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ABSOLUTE FIXED POINT SETS FOR CONTINUUM-VALUED MAPS

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ABSTRACT. The notion of an absolute fixed point set in the setting of continuum-valued maps will be defined and characterized.

1. INTRODUCTION

The notion of an absolute fixed point set, in the context of single-valued maps, was defined and studied in [3] and [4]. Except for a few special cases, intrinsic characterizations of absolute fixed point sets have yet to be determined even in the setting of compact metric spaces. In this paper, we define the notion of an absolute fixed point set in the context of continuum-valued maps and obtain a completely inherent characterization for the notion. We will state our theorem in 1.1 after giving the pertinent terminology and notation.

A *compactum* is a nonempty compact metric space, and a *continuum* is a connected compactum. If Z is a compactum, then $C(Z)$ denotes the space of all subcontinua of Z with the Hausdorff metric [8]. A *map* is a continuous function, and a map into some $C(Z)$ will be referred to as a *continuum-valued map*.

A *fixed point of a continuum-valued map* F is a point, p, such that $p \in F(p)$; the *fixed point set of* F is the set of all fixed points of F and is denoted by $\mathcal{F}(F)$. We want to be able to talk about a fixed point set of a compactum, but there could easily be confusion over the type of map involved. Hence, for a compactum Z, we use the phrase *multi-valued fixed point set of* Z to refer to a subset, A, of Z such that A is the fixed point set of some continuum-valued map $F : Z \to C(Z)$. We remark that conditions under which every nonempty closed subset of a continuum, X, is a multi-valued fixed point set of X were investigated in [2].

The notion that we study may now be defined as follows. A *multi-valued absolute fixed point set* (MAFS) is a compactum, A, such that whenever A is embedded as a subspace, A', of any compactum, Z, then A' is a multi-valued fixed point set of Z.

Now, we state the characterization that we will prove.

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1.1 **Theorem.** *Let A be a compactum. Then, A is a MAFS if and only if A contains only finitely many components, all but at most one of which are locally connected.*

We assume that the reader is familiar with basic results and notions from continuum theory and from the theory of hyperspaces ([7] and [8]). We point out that the phrase *order arc* refers to the hyperspace notion as discussed in [8, pp. 57-81]. In this paper we adopt the point of view that for a continuum, X, an *order arc in* $C(X)$ is a homeomorphism $\alpha : [0,1] \to C(X)$ such that

$$\alpha(t_1) \subset \alpha(t_2) \text{ whenever } 0 \leq t_1 \leq t_2 \leq 1.$$

In addition to the hyperspace $C(Z)$ defined above, we will use the hyperspace 2^Z of all compacta in Z (with the Hausdorff metric).

2. Proof that the condition in 1.1 is sufficient for A being a MAFS

We will prove this in 2.5. It will be convenient to have the following lemma.

2.1 **Lemma.** *Let Y be a Peano continuum, and let L be a compactum in Y. Then there is a map $M : Y \to 2^Y$ such that $\mathcal{F}(M) = Y$ and $M(y) = L$ for each $y \in L$.*

Proof. Let ρ be a convex metric for Y ([1] or [5]). Define $M : Y \to 2^Y$ as follows: for each $y \in Y$, let $M(y)$ be the generalized closed ρ-ball about L of radius $\rho(y, L)$ where $\rho(y, L) = \inf\{\rho(y, l) : l \in L\}$; in other words,

$$M(y) = \{z \in Y : \rho(z, L) \leq \rho(y, L)\}.$$

It follows easily from properties of convex metrics that M is continuous (actually, for each $y \in Y$, $M(y) = \varphi_\rho(\rho(y, L), L)$ where φ_ρ is as in [6, 3.4]). Also, it is evident that $\mathcal{F}(M) = Y$ and that $M(y) = L$ for each $y \in L$. Therefore, we have proved 2.1.

2.2 *Remark.* The range of M in 2.1 is a Peano continuum in 2^Y (since Y is a Peano continuum and M is continuous).

The following result (lemma 2.3 of [2]) is the primary tool used in the proof of 2.4.

