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THE NILPOTENCE HEIGHT OF Sq^{2^n}

G. WALKER AND R. M. W. WOOD

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Abstract. A 20-year-old conjecture about the mod 2 Steenrod algebra A, namely that the element Sq^{2^n} has nilpotence height $2n+2$, is proved. The proof uses formulae of D. M. Davis and J. H. Silverman to obtain commutation relations involving 'atomic' Sq^i and the canonical antiautomorphism of A, together with a 'stripping' technique for obtaining new relations in A from old. This construction goes back to Kristensen [Math. Scand. 16 (1965), 97–115].

1. The main result

We shall prove the following relation in the mod 2 Steenrod algebra A.

Theorem 1.1. *For* $n \geq 0$, $\quad (Sq^{2^n})^{2n+2} = 0.$

This establishes a conjecture made by Steve Wilson in 1975, namely that the nilpotence height of Sq^{2^n} is exactly $2n+2$. In an 'informal note' [5] Don Davis showed that $(Sq^{2^n})^{2n+1} \neq 0$; another proof of this can be found in [3]. At the same time, Davis verified by computer calculation that $(Sq^{2^n})^{2n+2} = 0$ for $n \leq 5$: this has also been verified for $n = 6, 7$ by Ken Monks [9].

The proof we give is based on the work of Davis [4], as extended by Judith Silverman [12]. Following [12], we say that if $a = \sum a_i 2^i$ and $b = \sum b_i 2^i$ are integers ≥ 0, where a_i, $b_i \in \{0, 1\}$, then *a dominates b* if $a_i \geq b_i$ for all i, and write $a \succeq b$. This is equivalent to the condition that the binomial coefficient $\binom{a}{b}$ is odd.

If $R = (r_1, r_2, \dots)$ is a sequence of integers ≥ 0, we write $Sq\langle R\rangle$ for the corresponding Milnor basis element. The degree of $Sq\langle R\rangle$ is $|R| = \sum(2^i - 1)r_i$, and the excess of $Sq\langle R\rangle$ is $\sum r_i$. The antiautomorphism χ of A plays a fundamental part in our argument, and we find it convenient to write

$$\widehat{Sq^a} = \chi(Sq^a).$$

Then [12, Propositions 2.1, 2.2] the following basic formulae translate products of one Steenrod square by χ of another into the Milnor basis.

$$Sq^u \widehat{Sq^v} = \sum Sq\langle R\rangle : \ |R| = u + v; \ |R| + \sum r_i \succeq 2u, \tag{1}$$

$$\widehat{Sq^u} Sq^v = \sum Sq\langle R\rangle : \ |R| = u + v; \qquad \sum r_i \succeq v. \tag{2}$$

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From (1), we obtain Straffin's formula [13]

$$\widehat{Sq^{2^n}} = Sq^{2^n} + Sq^{2^{n-1}}\widehat{Sq^{2^{n-1}}}, \tag{3}$$

since $\widehat{Sq^u}$ is the sum of all Milnor basis elements in degree u [11].

Using (1) and (2), it is an elementary exercise to verify the following relation.

Proposition 1.2. *Let a and b be positive integers. Then*

$$\widehat{Sq^{2^a}}\, Sq^{2^a(2^b-1)} = Sq^{2^{a-1}(2^{b+1}-1)}\widehat{Sq^{2^{a-1}}}.$$

Following [14], we shall refer to integers of the form $2^a(2^b-1)$ as *atomic* numbers and to the corresponding squaring operations as *atomic* squares. For brevity we write $S^a_b = Sq^{2^a(2^b-1)}$ for an atomic square. In this notation, Proposition 1.2 can be written concisely as

$$\widehat{S^a_1} S^a_b = S^{a-1}_{b+1}\widehat{S^{a-1}_1}. \tag{4}$$

The proof of Theorem 1.1 will proceed by way of a sequence of auxiliary relations. To state these, we follow the notation of Dan Arnon [2], and write for $n \geq k$

$$X^n_k = Sq^{2^n} Sq^{2^{n-1}} \dots Sq^{2^k}.$$

In the case $k = 0$, Davis's result [4, Theorem 1] gives

$$X^n_0 = Sq^{2^n} Sq^{2^{n-1}} \dots Sq^2 Sq^1 = \chi(Sq^{2^{n+1}-1}). \tag{5}$$

Thus the Adem relation $Sq^{2^{n+1}-1}Sq^{2^n} = 0$ may be written in the form

$$\widehat{X}^n_0 Sq^{2^n} = 0. \tag{6}$$

Proposition 1.3. (i) *For $a > c$, $\widehat{X}^a_{c+1} S^a_b = S^c_{a+b-c}\widehat{X}^{a-1}_c$.*

(ii) *For $k \geq 1$, $(\widehat{S^a_1} S^a_b)^k = S^{a-1}_{b+1} S^{a-2}_{b+2} \dots S^{a-k}_{b+k}\widehat{X}^{a-1}_{a-k}$.*

Proof. (i) Proposition 1.2 allows us to pass a factor $\widehat{Sq^{2^a}}$ across to the right of an atomic square S^a_b with the same 2-power a. The new factors again have the same 2-power $a-1$. Thus we can pass a factor $\widehat{Sq^{2^{a-1}}}$ in the same way across the new atomic square on the right of the preceding formula, and so on. This proves (i).

