= \frac{1}{2k^2} \sum_{m,n=1}^{k} \eta^{i(m-n)}\, a_{m+j,n+j}\, e_{mn}. \end{aligned}$$

(iii) follows from a similar calculation.

(iv) Since

$$\begin{aligned}
& \langle X, A * B\rangle \\
= \; & (\varphi\otimes\varphi)(\Gamma(X)(A\otimes B)) \\
= \; & \frac{1}{k}\sum_{(i,j)\in G} x_{(i,j)} \sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r}\,\overline{(p_{(i,j)})_{q,s}}\,(\varphi\otimes\varphi)((e_{pq}\otimes e_{rs})(A\otimes B)) \\
= \; & \sum_{(i,j)\in G} \frac{1}{2k^2}\, x_{(i,j)} \left(\frac{1}{2k}\sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r}\,\overline{(p_{(i,j)})_{q,s}}\, a_{qp}\, b_{sr}\right) \\
= \; & \left\langle X, \sum_{(i,j)\in G}\left(\frac{1}{2k}\sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r}\,\overline{(p_{(i,j)})_{q,s}}\, a_{qp}\, b_{sr}\right) e_{(i,j)}\right\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
A * B &= \sum_{(i,j)\in G}\left(\frac{1}{2k}\sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r}\,\overline{(p_{(i,j)})_{q,s}}\, a_{qp}\, b_{sr}\right) e_{(i,j)} \\
&= \sum_{(i,j)\in G}\left(\frac{1}{2k}\sum_{m,n=1}^{k} \eta^{-i(m-n)}\, a_{m,n}\, b_{m+j,n+j}\right) e_{(i,j)}.
\end{aligned}$$

□

Lemma 2. *Let* $X = \sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \in M$.

(i) $X * X$ *if and only if*

$$\begin{cases}
x_{(i,j)} = \dfrac{1}{2k^2}\displaystyle\sum_{(m,n)\in G} x_{(i-m,j-n)}\, x_{(m,n)} + \dfrac{1}{2k}\sum_{m,n=1}^{k} \eta^{-i(m-n)}\, x_{m,n}\, x_{m+j,n+j}, \\
x_{m,n} = \dfrac{1}{2k^2}\left(\displaystyle\sum_{(i,j)\in G} \eta^{i(m-n)}\, x_{(i,j)}\,(x_{m+j,n+j} + x_{m-j,n-j})\right).
\end{cases}$$

(ii) $X^{\#} = X$ *if and only if*

$$\begin{cases} x_{(-i,-j)} = \overline{x_{(i,j)}}, \\ x_{ij} \in \mathbf{R}. \end{cases}$$

Proof. (i) follows from Lemma 1.
(ii) follows from the definition of the involution $^{\#}$. □

3.1. **The case k : odd.** We first compute the center of M with respect to the convolution product.

Lemma 3. *Let $Z(M,*)$ denote the center of M with respect to the convolution $*$. Then we have*

$$\begin{aligned}
& Z(M,*) \\
= \; & \left\{ \sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \;\middle|\; \begin{array}{l} x_{(i,-j)} = x_{(i,j)},\ i,j = 0,1,\cdots,k-1 \\ x_{i+n,j+n} = x_{i,j},\ i,j,n = 1,2,\cdots,k \end{array} \right\}.
\end{aligned}$$

In particular,

$$\dim Z(M,*) = \frac{k(k+3)}{2}.$$

Proof. Let $X = \sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \in Z(M,*)$. Let $(m,n) \in G$ be fixed. Then the equation $e_{(m,n)} * X = X * e_{(m,n)}$ implies

$$x_{i+n,j+n} = x_{i,j},\ i,j,n = 1,2,\cdots,k.$$

Moreover, since $A * X = X * A$ for any $A \in M_k(\mathbf{C})$, we obtain

$$x_{(i,-j)} = x_{(i,j)},\ i,j = 0,1,\cdots,k-1.$$

□

Lemma 4. *Let $l \in \{0,1,\cdots,k-1\}$. If we set*

$$p_{l,+} = \sum_{(i,j)\in G} \eta^{il}\, e_{(i,j)} + \sum_{m=1}^{k} e_{m,m+l}$$

and

$$p_{l,-} = \sum_{(i,j)\in G} \eta^{il}\, e_{(i,j)} - \sum_{m=1}^{k} e_{m,m+l},$$

then $p_{l,\pm}$'s are 1-dimensional central projections.