2.3 **Lemma.** *Let X be a continuum and let Y be a proper subcontinuum of X. Let $G : Y \to C(Y)$ be a mapping with $\mathcal{F}(G) = A$ (possibly, $A = \varnothing$) such that $\{G(y) : y \in Y\}$ is contained in a Peano subcontinuum of $C(Y)$. Let $\alpha : [0,1] \to C(X)$ be an order arc in $C(X)$ from Y to X, and let K be a compact subset of $X - \bigcup\{\alpha(t) : t < 1\}$. Then, G can be extended to a mapping $F : X \to C(X)$ such that $\mathcal{F}(F) = A \cup K$, $F(x) = X$ for all $x \in K$, and $\{F(x) : x \in X\}$ is contained in a Peano subcontinuum of $C(X)$.*

The following lemma proves the sufficiency of the condition in 1.1 for the case of embeddings in continua.

2.4 **Lemma.** *Let X be a continuum. Let A be a compactum in X such that A has only finitely many components $A_1, ..., A_n$. If A_i is locally connected for each $i \geq 2$, then A is the fixed point set of some map $F : X \to C(X)$. Furthermore, the range of F is contained in a Peano subcontinuum of $C(X)$.*

Proof. The proposition is evident if $n = 1$ (by defining $F(x) = A_1$ for each $x \in X$). Hence, for the purpose of proof, we assume that $n \geq 2$. Let $G : A_1 \to C(A_1)$ be the constant map defined by letting $G(y) = A_1$ for each $y \in A_1$. Since $n \geq 2$, there is an order arc α_1 in $C(X)$ such that $\alpha_1(0) = A_1$, $\alpha_1(1) \cap (A - A_1) \neq \varnothing$, and $\alpha_1(t) \cap A = A_1$ when $t < 1$. Let

$$Z_1 = \alpha_1(1) \text{ and } K_1 = Z_1 \cap (A - A_1),$$

and note that $K_1 \subset Z_1 - \cup\{\alpha_1(t) : t < 1\}$. Hence, we may apply 2.3 to see that there is a map $G_1 : Z_1 \to C(Z_1)$ satisfying (1)-(3) below:

(1) $\mathcal{F}(G_1) = A_1 \cup K_1$;

(2) $G_1(z) = Z_1$ for all $z \in K_1$;

(3) $\mathcal{G}_1 = \{G_1(z) : z \in Z_1\}$ is contained in a Peano subcontinuum, $\mathcal{L}_1$, of $C(Z_1)$.

Now, let $I_1 = \{i \geq 2 : Z_1 \cap A_i \neq \varnothing\}$ (observe that $I_1 \neq \varnothing$ since $K_1 \neq \varnothing$). For each $i \in I_1$, A_i is a Peano continuum (by assumption). Hence, for each $i \in I_1$, 2.1 gives us a map $M_i : A_i \to 2^{A_i}$ such that $\mathcal{F}(M_i) = A_i$ and $M_i(y) = Z_1 \cap A_i$ for each $y \in Z_1 \cap A_i$. Now, letting $C_1 = \bigcup\{A_i : i \in I_1\}$, define F_1 on $Z_1 \cup C_1$ (which is a continuum) as follows:

$$F_1(y) = \begin{cases} G_1(y), & \text{if } y \in Z_1, \\ Z_1 \cup M_i(y), & \text{if } y \in A_i \text{ (for } i \in I_1). \end{cases}$$