(ii) The case $k = 1$ is equation (4). The general case follows by induction on k, since

$$(\widehat{S^a_1} S^a_b)^k = (S^{a-1}_{b+1}\widehat{S^{a-1}_1})^k = S^{a-1}_{b+1}(\widehat{S^{a-1}_1} S^{a-1}_{b+1})^{k-1}\widehat{S^{a-1}_1}. \quad \square$$

One more preliminary result is needed before we embark on the main proof. This is an example of the technique of deriving a relation in A in degree u from a relation in a higher degree $u+v$ by 'stripping' it by an 'allowable vector of degree v'. For a general discussion and explanation of this method, we refer the reader to [3]. Here we require only the special case where the allowable vector consists of a single power of 2, say 2^a. In this case, the stripping operation consists of subtracting 2^a in turn from each exponent $\geq 2^a$ in the given relation (so that Sq^{u+2^a} is replaced by Sq^u), and adding the resulting terms. Note in particular that the result of stripping Sq^{2^a} by 2^a is $Sq^0 = 1$.

As a simple example, the relation $Sq^{15}Sq^8 = 0$ yields the relations $Sq^{14}Sq^8 = Sq^{15}Sq^7$, $Sq^{13}Sq^8 = Sq^{15}Sq^6$, $Sq^{11}Sq^8 = Sq^{15}Sq^4$ and $Sq^7Sq^8 = Sq^{15}$ on stripping it by 1, 2, 4 and 8 respectively.

We recall that for $n \geq 1$, $A(n-1)$ is the subalgebra of A generated by the Sq^u with $u < 2^n$. Hence the result of stripping any element $\Theta \in A(n-1)$ by 2^n is zero. (Elements such as $Sq^{2^n} Sq^{2^n}$ also have this property.)

In certain cases, this stripping technique can be used strategically to 'remove hats' in relations in A. The following lemma is an example.

Lemma 1.4. *Let $\Theta \in A$ be an element which gives 0 when stripped by 2^n. Then*

$$\Theta \widehat{Sq^{2^n}} (Sq^{2^n})^{2k+1} = 0 \qquad \Longrightarrow \qquad \Theta (Sq^{2^n})^{2k+2} = 0.$$

Proof. Strip the given relation by 2^n. In Straffin's formula (3), the second term on the right lies in $A(n-1)$. Hence the result of stripping is the relation

$$\Theta (Sq^{2^n})^{2k+1} = (2k+1) \Theta \widehat{Sq^{2^n}} (Sq^{2^n})^{2k}.$$

Thus $\Theta (Sq^{2^n})^{2k+2} = \Theta \widehat{Sq^{2^n}} (Sq^{2^n})^{2k+1} = 0$. □

Proof of Theorem 1.1. We shall prove the following two statements by induction on k for $0 \leq k \leq n$:

$$\widehat{X}^n_k (Sq^{2^n})^{2k+1} = 0; \tag{7}$$

$$\widehat{X}^{n-1}_k (Sq^{2^n})^{2k+2} = 0. \tag{8}$$

The case $k = 0$ of (7) is equation (6). This starts the induction. We shall show that (7) for k implies (8) for k, and that (8) for k implies (7) for $k+1$. If $k = n$, we interpret $\widehat{X}^{n-1}_k$ as 1, so that (8) is the required relation $(Sq^{2^n})^{2k+2} = 0$.

Statement (7) for k is equivalent to the relation

$$\widehat{X}^{n-1}_k \widehat{Sq^{2^n}} (Sq^{2^n})^{2k+1} = 0,$$

and on removing the hat by Lemma 1.4 we obtain statement (8).

Now we use Proposition 1.3(i) with $a = n$, $b = 1$ and $c = k$. This gives

$$\widehat{X}^n_{k+1} (Sq^{2^n})^{2k+3} = \widehat{X}^n_{k+1} Sq^{2^n} (Sq^{2^n})^{2k+2} = Sq^{2^k(2^{n-k+1}-1)} \widehat{X}^{n-1}_k (Sq^{2^n})^{2k+2}.$$

Thus (8) for k implies (7) for $k+1$. This completes the proof of Theorem 1.1 by induction. □

2. Related results

In this section we shall prove the following result.

Theorem 2.1. *For $n \geq 0$, the nilpotence height of the elements $Sq^{2^n} \widehat{Sq^{2^n}}$ and $\widehat{Sq^{2^n}} Sq^{2^n}$ is exactly $n+1$.*

Proof. To show that the nilpotence order of both elements is at least $n+1$, it suffices to prove that $Sq^{2^n} (\widehat{Sq^{2^n}} Sq^{2^n})^n \neq 0$.

