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THE EXISTENCE OF A MAXIMIZING VECTOR FOR THE NUMERICAL RANGE OF A COMPACT OPERATOR

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Abstract. Let X be a complex Lebesgue space with a unique duality map J from X to X^*, the conjugate space of X. Let A be a compact operator on X. This paper focuses on properties of $W(A) = \{J(x)(A(x)) : \|x\| = 1\}$ and $\Lambda(A) = \sup\{\operatorname{Re}\alpha : \alpha \in W(A)\}$. We adapt an example due to Halmos to show that for $X = l_p, 1 < p < \infty$, there is a compact operator A on l_p with $W(A)$ the semi-open interval $[-1, 0)$. So $\Lambda(A)$ is not attained as a maximum. A corollary of the main result in this paper is that if $X = l_p, 1 < p < \infty$, and $\Lambda(A) \neq 0$, then $\Lambda(A)$ is attained as a maximum.

Introduction

A map $J : X \to X^*$ that satisfies $\|J(x)\| = \|x\| = \langle x, J(x)\rangle$ for all $x \in X$ is called a *duality map* for X. Here $\|x\|$ is the norm and $\langle\ ,\ \rangle$ is the pairing defined by $\langle x, f\rangle = f(x)$ for $x \in X, f \in X^*$. We shall need a precise formula for the duality map for $L_p = L_p(\Omega, \Sigma, \mu)$, $1 < p < \infty$, where Σ is a σ-field of subsets of Ω and μ is an extended nonnegative countably additive measure on Σ. For such p we identify L_p^* with L_q, $q = p/(p-1)$. From Theorem IV, 8.1 of [6], we see that the linear functional on L_p given by $x \in L_q$ is

$$y \mapsto \int_\Omega yx\,d\mu, \qquad y \in L_p. \tag{1}$$

We write $\langle y, x\rangle$ for $\int_\Omega yx\,d\mu$. For a nonzero complex number α, $\operatorname{sgn}\alpha = \alpha/|\alpha|$ while $\operatorname{sgn} 0 = 0$. For $1 < p < \infty$, put $K_p(\alpha) = |\alpha|^{p-1}\operatorname{sgn}\overline{\alpha}$. We extend this notation to functions $x : \Omega \to \mathbf{C}$ by $K_p(x)(\omega) = K_p(x(\omega))$ for all $\omega \in \Omega$. Since

$$|K_p(x)|^q = |x|^p = xK_p(x), \tag{2}$$

it follows that if $x \in L_p$, then $K_p(x) \in L_q$ and

$$\|K_p(x)\|_q^q = \|x\|_p^p = \langle x, K_p(x)\rangle \tag{3}$$

where $\|x\|_p$ is the norm of L_p. We define a map $J_p : L_p \to L_q$ by

$$J_p(x) = \|x\|_p^{2-p} K_p(x). \tag{4}$$

From (3) we get

$$\|J_p(x)\|_q^2 = \|x\|_p^2 = \langle x, J_p(x)\rangle. \tag{5}$$

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Therefore, J_p is a duality map for L_p. It follows from Corollary 3.19 of [4] or [14] that J_p is the only duality map for L_p. It is shown in [13] that neither L_1 nor L_∞ has a unique duality map. Let $S(X) = \{x \in X : \|x\| = 1\}$. For $x \in S(L_p)$ we have

$$K_p(x) = J_p(x) \tag{6}$$

and for $x \in L_p$,

$$K_q(K_p(x)) = J_q(J_p(x)) = x. \tag{7}$$

Let A be a bounded linear operator on X. Put

$$\psi_A(x) = \langle Ax, K_p(x)\rangle. \tag{8}$$

From (6) we get

$$\Lambda(A) = \sup\{\operatorname{Re}\psi_A(x) : x \in S(L_p)\}. \tag{9}$$

In Theorem 11 of [11] it is shown that

$$\Lambda(A) = \lim_{t\to 0^+} t^{-1}(\|I + tA\| - 1)$$

where I is the identity operator on X. (See also Theorem 2 of [10].) When $\lambda = \Lambda(A)$ is attained as a maximum, a nonlinear eigenvalue problem for λ is developed in [7]. Let $\rho(A)$ be the numerical radius of A. Then $\Lambda(A) \leq \rho(A)$. There are many papers on operators that attain their numerical radii. In some cases the set of such numerical radius attaining operators is norm-dense in the operator space of X; see [1], [2], and [12]. The reference for background is [3].

1. $\Lambda(A)$ as a maximum

We begin with an example essentially due to Halmos in Problem 168, p. 111, of [9]. Let $(e_j)_{j=1}^\infty$ be the standard basis of l_p, $1 < p < \infty$. Let A be the operator on l_p whose matrix relative to this basis is diagonal with entries $-1/n^2$, $n = 1, 2, \ldots$. One sees that A is compact and $W(A)$ is the semi-open interval $[-1, 0)$. Hence $\Lambda(A) = 0$ and it is not attained as a maximum.

