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# AN EXAMPLE OF FINITE DIMENSIONAL KAC ALGEBRAS OF KAC-PALJUTKIN TYPE

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ABSTRACT. An example of finite dimensional Kac algebras of Kac-Paljutkin type is given.

## 1. INTRODUCTION

Motivated by Pontryagin-Tannaka-Krein-Stinespring-Tatsuuma duality for locally compact groups, M.Enock and J.M.Schwartz introduced the notion of a Kac algebra and extended the duality to that of the category of Kac algebras. Any locally compact group is realized as a commutative Kac algebra and a cocommutative one, and conversely, any commutative (resp. cocommutative) Kac algebra comes from a locally compact group. However, to the author's knowledge, except for recent works of Majid [7, 8, 9] based on the matched pair introduced by Takeuchi [16], a number of examples of non-commutative and non-cocommutative Kac algebras do not seem to be known.

Before the general theory of Kac algebras, G.I.Kac and V.G.Paljutkin [5] considered a finite dimensional Kac algebra of a certain type. (They used the terminology "ring group".) They showed that the problem of constructing examples of the type is reduced to that of finding a family of unitary matrices satisfying certain conditions and gave a non-commutative and non-cocommutative 8 dimensional Kac algebra. (See also Masuoka [10].)

In the theory for subfactors initiated by V.Jones [4], A.Ocneanu [11] introduced the concept of a paragroup, which can be regarded as a quantization of groups, to classify subfactors. By using the so-called standard invariant equivalent to the paragroup, S.Popa classified subfactors under more general conditions. (See [13, 14].) It is known that the finite dimensional Kac algebras are characterized as special paragroups. (See David [1], Longo [6], Szymanski [15], and also Yamanouchi [17].)

Motivated by constructing non-commutative and non-cocommutative finite dimensional Kac algebras (as examples of paragroups), we shall give an example of finite dimensional Kac algebras of Kac-Paljutkin type by using the representation theory of extra-special p-groups and compute the dual structure.

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## 2. Construction of finite dimensional Kac algebras

In order to fix notation, we first recall the definition of a finite dimensional Kac algebra. (We consider only the finite dimensional case.) For the general theory of Kac algebras, we refer to Enock and Schwartz [2, 3].

- A finite dimensional Kac algebra **K** is a quadruple  $(M, \Gamma, \kappa, \varphi)$  such that
- (i)  $(M, \Gamma, \kappa)$  is an involutive finite dimensional Hopf-von Neumann algebra,
- (ii)  $\varphi$  is a faithful positive trace on M satisfying

$$(\iota_M \otimes \varphi)((1 \otimes y^*)\Gamma(x)) = \kappa((\iota_M \otimes \varphi)(\Gamma(y^*)(1 \otimes x))), \ x, y \in M.$$

Here  $\iota_M$  is the identity mapping of M. The linear functional  $\varphi$  is called a Haar weight of  $\mathbf{K}$ . Given a Kac algebra  $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$ , the dual Kac algebra  $\hat{\mathbf{K}} = (\hat{M}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$  of  $\mathbf{K}$  is canonically constructed as follows. M is assumed to be represented on the standard Hilbert space  $L^2(\varphi)$ . Let  $\lambda$  be the left regular representation of M on  $L^2(\varphi)$  with respect to the convolution product in M, W the fundamental operator of  $\mathbf{K}$ , and  $\Sigma$  the flip on  $L^2(\varphi) \otimes L^2(\varphi)$ . Then  $\hat{\mathbf{K}} = (\hat{M}, \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$  is defined by

$$\begin{split} \hat{M} &= \lambda(M), \\ \hat{\Gamma}(\lambda(f)) &= \Sigma \, W^* \left(\lambda(f) \otimes 1\right) W \, \Sigma, \ f \in M, \\ \hat{\kappa}(\lambda(f)) &= \lambda(\kappa(f)), \ f \in M, \\ \hat{\varphi}(\lambda(f)) &= \varphi(ef), \ f \in M, \end{split}$$

where e is the unit of M with respect to the convolution product.