Using Proposition 1.3(ii) and equation (5), this expression can be written as

$$S^n_1 S^{n-1}_2 \ldots S^0_{n+1} Sq^{2^n-1} \neq 0. \tag{9}$$

This monomial in the Sq^i is a Z-basis element [14, Theorem 2], and so (9) is true.

Next we prove $(\widehat{Sq^{2^n}}Sq^{2^n})^{n+1} = 0$. This follows directly from Proposition 1.2. The method collects atomic squares on the left and $\widehat{X}_0^{n-1}$ on the right, as will be clear from the following example:

$$\begin{aligned}(\widehat{Sq^8}Sq^8)^4 &= Sq^{12}(\widehat{Sq^4}Sq^{12})^3\widehat{Sq^4}\\ &= Sq^{12}Sq^{14}(\widehat{Sq^2}Sq^{14})^2\widehat{Sq^2}\widehat{Sq^4}\\ &= Sq^{12}Sq^{14}Sq^{15}(\widehat{Sq^1}Sq^{15})\widehat{Sq^1}\widehat{Sq^2}\widehat{Sq^4}\\ &= 0, \text{ since } Sq^1Sq^{15} = 0.\end{aligned}$$

It remains to prove that $(Sq^{2^n}\widehat{Sq^{2^n}})^{n+1} = 0$. The proof of this relation is another example of 'removal of hats'. We shall show that the expression $(Sq^{2^n}\widehat{Sq^{2^n}})^{n-k+1} \cdot (Sq^{2^n})^{2k}$ is independent of k for $0 \leq k \leq n+1$. That is,

$$(Sq^{2^n}\widehat{Sq^{2^n}})^{n-k}(Sq^{2^n})^{2k+2} + (Sq^{2^n}\widehat{Sq^{2^n}})^{n-k+1}(Sq^{2^n})^{2k} = 0, \text{ for } 0 \leq k \leq n. \tag{10}$$

The result then follows from this chain of identities using Theorem 1.1.

The left-hand side of (10) is

$$Sq^{2^n}(\widehat{Sq^{2^n}}Sq^{2^n})^{n-k}(Sq^{2^n} + \widehat{Sq^{2^n}})(Sq^{2^n})^{2k},$$

and using Straffin's relation (3) and Proposition 1.3(ii) with $a = n$ and $b = 1$, this is

$$S_1^nS_2^{n-1}\dots S_{n-k+1}^k\widehat{X}_k^{n-1}(Sq^{2^{n-1}}\widehat{Sq^{2^{n-1}}})(Sq^{2^n})^{2k}.$$

In the case $k = 0$, this expression is zero by equation (6), while for $1 \leq k \leq n$ we have

$$\widehat{X}_k^{n-1}(Sq^{2^{n-1}}\widehat{Sq^{2^{n-1}}})(Sq^{2^n})^{2k} = Sq^{2^{k-1}(2^{n-k+1}-1)}\widehat{X}_{k-1}^{n-2}\widehat{Sq^{2^{n-1}}}(Sq^{2^n})^{2k}.$$

Since $\widehat{X}_{k-1}^{n-2}\widehat{Sq^{2^{n-1}}}(Sq^{2^n})^{2k} = \widehat{X}_{k-1}^{n-1}(Sq^{2^n})^{2k} = 0$ by (8), this completes the proof of Theorem 2.1. □

Theorem 2.1 tells us that the Z-basis element (9) is annihilated by right multiplication by $\widehat{Sq^{2^n}}$. The case $k = 0$ of equation (10) shows that it is also annihilated by Sq^{2^n}. Since $Sq^{2^n-1}Sq^{2^n} = Sq^{2^{n+1}-1} = S_{n+1}^0$, this gives

$$S_1^nS_2^n\dots S_{n+1}^0S_{n+1}^0 = 0. \tag{11}$$

In the case $n = 3$, for example, (9) and (11) state that

$$Sq^8Sq^{12}Sq^{14}Sq^{15}Sq^7 \neq 0, \quad Sq^8Sq^{12}Sq^{14}Sq^{15}Sq^{15} = 0.$$

It is of interest to determine the nilpotence height of other elements of A. Monks [8, 10] has proved a number of results of this kind for the elements P_t^s and for Sq^u when u is not a power of 2. Of course, there also remains the question of finding corresponding results in the mod p Steenrod algebra for an odd prime p.

In conclusion, we cannot resist quoting from Frank Adams's reaction [1] to Don Davis's preprint [5] — a letter in which he suggested, among other things, the 'stripping' technique used above, as a cap product operation with an element of the dual algebra A^*. This letter (widely publicised by Davis) ends as follows: "Well, in my opinion this is an ill-chosen problem. (a) If you can solve it, it won't do the rest of algebraic topology much good, so you don't stand to gain much. (b) It's hard, so you stand to lose time and self-esteem. (c) Finally, it's addictive.

I propose to kick the habit now, and I advise you to do the same..."

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Department of Mathematics, University of Manchester, Manchester M13 9PL, United Kingdom

E-mail address: `grant@ma.man.ac.uk`

E-mail address: `reg@ma.man.ac.uk`