It is shown in [5] that if A is a compact operator on a Hilbert space and $0 \in W(A)$, then $W(A)$ is closed. So in this case $\Lambda(A)$ is attained as a maximum. Even though our methods work for l_2 we shall slight this case because our results are not new for Hilbert spaces. On the other hand no extra effort is required to prove our main result for measure spaces, (Ω, Σ, μ), in which every measurable subset of finite positive measure contains an atom. We recall that a measurable set E is an *atom* in case $\mu(E) > 0$ and for every measurable subset F of E, either $\mu(F) = 0$ or $\mu(E - F) = 0$. The following lemma can be proved by imitating the proof of the Bolzano-Weierstrass theorem.

Lemma 1.1. *Any measurable function on an atom is constant almost everywhere.*

Lemma 1.2. *Let $X = L_p(\Omega, \Sigma, \mu)$, $1 < p < \infty$. Suppose that every measurable subset of finite positive measure contains an atom. Then X has the following property: If $(x_k)_{k=1}^\infty$ is a sequence in X which converges weakly to x_0, then $(K_p(x_k))_{k=1}^\infty$ converges weakly to $K_p(x_0)$.*

Proof. We use the FACT that $(x_k)_{k=1}^\infty$ converges weakly to x_0 if and only if

$$\sup(\|x_k\|)_{k=1}^\infty < \infty \tag{10}$$

and for every $E \in \Sigma$ with $0 < \mu(E) < \infty$

$$\lim_{k\to\infty} \int_E x_k \, d\mu = \int_E x_0 \, d\mu. \tag{11}$$

See Ex. IV, 13.24 and Corollary III, 3.8 of [6].

Conditions (10) and (3) imply that

$$\sup(\|K_p(x_k)\|_q)_{k=1}^{\infty} = M < \infty. \tag{12}$$

Let $E \in \Sigma$ with $0 < \mu(E) < \infty$. We first show that in verifying that the $K(x_k)$ satisfy the analogue of (11) we may replace E by $E' = \bigcup_{j=1}^{\infty} F_j$, where $E' \subseteq E$ and F_j is either an atom or the empty set, and $\mu(E') = \mu(E)$: by hypothesis E contains an atom E_1. If $\mu(E_1) = \mu(E)$, then $E' = E_1$. Otherwise by Zorn's lemma there exists a maximal disjoint family $\{F_j : j \in \Gamma\}$ of atoms contained in E. Let $\Gamma_n = \{j \in \Gamma : \mu(F_j) > 1/n\}$, $n = 1, 2, \dots$. As the cardinality of Γ_n cannot exceed the finite number $n\mu(E)$ and $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, we conclude that Γ is countable, taking it to be the set of positive integers. Let $G = E - \bigcup_{j=1}^{\infty} F_j$. If $\mu(G) > 0$, then by our assumption, G contains an atom contradicting the maximality of $\{F_j : j \in \Gamma\}$. Thus $\mu(G) = 0$. Therefore in verifying the analogue of (11) for the $K(x_k)$ we may replace E by $\bigcup_{j=1}^{\infty} F_j$, where we allow the possibility that some F_j's are empty. By Lemma 1.1, x_k is a constant α_{kj} on F_j for each j. If F_j is empty, put $\alpha_{kj} = 0$. Thus we have for $k = 0, 1, \dots$

$$x_k = \sum_{j=1}^{\infty} \alpha_{kj} \chi_{F_j}. \tag{13}$$

Since x_k is integrable on E, $\alpha_{kj} < \infty$. Applying the FACT to each F_j, we obtain for $j = 1, 2, \dots$

$$\lim_{k\to\infty} \alpha_{kj} = \alpha_{0j}. \tag{14}$$

Now we can prove that for E as above,

$$\lim_{k\to\infty} \int_E K(x_k) \, d\mu = \int_E K(x_0) \, d\mu \tag{15}$$

($K = K_p$). Let $\varepsilon > 0$ be given. Since $\mu(E) < \infty$, we can choose l such that

$$\mu\left(\bigcup_{j=l+1}^{\infty} F_j\right) < (\varepsilon/(4M))^p. \tag{16}$$

Since K is a continuous function on $\mathbf{C}$ into $\mathbf{C}$, it follows from (14) that we can choose $m \geq 1$ so that for $j = 1, 2, \dots, l$ and all $k \geq m$

$$|K(\alpha_{kj}) - K(\alpha_{0j})| < \varepsilon/(2l\mu(F_j)). \tag{17}$$

Put $H = \bigcup_{j=l+1}^{\infty} F_j$. For all $k = 1, 2, \dots$ we have

$$\int_E (K(x_k) - K(x_0)) \, d\mu = \int_H (K(x_k) - K(x_0)) \, d\mu + \sum_{j=1}^{l} (K(\alpha_{kj}) - K(\alpha_{0j}))\mu(F_j).$$