In the following, we shall give an example of finite dimensional Kac algebras by using Kac-Paljutkin's method. For details, see Kac and Paljutkin [5].

Let  $k \in \mathbf{N}, k \geq 3$  be fixed and  $\eta$  a primitive k-th root of 1. We set

$$G = \mathbf{Z}_k \times \mathbf{Z}_k = \{(i, j) \mid i, j = 0, 1, \cdots, k-1\}$$

and define  $k\times k$  unitary matrices  $\{p_{(i,j)}\},\{u_{(i,j)}\}$  and  $\{v_{(i,j)}\}$  parametrized by G by

$$p_{(i,j)} = u_{(i,j)} = \sum_{m=1}^{k} \eta^{im} e_{m,m+j}, \quad v_{(i,j)} = \overline{u_{(i,j)}} = \sum_{m=1}^{k} \eta^{-im} e_{m,m+j}.$$

Here  $\{e_{ij}\}_{i,j=1}^{k}$  means the usual matrix units in  $M_k(\mathbf{C})$ . It is easy to check that the above matrices satisfy the conditions in Kac-Paljutkin [5], hence we have a  $2k^2$ dimensional Kac algebra  $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$  whose structure is given by the following:

$$M = \bigoplus_{(i,j)\in G} \mathbf{C}e_{(i,j)} \oplus M_k(\mathbf{C}),$$

where  $\{e_{(i,j)}\}_{(i,j)\in G}$  denotes the 1-dimensional central minimal projections,

$$\Gamma(e_{(i,j)}) = \sum_{(m,n)\in G} e_{(m,n)} \otimes e_{(i-m,j-n)} \\
+ \frac{1}{k} \sum_{m,n,s,t=1}^{k} (u_{(i,j)})_{m,s} \overline{(u_{(i,j)})_{n,t}} e_{mn} \otimes e_{st}, \ (i,j) \in G, \\
\Gamma(X) = \sum_{(i,j)\in G} e_{(-i,-j)} \otimes u_{(i,j)} X u_{(i,j)}^* \\
+ \sum_{(i,j)\in G} v_{(i,j)} X v_{(i,j)}^* \otimes e_{(i,j)}, \ X \in M_k(\mathbf{C}), \\
\kappa(e_{(i,j)}) = e_{(-i,-j)}, \ (i,j) \in G, \\
\kappa(X) = {}^tX, \ X \in M_k(\mathbf{C}),$$

where  ${}^{t}X$  means the transpose of X,

$$\varphi\left(\sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^k\right) = \frac{1}{2k^2} \left(\sum_{(i,j)\in G} x_{(i,j)} + k \sum_{i=1}^k x_{ii}\right).$$

## 3. The dual Kac Algebra

In this section, we shall compute the dual of the Kac algebra  $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$  constructed in the previous section. It is sufficient to calculate the convolution structure in M.

Let \* be the convolution product of M and # the involution of M, that is, they are defined by the following relations :

$$\begin{aligned} \langle a, x * y \rangle &= \langle \Gamma(a), x \otimes y \rangle \\ &= (\varphi \otimes \varphi)(\Gamma(a)(x \otimes y)), \ x, y, a \in M, \\ x^{\#} &= \kappa(x^*) = \kappa(x)^*, \ x \in M. \end{aligned}$$

**Lemma 1.** Let  $A = (a_{ij}), B = (b_{ij}) \in M_k(\mathbf{C})$ . Then we have

(i) 
$$e_{(i,j)} * e_{(i',j')} = \frac{1}{2k^2} e_{(i+i',j+j')},$$
  
(ii)  $e_{(i,j)} * A = \frac{1}{2k^2} \sum_{m,n=1}^{k} \eta^{i(m-n)} a_{m+j,n+j} e_{mn},$   
(iii)  $A * e_{(i,j)} = \frac{1}{2k^2} \sum_{m,n=1}^{k} \eta^{i(m-n)} a_{m-j,n-j} e_{mn},$   
(iv)  $A * B = \sum_{(i,j) \in G} c_{(i,j)} e_{(i,j)},$ 

where

$$c_{(i,j)} = \frac{1}{2k} \sum_{m,n=1}^{k} \eta^{-i(m-n)} a_{m,n} b_{m+j,n+j}.$$