Note that F_1 is a function by (2) and by the fact that, for each $y \in Z_1 \cap A_i$ $(i \in I_1)$, $M_i(y) \subset Z_1$. Also: F_1 is a map from $Z_1 \cup C_1$ into $C(Z_1 \cup C_1)$; $\mathcal{F}(F_1) = A_1 \cup C_1$ (use (1) and the fact that $\mathcal{F}(M_i) = A_i$ for each $i \in I_1$); and the range, $\mathcal{R}_1$, of F_1 is contained in a Peano subcontinuum of $C(Z_1 \cup C_1)$ (use (3) and 2.2, recalling that I_1 is finite). If $A_1 \cup C_1 \neq A$, then proceed as follows. Let α_2 be an order arc in $C(X)$ such that $\alpha_2(0) = Z_1 \cup C_1$, $\alpha_2(1) \cap [A - (A_1 \cup C_1)] \neq \varnothing$, and $\alpha_2(t) \cap A = A_1 \cup C_1$ when $t < 1$. Let

$$Z_2 = \alpha_2(1) \text{ and } K_2 = Z_2 \cap [A - (A_1 \cup C_1)],$$

and note that $K_2 \subset Z_2 - \cup\{\alpha_2(t) : t < 1\}$. Hence, recalling the properties of F_1, we may apply 2.3 again to obtain a map $G_2 : Z_2 \to C(Z_2)$ satisfying (4)-(6) below:

(4) $\mathcal{F}(G_2) = \mathcal{F}(F_1) \cup K_2 = A_1 \cup C_1 \cup K_2$;

(5) $G_2(z) = Z_2$ for all $z \in K_2$;

(6) $\mathcal{G}_2 = \{G_2(z) : z \in Z_2\}$ is contained in a Peano subcontinuum, $\mathcal{L}_2$, of $C(Z_2)$.

Now, let $I_2 = \{i \notin (I_1 \cup \{1\}) : Z_2 \cap A_i \neq \varnothing\}$. Repeat what we did before, that is: define $M_i : A_i \to 2^{A_i}$ for each $i \in I_2$ using 2.1; let $C_2 = \cup\{A_i : i \in I_2\}$; and define F_2 on $Z_2 \cup C_2$ (which is a continuum) by

$$F_2(y) = \begin{cases} G_2(y), & \text{if } y \in Z_2, \\ Z_2 \cup M_i(y), & \text{if } y \in A_i \text{ (for } i \in I_2). \end{cases}$$

Then (for reasons similar to the ones we gave for F_1, this time using (4)-(6)):F_2 is a map from $Z_2 \cup C_2$ into $C(Z_2 \cup C_2)$; $\mathcal{F}(F_2) = A_1 \cup C_1 \cup C_2$; and the range, $\mathcal{R}_2$, of F_2 is contained in a Peano subcontinuum of $C(Z_2 \cup C_2)$. Now, note that

$$\mathcal{F}(F_2) = \mathcal{F}(F_1) \cup C_2$$

and that $C_2 \neq \varnothing$ (since $K_2 \neq \varnothing$ and, thus, $I_2 \neq \varnothing$). Hence: not only does $\mathcal{F}(F_2)$ contain $\mathcal{F}(F_1)$, but $\mathcal{F}(F_2)$ contains at least one more component, A_i for $i \in I_2$, of

A than $\mathcal{F}(F_1)$ contains. Therefore, by repeating the above procedure only finitely many times, we arrive at a map

$$F_k : Z_k \cup C_k \to C(Z_k \cup C_k)$$

such that $\mathcal{F}(F_k) = A$ and the range, $\mathcal{R}_k$, of F_k is contained in a Peano subcontinuum of $C(Z_k \cup C_k)$ (where $Z_k \cup C_k$ is a subcontinuum of X). If $Z_k \cup C_k = X$, then we are done (by letting $F = F_k$). If $Z_k \cup C_k \neq X$, then (because of the properties of F_k) we may apply 2.3 one last time to extend F_k to a map $F : X \to C(X)$ such that $\mathcal{F}(F) = A$ and the range, $\mathcal{R}$, of F is contained in a Peano subcontinuum of $C(X)$ (note that this time we are applying 2.3 with $K = \varnothing$). This completes the proof of 2.4.