Hence from (17),

$$\left| \int_E (K(x_k) - K(x_0)) \, d\mu \right| \leq \varepsilon/2 + \int_H |K(x_k)| \, d\mu + \int_H |K(x_0)| \, d\mu. \tag{18}$$

From Hölder's inequality we get that for $k = 0, 1, 2, \ldots$

$$\int_H |K(x_k)|\, d\mu \le \left(\int_H |K(x_k)|^q\right)^{1/q} \left(\int_H 1^p\right)^{1/p}$$
$$\le \|K(x_k)\|_q \mu(H)^{1/p} \le M \cdot \varepsilon/(4M) = \varepsilon/4.$$

The last inequality is justified by (12) and (16). Therefore, we have from (18) that

$$\left|\int_E (K(x_k) - K(x_0))\, d\mu\right| < \varepsilon \quad \text{for } k \ge m.$$

This proves (15). By the FACT, (12) and (15) imply the required weak convergence of $K(x_k)$. □

Before applying Lemma 1.2 let us show the necessity of the restriction on the measure space.

Proposition 1.3. *Let $X = L_p(\Omega, \Sigma, \mu)$, where $1 < p < \infty$, $p \ne 2$. Suppose that Ω contains a measurable set E of finite positive measure such that E contains no atom. Then there exists a sequence $(x_k)_{k=1}^\infty$ in X converging weakly to x_0 such that $(K_p(x_k))_{k=1}^\infty$ converges weakly to an element distinct from $K_p(x_0)$.*

Proof. Since E does not contain an atom, it follows from [8, p. 174] that E can be partitioned into a disjoint union of two measurable subsets $E(1)$ and $E(2)$ with $\mu(E(1)) = \mu(E(2)) = \mu(E)/2$. These sets satisfy the same hypothesis as E. Thus we find inductively a sequence of partitions of E, $\{E(j_1, j_2, \ldots, j_k) : j_m = 1$ or $2\}$, $k = 1, 2, \ldots$, where the measure of a subset with k indices is $2^{-k}\mu(E)$ and $E(j_1, \ldots, j_k)$ is the disjoint union of $E(j_1, \ldots, j_k, 1)$ and $E(j_1, \ldots, j_k, 2)$.

For each positive integer k, let y_k be the function on E which has the value $(-1)^{j_1+\cdots+j_k}$ on $E(j_1, \ldots, j_k)$. Denote by Σ_1 the σ-field generated in E by all the sets $E(j_1, \ldots, j_k)$. Let μ_1 be the restriction of μ to Σ_1. The sequence $(y_k)_{k=1}^\infty$ is a bounded sequence in $L_p(E, \Sigma_1, \mu_1)$ because $\|y_k\|_p = \mu(E)^{1/p}$ for every k. If F is any of the sets $E(j_1, \ldots, j_m)$ and $k > m$, then $\int_F y_k\, d\mu = 0$. Thus,

$$\lim_{k\to\infty} \int_F y_k\, d\mu = 0. \tag{19}$$

The finite disjoint unions of the sets $E(j_1, \ldots, j_m)$ constitute a field Σ_0 in E, and (19) is valid for all $F \in \Sigma_0$. By [6, III, 8.3], the finite linear combinations of the characteristic functions of sets in Σ_0 are dense in $L_p(E, \Sigma_1, \mu_1)$. Hence it follows from [6, IV, 13.24] that $(y_k)_{k=1}^\infty$ converges weakly to 0 in $L_p(E, \Sigma_1, \mu_1)$. Now the map which sends a function f in $L_p(E, \Sigma_1, \mu_1)$ to its extension to Ω which vanishes on $\Omega - E$ is a linear isometry of $L_p(E, \Sigma_1, \mu_1)$ into $L_p(\Omega, \Sigma, \mu)$ (see, for example, [6, Lemma III, 8.1] and the following discussion). Denote the extension of y_k also by y_k. So $(y_k)_{k=1}^\infty$ converges weakly to 0 in $L_p(\Omega, \Sigma, \mu)$. Replacing p by q we get the same conclusion for $L_q(\Omega, \Sigma, \mu)$.