Hence, for  $X = \sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^k$  and  $Y = \sum_{(i,j)\in G} y_{(i,j)} e_{(i,j)} + (y_{ij})_{i,j=1}^k$ , we get

$$X * Y = \sum_{(i,j) \in G} z_{(i,j)} e_{(i,j)} + (z_{ij})_{i,j=1}^{k}$$

where

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$$z_{(i,j)} = \frac{1}{2k^2} \sum_{(m,n)\in G} x_{(i-m,j-n)} y_{(m,n)} + \frac{1}{2k} \sum_{m,n=1}^{k} \eta^{-i(m-n)} x_{m,n} y_{m+j,n+j},$$
$$z_{mn} = \frac{1}{2k^2} \left( \sum_{(i,j)\in G} \eta^{i(m-n)} \left( x_{(i,j)} y_{m+j,n+j} + y_{(i,j)} x_{m-j,n-j} \right) \right).$$

Proof. Let  $X = \sum_{(m,n)\in G} x_{(m,n)} e_{(m,n)} + (x_{mn})_{m,n=1}^k \in M.$ (i) We compute

$$\begin{aligned} \langle X, e_{(i,j)} * e_{(i',j')} \rangle &= \langle \Gamma(X), e_{(i,j)} \otimes e_{(i',j')} \rangle \\ &= (\varphi \otimes \varphi) \left( \Gamma(X) (e_{(i,j)} \otimes e_{(i',j')}) \right) \\ &= \left( \frac{1}{2k^2} \right)^2 x_{(i+i',j+j')} \\ &= \frac{1}{2k^2} \varphi \left( X e_{(i+i',j+j')} \right) \\ &= \left\langle X, \frac{1}{2k^2} e_{(i+i',j+j')} \right\rangle. \end{aligned}$$

Hence

$$e_{(i,j)} * e_{(i',j')} = \frac{1}{2k^2} e_{(i+i',j+j')}, \ (i,j), (i',j') \in G.$$

(ii) Since

$$\begin{aligned} \langle X, e_{(i,j)} * A \rangle &= (\varphi \otimes \varphi) \left( \Gamma(X) \left( e_{(i,j)} \otimes A \right) \right) \\ &= \frac{1}{2k^2} \varphi \left( u_{(-i,-j)} \left( x_{mn} \right) u_{(-i,-j)} A \right) \\ &= \frac{1}{2k^2} \varphi \left( (x_{mn}) u_{(-i,-j)} A u_{(-i,-j)} \right) \\ &= \left\langle X, \frac{1}{2k^2} u_{(-i,-j)} A u_{(-i,-j)} \right\rangle \\ &= \left\langle X, \frac{1}{2k^2} u_{(i,j)} A u_{(i,j)} \right\rangle, \end{aligned}$$

we get

$$e_{(i,j)} * A = \frac{1}{2k^2} u_{(i,j)} A u_{(i,j)}^{*}$$
  
=  $\frac{1}{2k^2} \sum_{m,n=1}^{k} \eta^{i(m-n)} a_{m+j,n+j} e_{mn}.$ 

(iii) follows from a similar calculation.