2.5 **Proposition.** *Let Z be a compactum. Let A be a compactum in Z such that A contains only finitely many components, all but at most one of which are locally connected. Then A is the fixed point set of some map $F : Z \to C(Z)$.*

Proof. Let K be a component of Z such that $K \cap A \neq \varnothing$. Let E be a closed and open subset of Z such that $E \cap A = K \cap A$. By 2.4, there is a map $G : K \to C(K)$ for which $\mathcal{F}(G) = K \cap A$ and such that the range of G is contained in a Peano subcontinuum, $\mathcal{L}$, of $C(K)$. Thus, since $C(\mathcal{L})$ is an absolute retract [8, p. 136], we may extend G to a map $\bar{G} : E \to C(\mathcal{L})$. We may repeat this procedure for each of the finitely many components, K_i, of Z intersecting A ($i = 1, ..., m$), and we may do so using mutually disjoint closed and open sets, E_i, for which $\bigcup_{i=1}^{m} E_i = Z$. We thereby obtain a map, G_i, defined on E_i such that $\mathcal{F}(G_i) = K_i \cap A$ for each $i = 1, ..., m$. It follows that the map $F : Z \to C(Z)$ given by

$$F(z) = G_i(z) \text{ if } z \in E_i$$

has the required properties. This proves 2.5.

3. Proof that every MAFS satisfies the condition in 1.1

We will prove this in 3.3 and 3.4.

3.1 **Lemma.** *Every retract of a MAFS is a MAFS.*

Proof. Let Y be a MAFS, and let A be a retract of Y. Let Z be a compactum containing a copy, A', of A. Let h be a homeomorphism from A onto A'. Now, assuming as we may that $Z \cap Y = \varnothing$, form the attaching space $Z \cup_h Y = W$ [7, 3.18], in which we consider Z, Y, and $A = A'$ as subspaces in the natural way. Note that W is a compactum [7, 3.19]. Thus, since Y is a MAFS, there is a map $G : W \to C(W)$ such that $\mathcal{F}(G) = Y$. Let r be a retraction from Y onto $A = A'$ (r exists by assumption), and let $\bar{r}$ be the retraction from W onto Z given by

$$\bar{r}(z) = \begin{cases} z, & \text{if } z \in Z, \\ r(z), & \text{if } z \in Y. \end{cases}$$

Finally, define $F : Z \to C(Z)$ by $F(z) = \bar{r}[G(z)]$ for each $z \in Z$. It is easy to see that F is a map and that $\mathcal{F}(F) = A'$. Therefore, we have proved that A is a MAFS. This proves 3.1.

3.2 **Lemma.** *Let A be a compactum with infinitely many components. Let $S = \{0\} \cup \{1/n : n = 1, 2, ...\}$ (with its usual topology). Then S can be embedded in A as a retract of A.*

Proof. Let $p \in A$ such that there is a sequence $\{p_i\}_{i=1}^{\infty}$ of points p_i of distinct components C_i of A with $p_i \to p$. Let C denote the component of A containing p. Let $\{H_j\}_{j=1}^{\infty}$ be a sequence of simultaneously closed and open subsets of A such that $\bigcap_{j=1}^{\infty} H_j = C$ [7, p. 82]. In addition, we assume that $H_1 = A$ and that, for each j, $H_j \supset H_{j+1}$ and $p_{i(j)} \in H_j - H_{j+1}$ for some $i(j)$. Then, for each $j = 1, 2, ...$, let r_j be the constant map defined on $H_j - H_{j+1}$ by letting $r_j(a) = p_{i(j)}$ for all $a \in H_j - H_{j+1}$. Now, define r on A as follows:

$$r(a) = \begin{cases} r_j(a), & \text{if } a \in H_j - H_{j+1}, \\ p, & \text{if } a \in C. \end{cases}$$

Evidently, r is a map. Furthermore, letting $S' = \{p\} \cup \{p_{i(j)} : j = 1, 2, ...\}$, we see that r is a retraction from A onto S'. Therefore, since S' is clearly homeomorphic to S, we have proved 3.2.

3.3 **Proposition.** *If A is a MAFS, then A has only finitely many components.*

Proof. Assume that A is a MAFS, and suppose that A has infinitely many components. Then, by combining 3.1 and 3.2, we see that S in 3.2 is a MAFS. But, the example in [2, 2.8] shows that S is not a MAFS. This proves 3.3.