Proof. The assertion follows from a direct computation. □

Lemma 5. *Let $u \in \{0,1,\cdots,k-1\}$ and $v \in \{1,2,\cdots,\frac{k-1}{2}\}$. Put*

$$p_{u,v} = 2 \sum_{(i,j)\in G} (\eta^{iu+jv} + \eta^{iu-jv})\, e_{(i,j)}.$$

Then $p_{u,v}$'s are 2-dimensional central minimal projections. Furthermore, a system of matrix units $\{e^{u,v}_{s,t}\}_{s,t=1}^{2}$ in $M * p_{u,v} \cong M_2(\mathbf{C})$ is given by

$$\begin{aligned} e^{u,v}_{1,1} &= 2 \sum_{(i,j)\in G} \eta^{iu+jv}\, e_{(i,j)}, \\ e^{u,v}_{1,2} &= 2 \sum_{m=1}^{k} \eta^{-mv}\, e_{m,m+u}, \\ e^{u,v}_{2,1} &= 2 \sum_{m=1}^{k} \eta^{mv}\, e_{m,m+u}, \\ e^{u,v}_{2,2} &= 2 \sum_{(i,j)\in G} \eta^{iu-jv}\, e_{(i,j)}. \end{aligned}$$

Proof. Direct computations. □

Summing up the calculations, we have

Theorem 6. *The algebraic structure of $\hat{M}$ is given by*

$$\hat{M} = \underbrace{\mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}}_{2k} \oplus \underbrace{M_2(\mathbf{C}) \oplus M_2(\mathbf{C}) \oplus \cdots M_2(\mathbf{C})}_{\frac{k(k-1)}{2}}.$$

3.2. **The case k : even.**

Lemma 7. *The center $Z(M,*)$ of M with respect to $*$ is given by*

$$Z(M,*)$$

$$= \left\{ \sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \;\middle|\; \begin{array}{l} x_{(i,-j)} = x_{(i,j)},\ i,j = 0,1,\cdots,k-1 \\ x_{i+2n,j+2n} = x_{i,j},\ i,j,n = 1,2,\cdots,k \end{array} \right\}.$$

In particular,

$$\dim Z(M,*) = \frac{k(k+6)}{2}.$$

Proof. Let $X = \sum_{(i,j)\in G} x_{(i,j)}\, e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \in Z(M,*)$. Then the equality $e_{(m,n)} * X = X * e_{(m,n)}$ for $(m,n)\in G$ implies

$$x_{i+2n,j+2n} = x_{i,j},\ i,j,n = 1,2,\cdots,k.$$

Since $A * X = X * A$ for $A \in M_k(\mathbf{C})$, we have

$$x_{(i,-j)} = x_{(i,j)},\ i,j = 0,1,\cdots,k-1.$$

□

Lemma 8. *Let $l \in \{0,1,\cdots,k-1\}$. We set*

$$\begin{aligned}
p_{l,1} &= \sum_{(i,j)\in G} \eta^{il}\, e_{(i,j)} + \sum_{m=1}^{k} e_{m,m+l}, \\
p_{l,2} &= \sum_{(i,j)\in G} \eta^{il}\, e_{(i,j)} - \sum_{m=1}^{k} e_{m,m+l}, \\
p_{l,3} &= \sum_{(i,j)\in G} (-1)^j\, \eta^{il}\, e_{(i,j)} + \sum_{m=1}^{k} (-1)^m\, e_{m,m+l}, \\
p_{l,4} &= \sum_{(i,j)\in G} (-1)^j\, \eta^{il}\, e_{(i,j)} - \sum_{m=1}^{k} (-1)^m\, e_{m,m+l}.
\end{aligned}$$

Then $p_{l,i}$'s are 1-dimensional central projections.

Proof. The result follows from a straightforward calculation. □

Lemma 9. *Let $u \in \{0,1,\cdots,k-1\}$ and $v \in \{1,2,\cdots,\frac{k}{2}-1\}$. Set*

$$p_{u,v} = 2 \sum_{(i,j)\in G} (\eta^{iu+jv} + \eta^{iu-jv})\, e_{(i,j)}.$$

Then it follows that $p_{u,v}$'s are 2-dimentional central minimal projections. Furthermore, a system of matrix units $\{e^{u,v}_{s,t}\}^2_{s,t=1}$ in $M * p_{u,v}$ is given by

$$\begin{aligned}
e^{u,v}_{1,1} &= 2 \sum_{(i,j)\in G} \eta^{iu+jv}\, e_{(i,j)}, \\
e^{u,v}_{1,2} &= 2 \sum_{m=1}^{k} \eta^{-mv}\, e_{m,m+u}, \\
e^{u,v}_{2,1} &= 2 \sum_{m=1}^{k} \eta^{mv}\, e_{m,m+u}, \\
e^{u,v}_{2,2} &= 2 \sum_{(i,j)\in G} \eta^{iu-jv}\, e_{(i,j)}.
\end{aligned}$$

Proof. The assertions follow from a direct computation. □

Theorem 10. *The algebraic structure of $\hat{M}$ is given by*

$$\hat{M} = \underbrace{\mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}}_{4k} \oplus \underbrace{M_2(\mathbf{C}) \oplus M_2(\mathbf{C}) \oplus \cdots M_2(\mathbf{C})}_{\frac{k(k-2)}{2}}.$$

Remark 11. We can compute the paragroup structure of the inclusion associated with the Kac algebra $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$. (See Ocneanu [12].)

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