(iv) Since

$$\begin{aligned} &\langle X, A * B \rangle \\ &= (\varphi \otimes \varphi)(\Gamma(X)(A \otimes B)) \\ &= \frac{1}{k} \sum_{(i,j) \in G} x_{(i,j)} \sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r} \overline{(p_{(i,j)})}_{q,s} (\varphi \otimes \varphi)((e_{pq} \otimes e_{rs})(A \otimes B)) \\ &= \sum_{(i,j) \in G} \frac{1}{2k^2} x_{(i,j)} \left( \frac{1}{2k} \sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r} \overline{(p_{(i,j)})}_{q,s} a_{qp} b_{sr} \right) \\ &= \left\langle X, \sum_{(i,j) \in G} \left( \frac{1}{2k} \sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r} \overline{(p_{(i,j)})}_{q,s} a_{qp} b_{sr} \right) e_{(i,j)} \right\rangle, \end{aligned}$$

we have

$$A * B = \sum_{(i,j)\in G} \left( \frac{1}{2k} \sum_{p,q,r,s=1}^{k} (p_{(i,j)})_{p,r} \overline{(p_{(i,j)})}_{q,s} a_{qp} b_{sr} \right) e_{(i,j)}$$
$$= \sum_{(i,j)\in G} \left( \frac{1}{2k} \sum_{m,n=1}^{k} \eta^{-i(m-n)} a_{m,n} b_{m+j,n+j} \right) e_{(i,j)}.$$

Lemma 2. Let  $X = \sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \in M.$ (i) X \* X if and only if  $\begin{cases} x_{(i,j)} = \frac{1}{2k^2} \sum_{(m,n)\in G} x_{(i-m,j-n)} x_{(m,n)} + \frac{1}{2k} \sum_{m,n=1}^{k} \eta^{-i(m-n)} x_{m,n} x_{m+j,n+j}, \\ x_{m,n} = \frac{1}{2k^2} \left( \sum_{(i,j)\in G} \eta^{i(m-n)} x_{(i,j)} (x_{m+j,n+j} + x_{m-j,n-j}) \right).$ (ii)  $X^{\#} = X$  if and only if

$$\begin{cases} x_{(-i,-j)} = \overline{x_{(i,j)}}, \\ x_{ij} \in \mathbf{R}. \end{cases}$$

*Proof.* (i) follows from Lemma 1.

(ii) follows from the definition of the involution  $^{\#}.$ 

3.1. The case k : odd. We first compute the center of M with respect to the convolution product.

**Lemma 3.** Let Z(M, \*) denote the center of M with respect to the convolution \*. Then we have

$$= \left\{ \sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \middle| \begin{array}{c} x_{(i,-j)} = x_{(i,j)}, \ i,j = 0, 1, \cdots, k-1 \\ x_{i+n,j+n} = x_{i,j}, \ i,j,n = 1, 2, \cdots, k \end{array} \right\}.$$

In particular,

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$$\dim Z(M,*) = \frac{k(k+3)}{2}.$$

*Proof.* Let  $X = \sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^k \in Z(M,*)$ . Let  $(m,n)\in G$  be fixed. Then the equation  $e_{(m,n)} * X = X * e_{(m,n)}$  implies

$$x_{i+n,j+n} = x_{i,j}, \ i, j, n = 1, 2, \cdots, k$$

Moreover, since A \* X = X \* A for any  $A \in M_k(\mathbf{C})$ , we obtain

$$x_{(i,-j)} = x_{(i,j)}, \ i, j = 0, 1, \cdots, k-1.$$

**Lemma 4.** Let  $l \in \{0, 1, \dots, k-1\}$ . If we set

$$p_{l,+} = \sum_{(i,j)\in G} \eta^{il} e_{(i,j)} + \sum_{m=1}^{k} e_{m,m+l}$$

and

$$p_{l,-} = \sum_{(i,j)\in G} \eta^{il} e_{(i,j)} - \sum_{m=1}^{k} e_{m,m+l},$$

then  $p_{l,\pm}$ 's are 1-dimensional central projections.

*Proof.* The assertion follows from a direct computation.