Put $x_k = y_k + \chi_E$, $k = 1, 2, \ldots$. Then we have that $(x_k)_{k=1}^\infty$ converges weakly in $L_p(\Omega, \Sigma, \mu)$ to χ_E. On the other hand, $K_p(x_k) = 2^{p-2}(y_k + \chi_E)$ converges weakly to $2^{p-2}\chi_E$ which is distinct from $K_p(\chi_E) = \chi_E$. □

Theorem 1.4. *Let $X = L_p(\Omega, \Sigma, \mu)$ where $1 < p < \infty$. Suppose that every measurable subset of Ω of finite positive measure contains an atom. Let A be a compact operator on X with $\Lambda(A) \ne 0$. Then there exists a vector x in $S(X)$ such that $\operatorname{Re}\psi_A(x) = \Lambda(A)$.*

Proof. If X is finite dimensional, then $S(X)$ is compact, by [6, p. 245]. Since $\operatorname{Re}\psi_A$ is continuous on $S(X)$, the conclusion holds in this case. If X is infinite dimensional, then its identity operator is not compact and therefore A cannot have a bounded inverse on X. So 0 is in the spectrum of A. According to [15, p. 217], the spectrum of a bounded linear operator, A, is contained in the closure of $W(A)$. Thus in the present case $\Lambda(A) > 0$.

Given ε such that $0 < \varepsilon < \Lambda(A)$, there exists $y \in S(X)$ with $\Lambda(A) - \varepsilon \leq \operatorname{Re}\psi_A(y)$. Put $\operatorname{Re}\psi_A(y) = \alpha > 0$ and $z = \alpha^{-1/p}y$. As K_p is positively homogeneous of degree $p-1$, ψ_A is positively homogeneous of degree p. Thus $\operatorname{Re}\psi_A(z) = 1$, and the set $E = \{u \in X : \operatorname{Re}\psi_A(u) = 1\}$ is not empty. If we put

$$\inf\{\|u\| : u \in E\} = \mu \tag{20}$$

we have $\mu \leq \|z\| = \alpha^{-1/p} \leq (\Lambda(A) - \varepsilon)^{-1/p}$. Letting ε tend to 0, we get

$$\mu \leq \Lambda(A)^{-1/p}. \tag{21}$$

We can obtain a sequence $(x_k)_{k=1}^{\infty}$ in E such that $\lim_{k\to\infty} \|x_k\| = \mu$. Since X is reflexive, the bounded sequence $(x_k)_{k=1}^{\infty}$ has a subsequence that converges weakly to some function x_0; see Lemma 28, p. 68 of [6]. We continue to denote this subsequence by $(x_k)_{k=1}^{\infty}$. For every $k = 1, 2, \dots$ we have

$$|\operatorname{Re}\psi_A(x_0) - 1| = |\operatorname{Re}\langle Ax_0, K(x_0)\rangle - \operatorname{Re}\langle Ax_k, K(x_k)\rangle| = R, \tag{22}$$

$$R \leq |\langle Ax_0, K(x_0)\rangle - \langle Ax_k, K(x_k)\rangle| = S, \tag{23}$$

$$S \leq |\langle Ax_0, K(x_0) - K(x_k)\rangle| + |\langle Ax_0 - Ax_k, K(x_k)\rangle|. \tag{24}$$

By Lemma 1.2, $K(x_0) - K(x_k)$ converges weakly to 0 as $k \to \infty$, which shows that the first summand on the right-hand side of (24) tends to 0. Since A is compact, $Ax_k \to Ax_0$ strongly. Hence the second summand of (24), which by Hölder's inequality and (3) is dominated by $\|Ax_0 - Ax_k\|_p \|x\|_p^{p-1}$, tends to 0 too. Therefore $x_0 \in E$. So $x_0 \neq 0$ and by (20), $\|x_0\| \geq \mu$. Since $(x_k)_{k=1}^{\infty}$ converges weakly to x_0, we have that $\|x_0\| \leq \lim_{k\to\infty} \inf \|x_k\|$; see Lemma 27, p. 68 of [6]. We get from $\lim_{k\to\infty} \|x_k\| = \mu$ that $\|x_0\| \leq \mu$. Hence, $\|x_0\| = \mu$ and $\mu > 0$. Put $x = \mu^{-1}(x_0)$ so that $x \in S(X)$. We have that $\operatorname{Re}\psi_A(x) = \mu^{-p}\operatorname{Re}\psi(x_0) = \mu^{-p} \geq \Lambda(A)$, where the last inequality comes from (21). From (9) we now conclude that $\operatorname{Re}\psi_A(x) = \Lambda(A)$. ☐

Remark. (a) We do not know whether the restriction on the measure, which is necessary in Lemma 1.2, is also necessary in Theorem 1.4.

(b) There exist noncompact operators B on l_p with $\Lambda(B) > 0$ for which $\Lambda(B)$ is attained as the maximum of $\operatorname{Re}\psi_B$ in $S(l_p)$, and also such that it is not attained. For the former it suffices to take $B = A + \alpha I$, where $\alpha > 0$ and A satisfies the assumptions of Theorem 1.4. Examples of such A abound, see for instance [5]. For the latter take $B = A + \alpha I$, where $\alpha > 0$ and A is the operator mentioned in the beginning of the section.

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