The idea for the proof of the following proposition comes from Example 2.7 of [2, p. 90].

3.4 **Proposition.** *If A is a MAFS, then all but at most one component of A must be locally connected.*

Proof. By 3.1 and 3.3, it suffices to prove the following: $(*)$ If K is the disjoint union of two nonlocally connected continua, K_1 and K_2, then K is not a MAFS.

We prove $(*)$ by making use of the auxiliary continuum, Y, defined (in the plane) as follows: $Y = W_1 \cup J_1 \cup W_2 \cup J_2$ where

$$W_1 = \{(x, \sin(1/x)) : 0 < x \leq 1\}, \quad J_1 = \{(0, y) : -1 \leq y \leq 1\},$$

$$W_2 = \{(2 - x, y) : (x, y) \in W_1\}, \quad J_2 = \{(2, y) : -1 \leq y \leq 1\}.$$

Now, let $K = K_1 \cup K_2$ be as in $(*)$. For each $i = 1$ and 2, let $p^i \in K_i$ such that K_i is not connected im kleinen (cik) at p^i. Let X be the continuum obtained from the disjoint union of K and Y by identifying p^1 with $(0, -1)$ and p^2 with $(2, -1)$. Since K_i is not cik at p^i, there is an $\varepsilon > 0$ and a sequence $\{q_j^i\}_{j=1}^{\infty}$ in K_i such that $q_j^i \to p^i$ as $j \to \infty$ and such that

(1) any subcontinuum of K_i containing p^i and q_j^i for some j must be of diameter $\geq \varepsilon$.

Now, suppose that there is a map $F : X \to C(X)$ such that $\mathcal{F}(F) = K_1 \cup K_2$. Then, since $w \notin F(w)$ for any $w \in W_1 \cup W_2$, we may assume without loss of generality that

(2) w lies to the left of $F(w)$ for each $w \in W_1 \cup W_2$.

Since $p^1 \in F(p^1)$, we see from (2) and the continuity of F that

(3) $F(p^1) \cap K_1 = \{p^1\}$.

Now, suppose that $F(p^1) \neq \{p^1\}$. Then, by (3), $F(p^1)$ is a nondegenerate subcontinuum of $Y \cup K_2$. Thus, since $q_j^1 \to p^1$ and $q_j^1 \in F(q_j^1)$, there exists N such that $p^1 \in F(q_j^1)$ for all $j \geq N$. Hence, observing that the intersection of any subcontinuum of X with K_1 is connected, we have that $F(q_j^1) \cap K_1$ is a continuum containing p^1 and q_j^1 for all $j \geq N$. Thus, by (1), $F(q_j^1) \cap K_1$ has diameter $\geq \varepsilon$ for all $j \geq N$. Hence, since $q_j^1 \to p^1$ and F is continuous, we see that $F(p^1) \cap K_1$ has diameter $\geq \varepsilon$, which contradicts (3). Therefore, we have proved that

(4) $F(p^1) = \{p^1\}$.

Since $\mathcal{F}(F) = K_1 \cup K_2$, it follows using (4) that $(0,1) \notin F(y)$ for any $y \in J_1$. Hence, by (2), we have that

(5) $(0,1) \notin F(y)$ for any $y \in J_1 \cup W_1$.

Now, let

$$M = \bigcup\{F(y) : y \in J_1 \cup W_1\}.$$

Then: M is a continuum (by the continuity of F and [8, 1.49]); $p^1 \in M$ (by (4)); and $M \cap W_1 \neq \varnothing$ (by (2), (4), and the continuity of F). Hence, it must be true that $(0,1) \in M$. But, this contradicts (5). Thus, our supposition immediately following (1) is false. Hence, we have proved $(*)$. Therefore, we have proved 3.4.

The theorem in 1.1 has now been proved by 2.5, 3.3, and 3.4.

3.5 *Remark.* Suppose that we revise the definition of a MAFS by only requiring that the embeddings of A be in continua. Then, 2.5 and the proofs in section 3 show that this revised definition is equivalent to the original one.

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