**Lemma 5.** Let  $u \in \{0, 1, \dots, k-1\}$  and  $v \in \{1, 2, \dots, \frac{k-1}{2}\}$ . Put

$$p_{u,v} = 2 \sum_{(i,j)\in G} \left( \eta^{iu+jv} + \eta^{iu-jv} \right) e_{(i,j)}.$$

Then  $p_{u,v}$ 's are 2-dimensional central minimal projections. Furthermore, a system of matrix units  $\{e_{s,t}^{u,v}\}_{s,t=1}^2$  in  $M * p_{u,v} \cong M_2(\mathbf{C})$  is given by

$$e_{1,1}^{u,v} = 2 \sum_{(i,j)\in G} \eta^{iu+jv} e_{(i,j)},$$

$$e_{1,2}^{u,v} = 2 \sum_{m=1}^{k} \eta^{-mv} e_{m,m+u},$$

$$e_{2,1}^{u,v} = 2 \sum_{m=1}^{k} \eta^{mv} e_{m,m+u},$$

$$e_{2,2}^{u,v} = 2 \sum_{(i,j)\in G}^{k} \eta^{iu-jv} e_{(i,j)}.$$

*Proof.* Direct computations.

Summing up the calculations, we have

**Theorem 6.** The algebraic structure of  $\hat{M}$  is given by

$$\hat{M} = \underbrace{\mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}}_{2k} \oplus \underbrace{M_2(\mathbf{C}) \oplus M_2(\mathbf{C}) \oplus \cdots M_2(\mathbf{C})}_{\frac{k(k-1)}{2}}.$$

3.2. The case k : even.

**Lemma 7.** The center Z(M, \*) of M with respect to \* is given by

$$Z(M,*) = \left\{ \sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^{k} \middle| \begin{array}{l} x_{(i,-j)} = x_{(i,j)}, \ i,j = 0, 1, \cdots, k-1 \\ x_{i+2n,j+2n} = x_{i,j}, \ i,j,n = 1, 2, \cdots, k \end{array} \right\}.$$

In particular,

$$\dim Z(M,*) = \frac{k(k+6)}{2}.$$

*Proof.* Let  $X = \sum_{(i,j)\in G} x_{(i,j)} e_{(i,j)} + (x_{ij})_{i,j=1}^k \in Z(M, *)$ . Then the equality  $e_{(m,n)} * X = X * e_{(m,n)}$  for  $(m,n) \in G$  implies

$$x_{i+2n,j+2n} = x_{i,j}, \ i, j, n = 1, 2, \cdots, k.$$

Since A \* X = X \* A for  $A \in M_k(\mathbf{C})$ , we have

$$x_{(i,-j)} = x_{(i,j)}, \ i, j = 0, 1, \cdots, k-1.$$

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**Lemma 8.** Let  $l \in \{0, 1, \dots, k-1\}$ . We set

$$p_{l,1} = \sum_{(i,j)\in G} \eta^{il} e_{(i,j)} + \sum_{m=1}^{k} e_{m,m+l},$$

$$p_{l,2} = \sum_{(i,j)\in G} \eta^{il} e_{(i,j)} - \sum_{m=1}^{k} e_{m,m+l},$$

$$p_{l,3} = \sum_{(i,j)\in G} (-1)^{j} \eta^{il} e_{(i,j)} + \sum_{m=1}^{k} (-1)^{m} e_{m,m+l},$$

$$p_{l,4} = \sum_{(i,j)\in G} (-1)^{j} \eta^{il} e_{(i,j)} - \sum_{m=1}^{k} (-1)^{m} e_{m,m+l}.$$

Then  $p_{l,i}$ 's are 1-dimensional central projections.

*Proof.* The result follows from a straightforward calculation.

**Lemma 9.** Let  $u \in \{0, 1, \dots, k-1\}$  and  $v \in \{1, 2, \dots, \frac{k}{2} - 1\}$ . Set

$$p_{u,v} = 2 \sum_{(i,j)\in G} \left( \eta^{iu+jv} + \eta^{iu-jv} \right) e_{(i,j)}.$$

Then it follows that  $p_{u,v}$ 's are 2-dimensional central minimal projections. Furthermore, a system of matrix units  $\{e_{s,t}^{u,v}\}_{s,t=1}^2$  in  $M * p_{u,v}$  is given by

$$\begin{aligned} e_{1,1}^{u,v} &= 2 \sum_{(i,j)\in G} \eta^{iu+jv} e_{(i,j)}, \\ e_{1,2}^{u,v} &= 2 \sum_{m=1}^{k} \eta^{-mv} e_{m,m+u}, \\ e_{2,1}^{u,v} &= 2 \sum_{m=1}^{k} \eta^{mv} e_{m,m+u}, \\ e_{2,2}^{u,v} &= 2 \sum_{(i,j)\in G} \eta^{iu-jv} e_{(i,j)}. \end{aligned}$$

*Proof.* The assertions follow from a direct computation.

**Theorem 10.** The algebraic structure of  $\hat{M}$  is given by

$$\hat{M} = \underbrace{\mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}}_{4k} \oplus \underbrace{M_2(\mathbf{C}) \oplus M_2(\mathbf{C}) \oplus \cdots M_2(\mathbf{C})}_{\frac{k(k-2)}{2}}.$$

*Remark* 11. We can compute the paragroup structure of the inclusion associated with the Kac algebra  $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$ . (See Ocneanu [12].)

### References

- M.-C.David, Paragroupe d'Adrian Ocneanu et algebre de Kac, Pacific J. Math. (to appear).
   M.Enock and J.M.Schwartz, Une dualité dans les algèbres de von Neumann, Bull. Soc. Math.
- France Suppl. Mem., 44(1975),1-144. MR **56**:1091
- M.Enock and J.M.Schwartz, Kac algebras and duality of locally compact groups, 1992, Springer. MR 94e:46001
- 4. V.Jones, Index for subfactors, Invent. Math., 72(1983),1-25. MR 84d:46097
- G.I.Kac and V.G.Paljutkin, Finite ring groups, Trans. Moscow Math. Soc., (1966),251-294. MR 34:8211
- R.Longo, A duality for Hopf algebras and for subfactors, Comm. Math. Phys., 159(1994), 133-150. CMP 94:07
- S.Majid, Physics for algebraists : non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J. Algebra, 130(1990),17-64. MR 91j:16050
- S.Majid, Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations, Pacific J. Math., 141(1990),311-332. MR 91a:17009
- 9. S.Majid, Hopf-von Neumann algebra bicrossproducts, Kac algebra bicrossproducts, and the classical Yang-Baxter equations, J. Funct. Anal., 95(1991),291-319. MR **92b**:46088
- 10. A.Masuoka, Semisimple Hopf algebras of dimension 6, 8, Israel J. Math. (to appear).
- A.Ocneanu, Quantized groups, string algebras and Galois theory for algebras, Operator Algebras and Applications, vol.2, London Math. Soc. Lecture Note Series Vol.136, Cambridge Univ. Press, 119-172(1988). MR 91k:46068
- A.Ocneanu, Quantum symmetry, differential geometry of finite graphs and classification of subfactors, University of Tokyo Seminary Notes 45, (Notes recorded by Y.Kawahigashi), 1991.
- S.Popa, Classification of subfactors : the reduction to commuting squares, Invent. Math., 101(1990),19-43. MR 91h:46109
- S.Popa, Classification of amenable subfactors of type II, Acta Math., 172(1994), 163-255. MR 95f:46105
- W.Szymanski, Finite index subfactors and Hopf algebra crossed products, Proc. Amer. Math. Soc., 120(1994),519-528. MR 94d:46061

- 16. M.Takeuchi, Matched pairs of groups and bismash products of Hopf algebras, Comm. Algebra, 9(1981),841-882. MR ${\bf 83f:}16013$
- 17. T.Yamanouchi, Construction of an outer action of a finite-dimensional Kac algebra on the AFD factor of type  $II_1$ , Inter. J. Math., 4(1993),1007-1045. MR **95c:**46108

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